Step potential and Ramsauer-Townsend effect in Wigner-Dunkl quantum mechanics

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In this paper the continuity equation for Wigner-Dunkl-Schrödinger equation is studied. Some properties of ν-deformed functions related to Dunkl derivative are also studied. Based on these, the step potential and Ramsauer-Townsend effect are discussed in Wigner-Dunkl quantum mechanics.

Keywords: Wigner-Dunkl-Schrödinger equation; Ramsauer-Townsend effect.

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1. Introduction

In 1950 Wigner [1] proposed a deformed Heisenberg algebra including reflection operator in the form,

$$[\hat{x},\hat{p}] = i(1+2\nu P), \tag{1}$$

where P denotes the reflection operator $Pf(x) = f(-x)$. The realization of this algebra [2,3] was given by

$$\hat{p} = \frac{1}{i}D_x, \quad \hat{x} = x, \tag{2}$$

where we set $\hbar = 1$ and the Dunkl derivative [3] is defined as

$$D_x = \partial_x + \frac{\nu}{x}(1-P). \tag{3}$$

In fact, Wigner was the first who discussed the question on supreme level of the Heisenberg-Lie equations on the commutation relation between the momentum and position operator. In 1951 Yang simply developed the problem treated by Wigner and obtained the well-known non-canonical description of the non-relativistic momentum operator [2]. In 1980 N. Mukunda et al investigated Energy position and momentum eigenstates of para-Bose oscillator operators. They found that the two apparently different solutions obtained by Ohnuki and Kamefuchi in this context are actually unitarily equivalent [3]. Also in next year coherent states and the minimum uncertainty states of para-Bose oscillator operators investigate by J. K. Sharma *et al.* [4]. In 1989 Dunkl constructed a commutative set of first-order differential difference operators associated to the second-order operator [5]. The canonical approach to the non-relativistic quantum mechanics with $\nu = 0$ is a special case of the non-canonical approach for arbitrary ν that was proposed by Wigner. Reference [6] shows the Wigner function of the ground state for the para-Bose oscillator (it is an oscillator that was discussed in Refs. [3,4]) that generalizes Gaussian distribution almost overlaps with the Wigner function of the canonical non-relativistic bosonic harmonic oscillator. If there is overlap of the Wigner function of non-canonical ground state with the any excited canonical state, then it means that not only Ramsauer-Townsend effect, but also a lot of other physical effects can be obtained theoretically if to replace canonical momentum operator with non-canonical one. Some studies related to the Wigner-Dunkl quantum mechanics have been accomplished in Refs. [7-12].

In one-dimensional Wigner-Dunkl quantum mechanics, the inner product is given by [7,12]

$$\langle f|g\rangle = \int_{-\infty}^{\infty} g^*(x)f(x)|x|^{2\nu}dx, \tag{4}$$

where $|x|^{2\nu}$ is a weight function. The expectation value of a physical operator $\mathcal{O}$ with respect to the state $\psi(x,t)$ is defined by

$$\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\mathcal{O}\psi(x,t)|x|^{2\nu}dx, \tag{5}$$

and $\mathcal{O}$ is a Hermitian operator if it obeys

$$\langle\mathcal{O}\psi|\psi\rangle = \langle\psi|\mathcal{O}\psi\rangle. \tag{6}$$

For the weight function (6) the momentum operator $\hat{p} = 1/iD_x$ is a Hermitian operator.

In this paper we study the continuity Equation for Wigner-Dunkl-Schrödinger equation. We discuss some properties of ν-deformed functions related to Dunkl derivative. Using these we discuss the step potential and Ramsauer-Townsend effect in Wigner-Dunkl quantum mechanics. This paper is organized as follows: In Sec. 2 we discuss continuity equation for Wigner-Dunkl-Schrödinger equation. In Sec. 3

we discuss the ν-deformed functions. In Sec. 4 we discuss Step potential. In Sec. 5 we discuss Ramsauer-Townsend effect.

2. Continuity equation for Wigner-Dunkl-Schrödinger equation

Now let us consider the time-dependent Wigner-Dunkl-Schrödinger equation,

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{1}{2m}D_x^2 + V(x)\right)\psi(x,t). \tag{7}$$

Now let us derive the continuity equation for the Wigner-Dunkl-Schrödinger equation. From Eq.(7) we have

$$i\frac{\partial|\psi(x,t)|^2}{\partial t} = \frac{1}{2m}\Big(\psi(x,t)D_x^2\psi^*(x,t) - \psi^*(x,t)D_x^2\psi(x,t)\Big). \tag{8}$$

The wave function can be split into the even part and odd part,

$$\psi_e = \frac{1}{2}(1+P)\psi = \frac{1}{2}\left(\psi(x)+\psi(-x)\right), \tag{9}$$

$$\psi_o = \frac{1}{2}(1-P)\psi = \frac{1}{2}\left(\psi(x)-\psi(-x)\right). \tag{10}$$

Now let us multiply the weight function $K(x) = |x|^{2\nu}$ by Eq.(8),

$$\frac{\partial K(x)|\psi(x,t)|^2}{\partial t} = \frac{1}{2mi}K(x)\Big(\psi(x,t)D_x^2\psi^*(x,t) - \psi^*(x,t)D_x^2\psi(x,t)\Big), \tag{11}$$

Let us set

$$\rho = K(x)|\psi(x,t)|^2, \tag{12}$$

where we set

$$\psi = \psi_e + \psi_o. \tag{13}$$

Then we have

$$\rho = \rho_e + \rho_o, \tag{14}$$

where

$$\rho_e = K(x)(|\psi_e|^2 + |\psi_o|^2), \tag{15}$$

$$\rho_o = K(x)(\psi_e\psi_o^* + \psi_o\psi_e^*). \tag{16}$$

Thus we have the continuity equation of the form,

$$\partial_t\rho = -\partial_x J + f(x), \tag{17}$$

where the flux is

$$J = \frac{1}{2mi}|x|^{2\nu}(\psi\partial_x\psi^* - \psi^*\partial_x\psi), \tag{18}$$

and the source is

$$f(x) = -\frac{\nu}{mix^2}|x|^{2\nu}(\psi_e^*\psi_o - \psi_e\psi_o^*). \tag{19}$$

The derivation of Eq.(17) is given in Appendix A. Then we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}|x|^{2\nu}|\psi|^2dx = 0, \tag{20}$$

where we have

$$\int_{-\infty}^{\infty}f(x)dx = 0, \tag{21}$$

because f is odd. Thus, $|x|^{2\nu}|\psi|^2$ can be interpreted as the probability density function.

Now let us consider the time-independent Wigner-Dunkl-Schrödinger equation,

$$\left(-\frac{1}{2m}D_x^2 + V(x)\right)\psi(x) = E\psi(x), \tag{22}$$

or

$$D_x^2\psi(x) = 2m(V(x)-E)\psi(x). \tag{23}$$

This means that $D_x^2\psi(x)$ is not continuous in general because the potential can be discontinuous.

To find the continuity condition in the Wigner-Dunkl-Schrödinger equation, we should find the inverse of Dunkl derivative. The Dunkl derivative can be written as

$$D_x = \left(1+\frac{\nu}{x}(1-P)\partial_x^{-1}\right)\partial_x. \tag{24}$$

The inverse of Dunkl derivative is

$$\begin{aligned} D_x^{-1} &= \partial_x^{-1}\left(1+\frac{\nu}{x}(1-P)\partial_x^{-1}\right)^{-1} \\ &= \partial_x^{-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\nu}{x}(1-P)\partial_x^{-1}\right)^n, \end{aligned} \tag{25}$$

where ∂_x^{-1} denotes the ordinary integration. Thus we have

$$D_x^{-1}x^N = \frac{x^{N+1}}{[N+1]_\nu}, \tag{26}$$

where Dunkl number is

$$[n]_\nu = n + \nu(1-(-1)^n). \tag{27}$$

Because the inverse of Dunkl derivative is expressed in terms of multiple of the ordinary integration, from Eq. (23) we know that both $D_x\psi$ and ψ are continuous.

3. The ν-deformed functions

Now let us find the ν-exponential function obeying

$$D_x e_\nu(ax) = a e_\nu(ax), \quad e_\nu(0) = 1. \tag{28}$$

We consider the ν-deformed differential equation

$$D_x y(x) = a y(x), \quad y(0) = 1. \tag{29}$$

Let us set

$$y(x) = y_e(x) + y_o(x), \tag{30}$$

where $y_e(x)$ is the even function obeying $P y_e(x) = y_e(x)$ while $y_o(x)$ is the odd function obeying $P y_o(x) = -y_o(x)$. Inserting Eq. (30) into Eq. (28) and splitting into the even part and odd part we get

$$\frac{dy_e(x)}{dx} = a y_o(x), \tag{31}$$

$$\frac{dy_o(x)}{dx} + \frac{2\nu}{x} y_o(x) = a y_e(x). \tag{32}$$

Let us set

$$y_e(x) = \sum_{n=0}^{\infty} a_n x^{2n}, \tag{33}$$

$$y_o(x) = \sum_{n=0}^{\infty} b_n x^{2n+1}. \tag{34}$$

Inserting Eqs. (33,34) into Eq. (31,32), we get

$$2(n+1)a_{n+1} = a b_n, \tag{35}$$

$$(2n+1+2\nu) b_n = a a_n. \tag{36}$$

From the above equations we have

$$a_{n+1} = \frac{a^2}{2(n+1)(2n+1+2\nu)} a_n. \tag{37}$$

Thus, we have

$$a_n = \frac{1}{n! \left(\nu + \frac{1}{2}\right)_n} \left(\frac{a}{2}\right)^{2n}, \tag{38}$$

$$b_n = \frac{1}{n! \left(\nu + \frac{1}{2}\right)_{n+1}} \left(\frac{a}{2}\right)^{2n+1}. \tag{39}$$

Thus, we have

$$y(x) = e_\nu(ax) = \cosh_\nu(ax) + \sinh_\nu(ax), \tag{40}$$

where

$$\begin{aligned} \cosh_\nu(ax) &= \sum_{n=0}^{\infty} \frac{1}{n! \left(\nu + \frac{1}{2}\right)_n} \left(\frac{ax}{2}\right)^{2n} \\ &= {}_0F_1\left(; \nu + \frac{1}{2}; \frac{a^2 x^2}{4}\right), \end{aligned} \tag{41}$$

$$\begin{aligned} \sinh_\nu(ax) &= \sum_{n=0}^{\infty} \frac{1}{n! \left(\nu + \frac{1}{2}\right)_{n+1}} \left(\frac{ax}{2}\right)^{2n+1} \\ &= \frac{ax}{2\nu+1} {}_0F_1\left(; \nu + \frac{3}{2}; \frac{a^2 x^2}{4}\right), \end{aligned} \tag{42}$$

and

$${}_0F_1(; a; x) = \sum_{n=0}^{\infty} \frac{1}{n! (a)_n} x^n, \tag{43}$$

and

$$(a)_0 = 1, \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1). \tag{44}$$

One can also express the ν-deformed hyperbolic functions as

$$\begin{aligned} \cosh_\nu(ax) &= \left(\frac{|ax|}{2}\right)^{-\nu+\frac{1}{2}} \\ &\quad \times \Gamma\left(\nu + \frac{1}{2}\right) I_{\nu-1/2}(|ax|), \end{aligned} \tag{45}$$

$$\begin{aligned} \sinh_\nu(ax) &= \frac{ax}{2}\left(\frac{|ax|}{2}\right)^{-\nu-\frac{1}{2}} \\ &\quad \times \Gamma\left(\nu + \frac{1}{2}\right) I_{\nu+1/2}(|ax|), \end{aligned} \tag{46}$$

where $I_\alpha(x)$ denotes the modified Bessel function. These deformed hyperbolic functions reduce to $\cosh(ax)$ and $\sinh(ax)$ in the limit $\nu \to 0$. The ν-deformed hyperbolic functions obey

$$P \cosh_\nu(ax) = \cosh_\nu(ax), \tag{47}$$

$$P \sinh_\nu(ax) = -\sinh_\nu(ax). \tag{48}$$

Action of the ν-derivative gives

$$D_x e_\nu(ax) = a e_\nu(ax), \tag{49}$$

$$D_x \cosh_\nu(ax) = a \sinh_\nu(ax), \tag{50}$$

$$D_x \sinh_\nu(ax) = a \cosh_\nu(ax). \tag{51}$$

If we replace $x \to ix$, we have the ν-deformed Euler relation

$$e_\nu(iax) = \cos_\nu(ax) + i \sin_\nu(ax), \tag{52}$$

where

$$\begin{aligned} \cos_\nu(ax) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \left(\nu + \frac{1}{2}\right)_n} \left(\frac{ax}{2}\right)^{2n} \\ &= {}_0F_1\left(; \nu + \frac{1}{2}; -\frac{a^2 x^2}{4}\right), \end{aligned} \tag{53}$$

$$\begin{aligned} \sin_\nu(ax) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \left(\nu + \frac{1}{2}\right)_{n+1}} \left(\frac{ax}{2}\right)^{2n+1} \\ &= \frac{ax}{2\nu+1} {}_0F_1\left(; \nu + \frac{3}{2}; -\frac{a^2 x^2}{4}\right). \end{aligned} \tag{54}$$

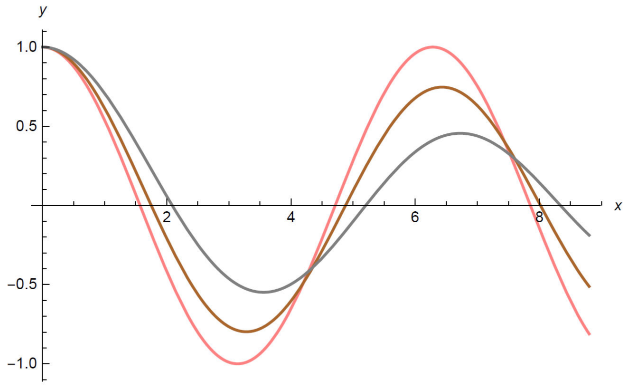

FIGURE 1. Plot of $y = \cos_\nu(x)$ for $\nu = 0$ (Pink), $\nu = 0.1$ (Brown) and for $\nu = -0.1$ (Gray).

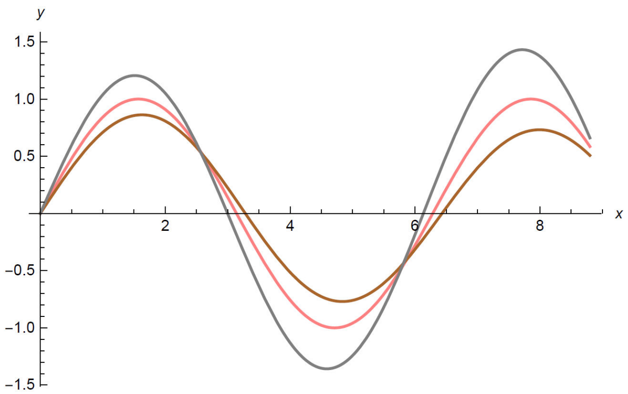

FIGURE 2. Plot of $y = \sin_\nu(x)$ for $\nu = 0$ (Pink), $\nu = 0.1$ (Brown) and for $\nu = -0.1$ (Gray).

One can also express the ν-deformed trigonometric functions as

$$\cos_\nu(ax) = \left(\frac{|ax|}{2}\right)^{-\nu+\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu-1/2}(|ax|), \quad (55)$$

$$\sin_\nu(ax) = \frac{ax}{2}\left(\frac{|ax|}{2}\right)^{-\nu-\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu+1/2}(|ax|), \quad (56)$$

where $J_\alpha(x)$ denotes the Bessel function. The ν-deformed trigonometric functions obey the following relations

$$D_x \cos_\nu(ax) = -a \sin_\nu(ax), \quad (57)$$

$$D_x \sin_\nu(ax) = a \cos_\nu(ax). \quad (58)$$

Figure 1 shows the plot of $y = \cos_\nu(x)$ for $\nu = 0$ (Pink), $\nu = 0.1$ (Brown) and for $\nu = -0.1$ (Gray). Figure 2 shows the plot of $y = \sin_\nu(x)$ for $\nu = 0$ (Pink), $\nu = 0.1$ (Brown) and for $\nu = -0.1$ (Gray).

4. Step potential

Now let us consider the step potential problem whose potential is given by

$$V(x) = \begin{cases} 0 & (x < 0) \\ V_0 & (x > 0) \end{cases} . \quad (59)$$

Now let us consider the case of $0 < V_0 < E$. Then, Wigner-Dunkl-Schrödinger equation reads

$$\left(-\frac{1}{2m} D_x^2\right) \psi_I(x) = E\psi_I(x), \quad (60)$$

for $x < 0$, while it reads

$$\left(-\frac{1}{2m} D_x^2 + V_0\right) \psi_{II}(x) = E\psi_{II}(x), \quad (61)$$

for $x > 0$. The solution is given by

$$\psi_I(x) = e_\nu(ik_0 x) + r e_\nu(-ik_0 x), \quad (62)$$

and

$$\psi_{II}(x) = t e_\nu(iqx), \quad (63)$$

where

$$k_0 = \sqrt{2mE}, \qquad q = \sqrt{2m(E - V_0)}. \quad (64)$$

From the continuity of ψ and $D_x\psi$ we have

$$1 + r = t, \qquad 1 - r = \frac{q}{k_0} t. \quad (65)$$

Thus we have

$$r = \frac{k_0 - q}{k_0 + q}, \qquad t = \frac{2k_0}{k_0 + q}, \quad (66)$$

which is the same as the case of $\nu = 0$. In this case the transmission and reflection flux are

$$R = |r|^2, \qquad T = \frac{q}{k_0} |t|^2. \quad (67)$$

We see that transmission and reflection flux are independent of the ν-deformed parameter. Figure 3 shows the plot of R and T versus E with $V_0 = 1$.

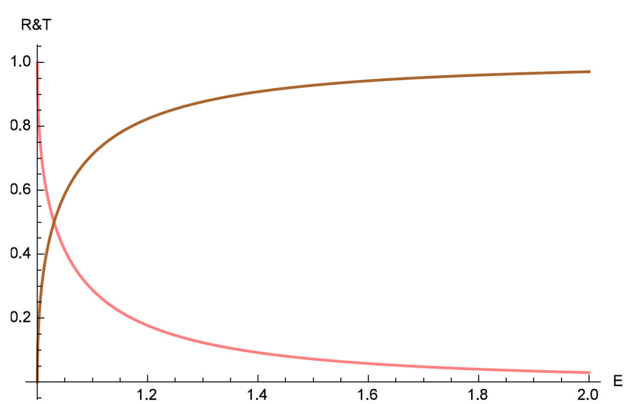

FIGURE 3. Plot of R and T versus E with$V_0 = 1$, (R:Pink, T:Brown).

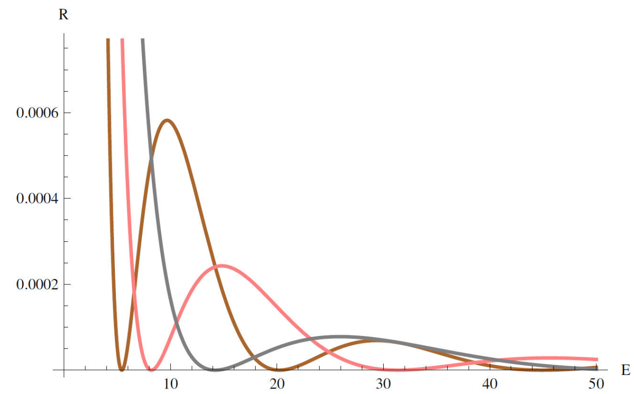

FIGURE 4. Plot of R versus E with $V_0 = 0.5, m = 1$ for $a = 0.8$ (Pink), $a = 1$ (Brown) and for $a = 0.6$ (Gray).

5. Ramsauer-Townsend effect

Let us consider the quantum well whose potential is given by

$$V(x) = \begin{cases} 0 \quad (x < 0, \quad \text{Region I}) \\ -V_0 \quad (0 < x < a, \quad \text{Region II}) \\ 0 \quad (x > a, \quad \text{Region III}) \end{cases}, \tag{68}$$

where V_0 is a positive constant and we assume $E > 0$.

Now let us consider the Wigner-Dunkl-Schrödinger equation for three cases:

$$-\frac{1}{2m} D_x^2 \psi_{\rm I} = E\psi_{\rm I}, \tag{69}$$

$$\left(-\frac{1}{2m} D_x^2 - V_0\right) \psi_{\rm II} = E\psi_{\rm II}, \tag{70}$$

$$-\frac{1}{2m} D_x^2 \psi_{\rm III} = E\psi_{\rm III}. \tag{71}$$

Solving three equations, we get

$$\psi_{\rm I} = e_\nu(ikx) + Ae_\nu(-ikx), \tag{72}$$

$$\psi_{\rm II} = Be_\nu(iqx) + Ce_\nu(-iqx), \tag{73}$$

$$\psi_{\rm III} = De_\nu(ikx), \tag{74}$$

where

$$k = \sqrt{2mE}, \qquad q = \sqrt{2m(E + V_0)}. \tag{75}$$

Now the boundary conditions are the continuity of the wave functions and their first Dunkl derivatives at the boundaries, which are

$$A - B - C = -1, \tag{76}$$

$$kA + qB - qC = k, \tag{77}$$

$$Be_\nu(iqa) + Ce_\nu(-iqa) = De_\nu(ika), \tag{78}$$

$$qBe_\nu(iqa) - qCe_\nu(-iqa) = kDe_\nu(ika). \tag{79}$$

Solving Eqs. (76-79) for A we get

$$A = \frac{i(k^2 - q^2)\sin_\nu(qa)}{-2qk\cos_\nu(qa) + i(k^2 + q^2)\sin_\nu(qa)}. \tag{80}$$

The reflection probability density is given by

$$\begin{aligned} R &= |A|^2 e_\nu(-ika)e_\nu(ika) \\ &= |A|^2(\sin_\nu^2(ka) + \cos_\nu^2(ka)), \end{aligned} \tag{81}$$

where

$$|A|^2 = \frac{(k^2 - q^2)^2 \sin_\nu^2(qa)}{(k^2 + q^2)^2 \sin_\nu^2(qa) + 4k^2q^2\cos_\nu^2(qa)}. \tag{82}$$

In Fig. 4 shows the behavior of the reflection probability density versus energy. Now let us investigate the Ramsauer-Townsend effect. This effect is a physical phenomenon involving the scattering of low-energy electrons by atoms of a noble gas. Ramsauer-Townsend effect is no reflection condition. From $|A|^2 = 0$, we have

$$\sin_\nu(qa) = 0. \tag{83}$$

Thus, we have

$$q = \frac{2\alpha_{\nu+1/2,p}}{a}, \quad p = 1, 2, \cdots, \tag{84}$$

where $\alpha_{\nu+1/2,p}$ denotes p-th zero of $J_{\nu+1/2}(x)$.

6. Conclusion

In this paper we derived the continuity equation for Wigner-Dunkl-Schrödinger equation. We found the flux and probability density. We found that the probability conserved in time. From the Dunkl integral (inverse of Dunkl derivative), we found that the wave function and first order Dunkl derivative are continuous although the potential is not continuous. We introduced the ν-deformed functions related to Dunkl derivative and investigated their mathematical properties. Using the continuity condition in the Wigner-Dunkl-Schrödinger equation we discussed two examples; the step potential problem and Ramsauer-Townsend effect.

Appendix A

In right side of Eq. (11), let us set

$$\begin{aligned} I = K(x)(&\psi(x,t)D_x^2\psi^*(x,t) \\ &- \psi^*(x,t)D_x^2\psi(x,t)). \end{aligned} \tag{A.1}$$

Then we have

$$
\begin{aligned}
I &= K\psi^*\left(\partial^2+\frac{2\nu}{x}\partial-\frac{\nu}{x^2}(1-P)\right)\psi - K\psi\left(\partial^2+\frac{2\nu}{x}\partial-\frac{\nu}{x^2}(1-P)\right)\psi^* = K(\psi_e^*+\psi_o^*)\\
&\times\left(\partial^2+\frac{2\nu}{x}\partial-\frac{\nu}{x^2}(1-P)\right)(\psi_e+\psi_o) - K(\psi_e+\psi_o)\left(\partial^2+\frac{2\nu}{x}\partial-\frac{\nu}{x^2}(1-P)\right)(\psi_e^*+\psi_o^*)\\
&= K\psi_e^*\left(\partial^2+\frac{2\nu}{x}\partial\right)\psi_e + K\psi_o^*\left(\partial^2+\frac{2\nu}{x}\partial\right)\psi_e + K\psi_e^*\left(\partial^2+\frac{2\nu}{x}\partial-\frac{2\nu}{x^2}\right)\psi_o\\
&+ K\psi_o^*\left(\partial^2+\frac{2\nu}{x}\partial-\frac{2\nu}{x^2}\right)\psi_o - K\psi_e\left(\partial^2+\frac{2\nu}{x}\partial\right)\psi_e^* - K\psi_o\left(\partial^2+\frac{2\nu}{x}\partial\right)\psi_e^*\\
&- K\psi_e\left(\partial^2+\frac{2\nu}{x}\partial-\frac{2\nu}{x^2}\right)\psi_o^* - K\psi_o\left(\partial^2+\frac{2\nu}{x}\partial-\frac{2\nu}{x^2}\right)\psi_o^*\\
&= -\partial_x\left(K(x)(\psi\partial_x\psi^*-\psi^*\partial_x\psi)\right) - \frac{2\nu}{x^2}K(x)(\psi_e^*\psi_o-\psi_e\psi_o^*),
\end{aligned} \tag{A.2}
$$

where we used

$$K'(x) = \frac{2\nu}{x}K(x). \tag{A.3}$$

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