$\mathrm{B-L}$ model with $\mathrm{D_5 \times Z_4}$ symmetry for lepton mass hierarchy and mixing

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We propose a gauge $B-L$ model with $D_5 \times Z_4$ for explaining the lepton mass and mixing through the type-I seesaw mechanism. The model can predict the neutrino masses and mixing angles including the Dirac and Majorana CP phases in good agreement with the experimental data. The model also predicts the effective neutrino parameters in highly consistent with the current constraints.

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1. Introduction

In particle physics, the hierarchy of fermion masses is one of the most exciting issues [1,2]. The basically experimental results related to flavour problem including the origin of charged-lepton mass hierarchy [3], $m_e : m_\mu : m_\tau \sim 10^{-4} : 10^{-1} : 1$, and the origins of two large and one small mixing angles of neutrinos, and of two neutrino squared mass differences [4], $7.50 \times 10^{-5}\,\mathrm{eV}^2 \sim \left(\Delta m^2_{21}\right)_{\rm bf} \ll \left(\Delta m^2_{31}\right)_{\rm bf} \sim 2.50 \times 10^{-3}\,\mathrm{eV}^2$.

A remarkable feature of discrete symmetries is that they can be combined with the Standard Model (SM) extensions to explain the neutrino mass and mixing data (see, for example, Refs. [5,8] and the references therein). D_5 symmetry [9] has been exploited in previous works [10-12] which differ from our current work[i] with the following basic properties:

(1) Reference [10] based on symmetry $SU(2)_L \times U(1)_Y \times U(1)_X \times D_5 \times T$ in which the first family of left handed lepton, right handed lepton and right handed neutrino are put in $\underline{1}_1$, the second and third families of left handed leptons are put in $\underline{2}_2$ while the second and third families of right handed leptons as well as right handed neutrinos are put in $\underline{2}_1$ of D_5. As a result, up to four $SU(2)_L$ doublet scalars are needed and only the normal mass hierarchy is satisfied.

(2) Reference [11] based on symmetry $SU(2)_L \times U(1)_Y \times U(1)_X \times D_5$ in which the first family of the left handed lepton is put in $\underline{1}_1$, the first family of the right handed lepton and right handed neutrino are put in $\underline{1}_2$, the two other families of left handed leptons, right handed leptons and right handed neutrinos are put in $\underline{2}_2$ of D_5. Furthermore, Ref. [10] also contains a non-minimal scalar sector with up to eight $SU(2)_L$ doublets and two singlets.

(3) Reference[12] based on symmetry $SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times D_5$ in which the first family of the left- and right-handed lepton are put in $\underline{1}_1$, the first family of the right handed neutrino is put in $\underline{1}_2$, the two other families of left handed leptons and right handed neutrinos are put in $\underline{2}_2$ while the two other families of right handed leptons are put in $\underline{2}_1$ of D_5. Furthermore, Ref. [12] contains up to six $SU(2)_L$ doublets and three singlets.

Thence, it is necessary to suggest another D_5-based model with fewer scalar fields and a simpler scalar potential. With such a motivation, we suggest a $B-L$ model with D_5 symmetry[ii] in which the first two generations of the left- and right-handed charged leptons ($\psi_{1,2L}, l_{1,2R}$) are assigned in $\underline{2}_1$ while the others (ψ_{3L}, l_{3R}) are in $\underline{1}_1$ of D_5. Furthermore, the first and second generations of the right-handed neutrinos are put in $\underline{1}_1$ while the third one is in $\underline{1}_2$ of D_5. As a consequence, only one $SU(2)_L$ doublets and six singlet scalars are needed to obtain the current neutrino mass and mixing data. As will be shown below, the considered model can also naturally explain the SM charged-lepton mass hierarchy problem.

The rest of this paper is as follows. In Sec. 2 we present the particle content of the model. Section 3 is devoted to the lepton sector. The numerical analysis is presented in Sec. 4. The conclusions are drawn in Sec. 5.

2. The model

The total symmetry of the model is $\mathbf{\Gamma} = G_{B-L} \times D_5 \times Z_4$ with G_{B-L} is the gauge symmetry of the $B-L$ model [13]. The particle content of the considered model under $\mathbf{\Gamma}$ symmetry is given in Tables I and II.

TABLE I. Lepton content of the model ($i = 1, 2$).

Fields	$\psi_{iL}\,(\psi_{3L})$	$l_{iR}\,(l_{3R})$	ν_{1R}, ν_{2R}	ν_{3R}
$\mathrm{SU}(2)_L$	**2**	**1**	**1**	**1**
$\mathrm{U}(1)_Y$	$-\frac{1}{2}$	-1	0	0
$\mathrm{U}(1)_{B-L}$	-1	-1	-1	-1
D_5	$\underline{2}_1\,(\underline{1}_1)$	$\underline{2}_1\,(\underline{1}_1)$	$\underline{1}_1$	$\underline{1}_2$
Z_4	$i\,(1)$	$1\,(1)$	$-i$	i

TABLE II. Scalar content of the model

Scalars	H	$\phi,\ \varphi,\ \eta$	ρ
$\mathrm{SU}(2)_L$	**2**	**1**	**1**
$\mathrm{U}(1)_Y$	$\frac{1}{2}$	0	0
$\mathrm{U}(1)_{B-L}$	0	0	2
D_5	$\underline{1}_1$	$\underline{1}_1,\ \underline{2}_2,\ \underline{2}_1$	$\underline{1}_1$
Z_4	1	$i,\ i,\ -1$	-1

The Yukawa interactions which are invariant under $\mathbf{\Gamma}$ symmetry take the form:

$$\begin{aligned}
-\mathcal{L}_Y^{lep} &= \frac{x_1}{\Lambda}(\bar{\psi}_{iL}l_{iR})_{\underline{1}_1}(H\phi)_{\underline{1}_1} + \frac{x_2}{\Lambda}\left(\bar{\psi}_{iL}l_{iR}\right)_{\underline{2}_2}(H\varphi)_{\underline{2}_2} \\
&+ x_3(\bar{\psi}_{3L}l_{3R})_{\underline{1}_1}H + \frac{y_1}{\Lambda}\left(\bar{\psi}_{iL}\nu_{1R}\right)_{\underline{2}_1}(\widetilde{H}\eta)_{\underline{2}_1} \\
&+ \frac{y_2}{\Lambda}\left(\bar{\psi}_{iL}\nu_{2R}\right)_{\underline{2}_1}(\widetilde{H}\eta)_{\underline{2}_1} + \frac{y_3}{\Lambda}\left(\bar{\psi}_{3L}\nu_{1R}\right)_{\underline{1}_1}(\widetilde{H}\phi)_{\underline{1}_1} \\
&+ \frac{y_4}{\Lambda}\left(\bar{\psi}_{3L}\nu_{2R}\right)_{\underline{1}_1}(\widetilde{H}\phi)_{\underline{1}_1} + \frac{z_1}{2}\left(\bar{\nu}^c_{1R}\nu_{1R}\right)_{1_1}\rho \\
&+ \frac{z_2}{2}\left(\bar{\nu}^c_{2R}\nu_{2R}\right)_{1_1}\rho + \frac{z_3}{2}\left(\bar{\nu}^c_{3R}\nu_{3R}\right)_{1_1}\rho \\
&+ \frac{z_4}{2}\left[\left(\bar{\nu}^c_{1R}\nu_{2R}\right)_{1_1}\rho + \left(\bar{\nu}^c_{2R}\nu_{1R}\right)_{1_1}\rho\right] + \text{H.c.} \qquad (1)
\end{aligned}$$

All other Yukawa terms, up to five-dimension, listed in Table III of the Appendix E are prevented by one or some of the model's symmetries; thus, they have not been included in the expression of $\mathcal{L}_Y^{lep}$ in Eq. (1).

The minimization condition of the Higgs potential, explicitly presented in Appendix B, provides a vacuum expectation value (VEV):

$$\begin{aligned}
&\langle H\rangle = (0\ v)^T, \quad \langle\phi\rangle = v_\phi, \quad \langle\varphi\rangle = (\langle\varphi_1\rangle,\ \langle\varphi_2\rangle), \\
&\langle\varphi_{1,2}\rangle = v_{\varphi_{1,2}}, \quad \langle\eta\rangle = (\langle\eta_1\rangle,\ \langle\eta_2\rangle), \\
&\langle\eta_{1,2}\rangle = v_{\eta_{1,2}}, \quad \langle\rho\rangle = v_\rho. \qquad (2)
\end{aligned}$$

We note that, for each D_5 doublet (ϕ,η), there may be four following alignments: i) $0 = \langle\Phi_1\rangle \neq \langle\Phi_2\rangle \neq 0$, D_5 is broken into {identity}; ii) $0 \neq \langle\Phi_1\rangle \neq \langle\Phi_2\rangle = 0$, D_5 is broken into Z_2 consisting of two elements $\{e, b\}$ where a being the $2\pi/5$ rotation and b is the reflection; iii) $\langle\Phi_1\rangle = \langle\Phi_2\rangle \neq 0$, D_5 is broken into {identity}; and iv) $0 \neq \langle\Phi_1\rangle \neq \langle\Phi_2\rangle \neq 0$, D_5 is broken into {identity}. Therefore, the VEV of φ and η in Eq. (2) will break D_5 into {identity} (*i.e.*, discrete symmetry D_5 is completely broken).

With the help of Eqs. (B.24)-(B.30), the expressions in Eqs. (B.18)-(B.22) reduce to

$$\lambda^H > 0, \quad \lambda^\varphi > 0, \quad \lambda^\rho > 0, \quad (3)$$

$$20\lambda^\phi v_{\varphi_1}^2 - \lambda^{\phi\varphi\eta}v_{\eta_1}^2 > 0, \ \ 3\lambda^\eta v_{\eta_1}^2 + 10\lambda^{\phi\varphi\eta}v_{\varphi_1}^2 > 0. \quad (4)$$

Expression (4) implies the following conditions:

$$\begin{aligned}
&\lambda^\eta < 0, \quad \lambda^\phi > 0, \quad 0 < \lambda^{\phi\varphi\eta} \leq \sqrt{6}\sqrt{-\lambda^\eta\lambda^\phi}, \\
&v_{\eta_1} < v_{\varphi_1}\sqrt{-10\lambda^{\phi\varphi\eta}/(3\lambda^\eta)}. \qquad (5)
\end{aligned}$$

3. Lepton mass and mixing

The lepton Yukawa terms invariant under $\mathbf{\Gamma}$ symmetry take the form:

$$\begin{aligned}
-\mathcal{L}_Y^{lep} &= \frac{x_1}{\Lambda}(\bar{\psi}_{1L}l_{2R} + \bar{\psi}_{2L}l_{1R})(H\phi) \\
&+ \frac{x_2}{\Lambda}\left(\bar{\psi}_{1L}l_{1R}H\varphi_2 + \bar{\psi}_{2L}l_{2R}H\varphi_1\right) + x_3(\bar{\psi}_{3L}l_{3R})H \\
&+ \frac{y_1}{\Lambda}\left(\bar{\psi}_{1L}\nu_{1R}\widetilde{H}\eta_2 + \bar{\psi}_{2L}\nu_{1R}\widetilde{H}\eta_1\right) \\
&+ \frac{y_2}{\Lambda}\left(\bar{\psi}_{1L}\nu_{2R}\widetilde{H}\eta_2 + \bar{\psi}_{2L}\nu_{2R}\widetilde{H}\eta_1\right) \\
&+ \frac{y_3}{\Lambda}\left(\bar{\psi}_{3L}\nu_{1R}\right)(\widetilde{H}\phi) + \frac{y_4}{\Lambda}\left(\bar{\psi}_{3L}\nu_{2R}\right)(\widetilde{H}\phi) \\
&+ \frac{z_1}{2}\left(\bar{\nu}^c_{1R}\nu_{1R}\right)\rho + \frac{z_2}{2}\left(\bar{\nu}^c_{2R}\nu_{2R}\right)\rho + \frac{z_3}{2}\left(\bar{\nu}^c_{3R}\nu_{3R}\right)\rho \\
&+ \frac{z_4}{2}\left[\left(\bar{\nu}^c_{1R}\nu_{2R}\right)_{1_1}\rho + \left(\bar{\nu}^c_{2R}\nu_{1R}\right)_{1_1}\rho\right] + \text{H.c.} \qquad (6)
\end{aligned}$$

After symmetry breaking, the mass Lagrangian for the charged leptons can be rewritten in the form:

$$-\mathcal{L}^{\text{mass}}_{\text{cl}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L})M_{cl}(l_{1R}, l_{2R}, l_{3R})^T + H.c, \quad (7)$$

where

$$\begin{aligned}
&M_{cl} = \begin{pmatrix} \epsilon^2 b_l & a_l & 0 \\ a_l & b_l & 0 \\ 0 & 0 & c_l \end{pmatrix}, \\
&a_l = \frac{x_1 v v_\phi}{\Lambda}, \ \ b_l = \frac{x_2 v v_{\varphi_1}}{\Lambda}, \ \ c_l = x_3 v. \qquad (8)
\end{aligned}$$

In general the Yukawa couplings $x_k\ (k = 1,2,3)$ are complex, *i.e.*, $a_l, b_{1,2l}, c_l$ and therefore M_{cl} in Eq. (8) are complex. We construct a Hermitian matrix whose real and positive eigenvalues:

$$M_{cl}^2 = M_{cl}M_{cl}^+ = \begin{pmatrix} \mathcal{A}_0 & \mathcal{D}_0 e^{i\psi} & 0 \\ \mathcal{D}_0 e^{-i\psi} & \mathcal{B}_0 & 0 \\ 0 & 0 & \mathcal{C}_0 \end{pmatrix}, \quad (9)$$

where[iii]

$$\begin{aligned}
&\mathcal{A}_0 = a_0^2 + b_0^2\epsilon^4, \quad \mathcal{B}_0 = a_0^2 + b_0^2, \quad \mathcal{C}_0 = c_0^2, \\
&\mathcal{D}_0 = a_0 b_0\sqrt{1 + \epsilon^4 + 2\epsilon^2\cos 2\psi}, \qquad (10)
\end{aligned}$$

$$\psi = \alpha - \beta, \quad \alpha = \arg a_l, \quad \beta = \arg b_l, \qquad (11)$$

$$\epsilon = \sqrt{\frac{v_{\varphi_2}}{v_{\varphi_1}}} = \frac{v_{\eta_2}}{v_{\eta_1}}. \qquad (12)$$

The matrix M_{cl}^2 in Eq. (9) can be diagonalized by biunitary transform $V_L^+ M_{cl}^2 V_R = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$, where[iv]

$$V_L = V_R = \begin{pmatrix} c_\theta & s_\theta.e^{i\psi} & 0 \\ -s_\theta.e^{-i\psi} & c_\theta & 0 \\ 0 & 0 & e^{i\psi} \end{pmatrix}, \tag{13}$$

$$m_{e,\mu}^2 = a_0^2 + \frac{1}{2}\left[b_0^2\left(\epsilon^4+1\right) \mp b_0\sqrt{\Delta}\right], \quad m_\tau^2 = c_0^2, \tag{14}$$

with

$$c_\theta = \frac{1}{\sqrt{\frac{\left(\sqrt{\Xi}+b_0^2(\epsilon^4-1)\right)^2}{4a_0^2b_0^2(2\epsilon^2c_\psi+\epsilon^4+1)}+1}}, \tag{15}$$

$$\Delta = 8a_0^2\epsilon^2c_{2\psi} + 4a_0^2\left(\epsilon^4+1\right) + b_0^2\left(\epsilon^4-1\right)^2, \tag{16}$$

$$\Xi = 4a_0^2b_0^2\left(2\epsilon^2c_{2\psi}+\epsilon^4+1\right) + b_0^4\left(\epsilon^4-1\right)^2, \tag{17}$$

The left-handed charged lepton mixing matrix V_L in Eq. (13) differs from unity; thus, it will contribute to the lepton mixing matrix in Eq. (31).

Combining Eqs. (8), (12), (14) and (15) yields[v]:

$$x_{01}^2 = \frac{\Lambda^2}{v^2v_\phi^2}\frac{\epsilon^2\left[2\left(\epsilon^4+1\right)c_\psi^2-(\epsilon^2-1)^2\right]\left(m_e^2+m_\mu^2\right)+\left(\epsilon^4+1\right)\sqrt{x_0}}{8\epsilon^2\left(\epsilon^4+1\right)c_\psi^2+(\epsilon^2-2)\left[(\epsilon^2-2)\epsilon^2+2\right]\epsilon^2+1}, \tag{18}$$

$$x_{02}^2 = \frac{\Lambda^2}{v^2v_\phi^2}\frac{\left[\epsilon^2\left(4c_\psi^2+\epsilon^2-2\right)+1\right]\left(m_e^2+m_\mu^2\right)-2\sqrt{x_0}}{(\epsilon^2-2)\left[2+(\epsilon^2-2)\epsilon^2\right]\epsilon^2+8\left(1+\epsilon^4\right)\epsilon^2c_\psi^2+1}, \quad x_{03} = \frac{m_\tau}{v}, \tag{19}$$

where

$$\begin{aligned} x_0 &= 4\epsilon^2\left[2\left(\epsilon^4-\epsilon^2+1\right)m_e^2m_\mu^2-(m_e^4+m_\mu^4)\epsilon^2\right]c_\psi^2 \\ &+ 4\epsilon^4\left(m_e^2+m_\mu^2\right)^2c_\psi^4+(\epsilon^2-1)^4m_e^2m_\mu^2. \end{aligned} \tag{20}$$

Equations (18)-(19) yields a ratio between x_{01} and x_{02}:

$$r_{12}^2 = \frac{x_{01}^2}{x_{02}^2} = \frac{v_{\varphi_1}^2}{v_\phi^2}\frac{\left(2c_\psi^2-1\right)\left(m_e^2+m_\mu^2\right)^2\epsilon^2+\left(m_e^2+m_\mu^2\right)\sqrt{x_0}+2\left(\epsilon^4+1\right)m_e^2m_\mu^2}{\left(m_e^2-m_\mu^2\right)^2}. \tag{21}$$

Furthermore, substituting Eq. (8) in Eq. (15), we get:

$$c_\theta = \frac{1}{\sqrt{\frac{\left(\sqrt{4v_\phi^2x_{01}^2\left[\left(4c_\psi^2+\epsilon^2-2\right)\epsilon^2+1\right]+(\epsilon^4-1)^2v_{\varphi_1}^2x_{02}^2}+(\epsilon^4-1)v_{\varphi_1}x_{02}\right)^2}{4v_\phi^2x_{01}^2\left[\left(4c_\psi^4+\epsilon^2-2\right)\epsilon^2+1\right]}+1}}. \tag{22}$$

Regarding the neutrino sector, when the scalar fields get VEVs as in Eq. (2), from Eq. (6), we get the mass matrices for Dirac and Majorana neutrinos:

$$M_D = \begin{pmatrix} a_{12D} & a_{22D} & 0 \\ a_{11D} & a_{21D} & 0 \\ a_{3D} & a_{4D} & 0 \end{pmatrix}, \tag{23}$$

$$M_R = \begin{pmatrix} a_{1R} & a_{4R} & 0 \\ a_{4R} & a_{2R} & 0 \\ 0 & 0 & a_{3R} \end{pmatrix}, \tag{24}$$

where

$$a_{ijD} = \frac{y_i}{\Lambda}vv_{\eta_j}, \quad a_{kD} = \frac{y_k}{\Lambda}vv_\phi, \quad a_{nR} = z_nv_\rho, \qquad (i,j=1,2; k=3,4; n=1,2,3,4). \tag{25}$$

By using the type-I seesaw mechanism, the light effective neutrino mass matrix is given by

$$\mathrm{M_{eff}} = \begin{pmatrix} \mathrm{A}_\nu & \mathrm{D}_\nu & \mathrm{G}_\nu \\ \mathrm{D}_\nu & \mathrm{B}_\nu & \mathrm{H}_\nu \\ \mathrm{G}_\nu & \mathrm{H}_\nu & \mathrm{C}_\nu \end{pmatrix}, \tag{26}$$

where $\mathrm{A}_\nu, \mathrm{B}_\nu, \mathrm{C}_\nu, \mathrm{D}_\nu, \mathrm{G}_\nu$ and H_ν are given in Appendix C.

The mass matrix $\mathrm{M_{eff}}$ in Eq. (26) possesses three eigenvalues and the corresponding mixing matrix as follows

$$\lambda_1 = 0, \quad \lambda_{2,3} = \Omega_1 \mp \Omega_2, \tag{27}$$

$$\mathrm{R}_\nu = \begin{pmatrix} \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_1}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_2}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_2}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{1}{\sqrt{r_1^2+r_2^2+1}} \end{pmatrix}, \tag{28}$$

where $\Omega_{1,2}$, $\kappa_{1,2}$, $\tau_{1,2}$ and $r_{1,2}$ are given in Appendix D. The neutrino mass spectrum can be normal ($m_1 < m_2 < m_3$) or inverted ($m_3 < m_1 < m_2$) hierarchy depends on the sign of Δm^2_{31} [3]. Note that the eigenvalue $\lambda_1 = 0$ corresponds to the first neutrino eigenvector,

$$\varphi_1 = \left(\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2+1}}, \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2+1}}, \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \right)^T.$$

Thus, the neutrino mass hierarchy should be either ($m_1 = \lambda_1 = 0,\ m_2 = \lambda_2,\ m_3 = \lambda_3$) or ($m_1 = \lambda_2,\ m_2 = \lambda_3,\ m_3 = \lambda_1 = 0$). As will see below, the model obtained results are in agreement with the current neutrino oscillation data for both normal and inverted orderings. The eigenvalues and corresponding vectors of $\mathrm{M_{eff}}$ in Eq. (26), for the two mass hierarchies, are defined by:

$$\mathrm{U}_\nu^T \mathrm{M_{eff}} \mathrm{U}_\nu = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ \mathrm{U}_\nu = \begin{pmatrix} \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_1}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_2}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_2}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{1}{\sqrt{r_1^2+r_2^2+1}} \end{pmatrix} & \text{for NH}, \\ \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathrm{U}_\nu = \begin{pmatrix} \frac{\tau_1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_1}{\sqrt{r_1^2+r_2^2+1}} & \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \\ \frac{\tau_2}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_2}{\sqrt{r_1^2+r_2^2+1}} & \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \\ \frac{1}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{1}{\sqrt{r_1^2+r_2^2+1}} & \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \end{pmatrix} & \text{for IH}. \end{cases} \tag{29}$$

It is easily to check that $\kappa_{1,2}, \tau_{1,2}$ and $r_{1,2}$ satisfy the following relations

$$\begin{cases} \tau_1 = -\frac{1+\kappa_1 r_1 + r_2^2}{r_1+\kappa_1 r_1^2+\kappa_1 r_2^2}, \quad \tau_2 = \frac{(r_1-\kappa_1) r_2}{r_1+\kappa_1 r_1^2+\kappa_1 r_2^2}, \quad \tau_2 = -\frac{1+\kappa_1 r_1}{r_2} \ \text{ for NH}, \\ r_1 = -\frac{1+\kappa_2^2+\kappa_1\tau_1}{\kappa_1+(\kappa_1^2+\kappa_2^2)\tau_1}, \quad r_2 = \frac{\kappa_2(\kappa_1-\tau_1)}{\kappa_1+(\kappa_1^2+\kappa_2^2)\tau 1}, \quad \tau_2 = -\frac{1+\kappa_1\tau_1}{\kappa_2} \ \text{ for IH}. \end{cases} \tag{30}$$

The leptonic mixing matrix, $\mathrm{U_{Lep}} = \mathrm{V}_L^\dagger \mathrm{U}_\nu$, reads:

$$\mathrm{U_{Lep}} = \begin{cases} \begin{pmatrix} \frac{\kappa_1 c_\theta - \kappa_2 s_\theta . e^{i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_1 c_\theta - \tau_2 s_\theta . e^{i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_1 c_\theta - r_2 s_\theta . e^{i\psi}}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{\kappa_2 c_\theta + \kappa_1 s_\theta . e^{-i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{\tau_2 c_\theta + \tau_1 s_\theta . e^{-i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_2 c_\theta + r_1 s_\theta . e^{-i\psi}}{\sqrt{r_1^2+r_2^2+1}} \\ \frac{e^{-i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} & \frac{e^{-i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{e^{-i\psi}}{\sqrt{r_1^2+r_2^2+1}} \end{pmatrix} & \text{for NH}, \\ \begin{pmatrix} \frac{\tau_1 c_\theta - \tau_2 s_\theta . e^{i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_1 c_\theta - r_2 s_\theta . e^{i\psi}}{\sqrt{r_1^2+r_2^2+1}} & \frac{\kappa_1 c_\theta - \kappa_2 s_\theta . e^{i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \\ \frac{\tau_2 c_\theta + \tau_1 s_\theta . e^{-i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{r_2 c_\theta + r_1 s_\theta . e^{-i\psi}}{\sqrt{r_1^2+r_2^2+1}} & \frac{\kappa_2 c_\theta + \kappa_1 s_\theta . e^{-i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \\ \frac{e^{-i\psi}}{\sqrt{\tau_1^2+\tau_2^2+1}} & \frac{e^{-i\psi}}{\sqrt{r_1^2+r_2^2+1}} & \frac{e^{-i\psi}}{\sqrt{\kappa_1^2+\kappa_2^2+1}} \end{pmatrix} & \text{for IH}. \end{cases} \tag{31}$$

In the three neutrino framework, the leptonic mixing matrix can be parameterized as

$$\mathrm{U_{PMNS}} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta_{CP}} s_{13} \\ -c_{12}s_{13}s_{23}e^{i\delta_{CP}} - c_{23}s_{12} & c_{12}c_{23} - e^{i\delta_{CP}} s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta_{CP}} s_{13} & -c_{23}s_{12}s_{13}e^{i\delta_{CP}} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{32}$$

where $c_{12} = \cos\theta_{12},\ c_{23} = \cos\theta_{23},\ c_{13} = \cos\theta_{13},\ s_{12} = \sin\theta_{12},\ s_{23} = \sin\theta_{23}$ and $s_{13} = \sin\theta_{13}$, δ_{CP} is the Dirac CP phase and $\eta_{1,2}$ are two Majorana phases.

By comparing Eqs. (31) and (32), with the aid of Eq.(30), we obtain:

$$s_{12}^2 = \begin{cases} \frac{\left(\kappa_1 r_1 + r_2^2 + 1\right)^2 c_\theta^2 - r_2(\kappa_1 - r_1)\left(\kappa_1 r_1 + r_2^2 + 1\right) s_{2\theta} c_\psi + r_2^2(\kappa_1 - r_1)^2 s_\theta^2}{\left[\left(\kappa_1^2+1\right) r_2^2 + (\kappa_1 r_1 + 1)^2\right]\left(r_1^2 s_\theta^2 + r_1 r_2 s_{2\theta} c_\psi + r_2^2 c_\theta^2 + 1\right)} & \text{for NH,} \\ \frac{\left(\kappa_1 \tau_1 + \kappa_2^2 + 1\right)^2 c_\theta^2 + \kappa_2(\kappa_1 - \tau_1)\left(\kappa_1 \tau_1 + \kappa_2^2 + 1\right) s_{2\theta} c_\psi + \kappa_2^2(\kappa_1 - \tau_1)^2 s_\theta^2}{\left[\left(\tau_1^2+1\right) \kappa_2^2 + (\kappa_1 \tau_1 + 1)^2\right]\left(\kappa_1^2 s_\theta^2 + \kappa_1 \kappa_2 s_{2\theta} c_\psi + \kappa_2^2 c_\theta^2 + 1\right)} & \text{for IH,} \end{cases} \tag{33}$$

$$s_{13}^2 = \begin{cases} \frac{r_1^2 c_\theta^2 - r_1 r_2 s_{2\theta} c_\psi + r_2^2 s_\theta^2}{r_1^2 + r_2^2 + 1} & \text{for NH,} \\ \frac{\kappa_1^2 c_\theta^2 - \kappa_1 \kappa_2 s_{2\theta} c_\psi + \kappa_2^2 s_\theta^2}{\kappa_1^2 + \kappa_2^2 + 1} & \text{for IH,} \end{cases} \tag{34}$$

$$s_{23}^2 = \begin{cases} \frac{r_2^2 c_\theta^2 + r_1 r_2 s_{2\theta} c_\psi + r_1^2 s_\theta^2}{r_1^2 + r_2^2 + 1 - r_1^2 c_\theta^2 + r_1 r_2 s_{2\theta} c_\psi - r_2^2 s_\theta^2} & \text{for NH,} \\ \frac{\kappa_2^2 c_\theta^2 + \kappa_1 \kappa_2 s_{2\theta} c_\psi + \kappa_1^2 s_\theta^2}{\kappa_1^2 + \kappa_2^2 + 1 - \kappa_1^2 c_\theta^2 + \kappa_1 \kappa_2 s_{2\theta} c_\psi - \kappa_2^2 s_\theta^2} & \text{for IH,} \end{cases} \tag{35}$$

$$\eta_1 = \begin{cases} -i \log\left(\frac{\kappa_1 c_\theta - \kappa_2 s_\theta . e^{i\psi}}{c_{12} c_{13}\sqrt{\kappa_1^2 + \kappa_2^2 + 1}}\right) & \text{for NH,} \\ -i \log\left(\frac{\tau_1 c_\theta - \tau_2 s_\theta . e^{i\psi}}{c_{12} c_{13}\sqrt{\tau_1^2 + \tau_2^2 + 1}}\right) & \text{for IH,} \end{cases} \tag{36}$$

$$\eta_2 = \begin{cases} -i \log\left(\frac{\tau_1 c_\theta - \tau_2 s_\theta . e^{i\psi}}{s_{12} c_{13}\sqrt{\tau_1^2 + \tau_2^2 + 1}}\right) & \text{for NH,} \\ -i \log\left(\frac{r_1 c_\theta - r_2 s_\theta . e^{i\psi}}{s_{12} c_{13}\sqrt{r_1^2 + r_2^2 + 1}}\right) & \text{for IH.} \end{cases} \tag{37}$$

Comparing the expressions of the Jalskog's CP-violation parameter, $J_{CP} = \mathrm{Im}(\mathrm{U}_{23}\mathrm{U}_{13}^*\mathrm{U}_{12}\mathrm{U}_{22}^*)$, in term of the model parameters in Eq. (31) and the standard parameterization in Eq. (32), we get

$$\sin\delta_{CP} = \begin{cases} \frac{\left[(r_1^2 + r_2^2)\kappa_1 + r_1\right] r_2 s_{2\theta} s_\psi}{2\left(r_1^2 + r_2^2 + 1\right)\left[(\kappa_1^2+1) r_2^2 + (\kappa_1 r_1 + 1)^2\right] s_{13} c_{13}^2 s_{12} c_{12} s_{23} c_{23}} & \text{(NH),} \\ \frac{\left[(\kappa_1^2 + \kappa_2^2)\tau_1 + \kappa_1\right] \kappa_2 s_{2\theta} s_\psi}{2\left(\kappa_1^2 + \kappa_2^2 + 1\right)\left[(\tau_1^2+1) \kappa_2^2 + (\kappa_1 \tau_1 + 1)^2\right] s_{13} c_{13}^2 s_{12} c_{12} s_{23} c_{23}} & \text{(IH).} \end{cases} \tag{38}$$

Equations (34)-(35) yield a solution:

- For NH:

$$\begin{aligned} \kappa_1 = &\left[r_1 r_2^2 c_{12}^2 s_\theta^2 + r_1(1 + r_1^2 + r_2^2) s_{12}^2 + r_2(1 + r_2^2 - r_1^2 c_{2\theta_{12}} s_\theta c_\theta c_\psi - \sqrt{\kappa_{01}}\right. \\ &\left. - r_1(1 + r_2^2 + r_1^2 s_{12}^2) c_\theta^2\right] / \left[r_1^2 c_\theta^2 + r_2^2 s_\theta^2 - r_1 r_2 c_\psi s_{2\theta} - \kappa_{02}(r_1^2 + r_2^2) s_{12}^2\right], \end{aligned} \tag{39}$$

$$r_1 = \frac{1}{c_{13}^2 c_{23}^2}\sqrt{\frac{r_{01} + 2\sqrt{r_{02}}}{c_\theta^4 + 2 s_\theta^2 c_\theta^2 \cos 2\psi + s_\theta^4}}, \tag{40}$$

$$r_2 = \frac{r_1\left[\sqrt{r_{02}} - c_{13}^2 c_{23}^2 (1 - c_{23}^2 c_{13}^2) s_\theta^2 c_\theta^2 c_\psi^2\right]}{c_{13}^2 c_{23}^2 (s_{13}^2 c_\theta^2 - c_{13}^2 s_{23}^2 s_\theta^2) s_\theta c_\theta c_\psi}, \tag{41}$$

where

$$\kappa_{01} = r_2^2\left(r_1^2 + r_2^2 + 1\right)\left[\kappa_{02}^2 s_{12}^2 c_{12}^2 - \left(r_1^2 + r_2^2 + 1\right) c_\theta^2 s_\theta^2 s_\psi^2\right], \tag{42}$$

$$\kappa_{02} = 1 + r_1^2 s_\theta^2 + r_1 r_2 c_\psi s_{2\theta} + r_2^2 c_\theta^2, \tag{43}$$

$$r_{01} = c_{13}^2 c_{23}^2\left[s_{13}^2 c_\theta^4 + c_{13}^2 s_{23}^2 s_\theta^4 + (s_{13}^2 + c_{13}^2 s_{23}^2) s_\theta^2 c_\theta^2 c_{2\psi}\right], \tag{44}$$

$$r_{02} = c_{13}^4 c_{23}^4\left[c_{13}^2 s_{13}^2 s_{23}^2 (c_\theta^4 + 2 c_\theta^2 c_\psi^2 s_\theta^2 + s_\theta^4) - (s_{13}^4 + c_{13}^4 s_{23}^4) s_\theta^2 c_\theta^2 s_\psi^2\right] s_\theta^2 c_\theta^2 c_\psi^2. \tag{45}$$

- For IH:

$$\tau_1 = \left[\kappa_1\kappa_2^2 s_\theta^2 + \kappa_1\left(1+\kappa_1^2+\kappa_2^2 c_\theta^2\right)s_{12}^2 + \frac{\kappa_2}{2}\left(1+\kappa_2^2-\kappa_1^2 c_{2\theta_{12}}\right)c_\psi s_{2\theta} - \sqrt{\tau_{01}}\right.$$
$$\left. - \kappa_1\left(1+\kappa_2^2+\kappa_1^2 s_{12}^2\right)c_\theta^2\right] \Big/ \left[\kappa_1^2 c_\theta^2 - \kappa_1\kappa_2 c_\psi s_{2\theta} + \kappa_2^2 s_\theta^2 - \tau_{02}\left(\kappa_1^2+\kappa_2^2\right)s_{12}^2\right], \tag{46}$$

$$\kappa_1 = \frac{1}{c_{13}^2 c_{23}^2}\sqrt{\frac{\kappa_{01}+2\sqrt{\kappa_{02}}}{c_\theta^4 + 2s_\theta^2 c_\theta^2 c_{2\psi} + s_\theta^4}}, \tag{47}$$

$$\kappa_2 = \frac{\kappa_1\left[\sqrt{\kappa_{02}} - c_{13}^2 c_{23}^2(1-c_{23}^2 c_{13}^2)s_\theta^2 c_\theta^2 c_\psi^2\right]}{c_{13}^2 c_{23}^2\left(s_{13}^2 c_\theta^2 - c_{13}^2 s_{23}^2 s_\theta^2\right)c_\theta s_\theta c_\psi}, \tag{48}$$

where

$$\tau_{01} = \kappa_2^2(1+\kappa_1^2+\kappa_2^2)\left[\tau_{02}^2 s_{12}^2 c_{12}^2 - c_\theta^2 s_\psi^2(1+\kappa_1^2+\kappa_2^2)s_\theta^2\right], \tag{49}$$

$$\tau_{02} = 1 + \kappa_2^2 c_\theta^2 + \kappa_1\kappa_2 c_\psi s_{2\theta} + \kappa_1^2 s_\theta^2, \tag{50}$$

$$\kappa_{01} = c_{13}^2 c_{23}^2\left[s_{13}^2 c_\theta^4 + c_{13}^2 s_{23}^2 s_\theta^4 + (1-c_{13}^2 c_{23}^2)s_\theta^2 c_\theta^2 c_\psi\right], \tag{51}$$

$$\kappa_{02} = c_{13}^4 c_{23}^4\left[c_{13}^2 s_{13}^2 s_{23}^2(c_\theta^4 + 2c_\theta^2 c_\psi^2 s_\theta^2 + s_\theta^4) - (c_{13}^4 s_{23}^4 + s_{13}^4)c_\theta^2 s_\theta^2 s_\psi^2\right] s_\theta^2 c_\theta^2 c_\psi^2. \tag{52}$$

Expressions (30), (36)-(52) imply that $k_{1,2}$, $\tau_{1,2}$, $r_{1,2}$, $\sin\delta_{CP}$ and $\eta_{1,2}$ depend on s_{13}, s_{12}, s_{23}, θ and ψ which will be presented in Sec. 4..

4. Numerical analysis

In fact, the electroweak symmetry breaking scale is low,

$$v = 2.46\times 10^2\,\text{GeV}, \tag{53}$$

The $B-L$ symmetry breaking is assumed to be TeV scale [14] and the cut-off scale is very high [15],

$$v_\rho \sim 10^3\,\text{GeV}, \quad \Lambda \sim 10^{13}\,\text{GeV}. \tag{54}$$

Further, the experimental values of $m_{e,\mu,\tau}$ at the weak scale is [3],

$$m_e \simeq 0.511\,\text{MeV}, \quad m_\mu \simeq 105.658\,\text{MeV},$$
$$m_\tau \simeq 1776.860\,\text{MeV}. \tag{55}$$

Expressions (19), (53) and (55) yield:

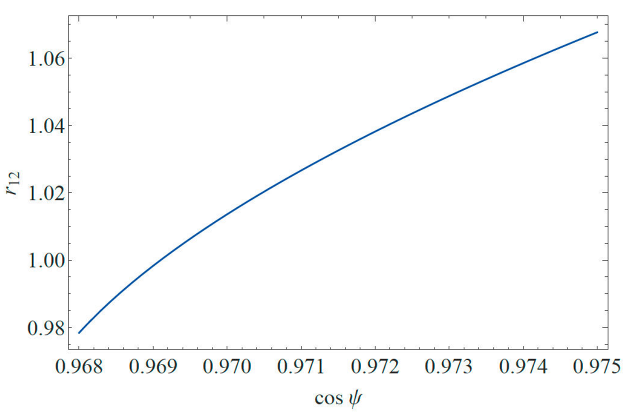

FIGURE 1. r_{12} versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$.

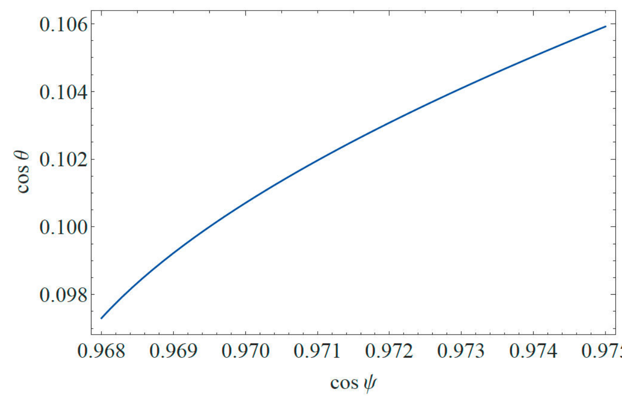

FIGURE 2. $\cos\theta$ versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$.

$$x_{03} = 0.0072. \tag{56}$$

To estimate the order of magnitude of x_{01} and x_{02}, we consider the case where v_ϕ is an order of magnitude lower than TeV scale and the $B-L$ symmetry breaking is assumed to be TeV scale:

$$v_\phi = 10 v_{\varphi_1}, \quad v_{\eta_2} = 10 v_{\eta_1} \quad (\epsilon = 10). \tag{57}$$

With the help of Eqs. (21) and (53)-(57), we find the possible region of $\cos\psi$ is

$$\cos\psi \in (0.968, 0.975). \tag{58}$$

The dependences of two parameters r_{12} and $\cos\theta$ on $\cos\psi$ are plotted in Figs. 1 and 2, respectively, which imply

$$\cos\theta \in (0.098, 0.106),\ i.e.,\ \theta^\circ \in (83.92, 84.38), \tag{59}$$

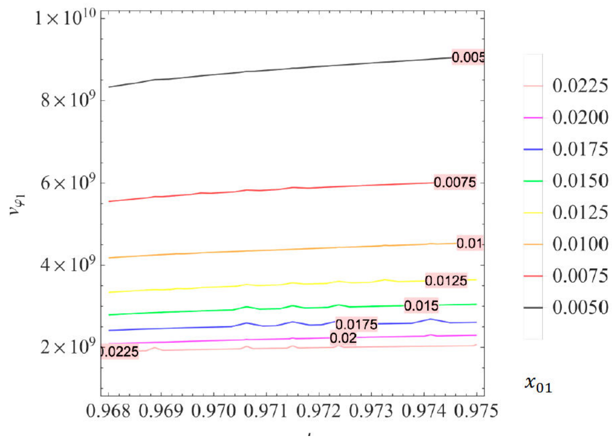

FIGURE 3. (color line) x_{01} versus $\cos\psi$ and v_{φ_1} with $\cos\psi \in (0.968, 0.975)$ and $v_{\varphi_1} \in (10^9, 10^{10})$ GeV.

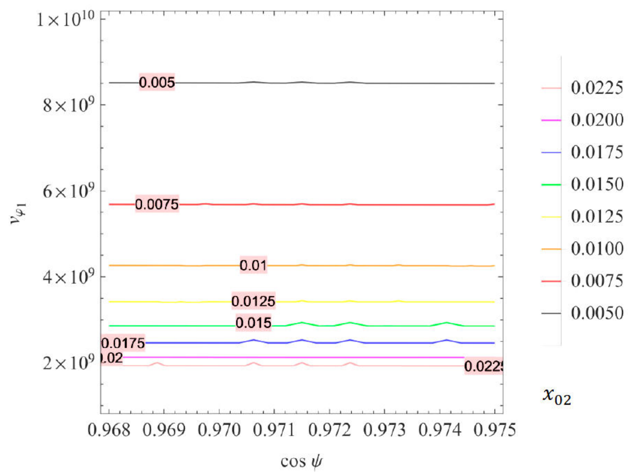

FIGURE 4. (color line) x_{02} versus $\cos\psi$ and v_{φ_1} with $\cos\psi \in (0.968, 0.975)$ and $v_{\varphi_1} \in (10^9, 10^{10})$ GeV.

and

$$r_{12} \in (0.98, 1.06),\ i.e.,\ \ x_{01} \sim (0.98 \div 1.06)x_{02}. \qquad (60)$$

Expression (60) shows that x_{01} and x_{02} are in the same scale of magnitude.

The dependences of x_{01} and x_{02} on $\cos\psi$ are presented in Figs. 3 and 4, respectively, which imply

$$x_{01} \sim x_{02} \in (0.0050, 0.0225). \qquad (61)$$

Expressions (56) and (61) show that three Yukawa couplings in the charged-lepton sector are in the same scale of magnitude, *i.e.*, the charged-lepton mass hierarchy is explained naturally.

As presented in Figs. 3 and 4, the possible region of v_{φ_1} is $v_{\varphi_1} \in (10^9, 10^{10})$ GeV. Thus, we can fix $v_{\varphi_1} = 5\times 10^9$ GeV. As a consequence, from the condition (5) one can estimate the range of the couplings $\lambda^\eta, \lambda^\phi, \lambda^{\phi\varphi\eta}$ and $v_{\eta_{1,2}}, v_{\varphi_2}$ as follows[vi]:

$$\lambda^\eta \sim -10^{-2},\ \ \lambda^\phi \sim 10^{-2},\ \ \lambda^{\phi\varphi\eta} \sim 10^{-3}, \qquad (62)$$

$$v_{\varphi_2} = 5\times 10^{11}\,\text{GeV},\ \ v_{\eta_1} \lesssim 2.89\times 10^9\,\text{GeV},$$
$$v_{\eta_2} \lesssim 2.89\times 10^{10}\,\text{GeV}. \qquad (63)$$

At the best-fit points of neutrino mass-squared differences given in Ref. [4], $\Delta m^2_{21} = 75.0\,\text{meV}^2$ and $\Delta m^2_{31} = 2.55\times 10^3\,\text{meV}^2$ for NH, and $\Delta m^2_{31} = -2.45\times 10^3\,\text{meV}^2$ for IH, with the help of Eq. (27) we obtain:

$$\begin{cases} \Omega_1 = 29.58\,\text{meV}, & \Omega_2 = 20.92\,\text{meV} \ \ \text{for} \ \ \text{NH} \\ \Omega_1 = 49.87\,\text{meV}, & \Omega_2 = 0.376\,\text{meV} \ \ \text{for} \ \ \text{IH} \end{cases}. \qquad (64)$$

Three neutrino masses and the sum of neutrino masses get the explicit values (in meV):

$$\begin{cases} m_1 = 0,\ m_2 = 8.66,\ m_3 = 50.50 \ \text{for} \ \ \text{NH} \\ m_1 = 49.50,\ m_2 = 50.52,\ m_3 = 0 \ \ \text{for} \ \ \text{IH} \end{cases}, \qquad (65)$$

$$\sum m_\nu = \begin{cases} 59.16 \text{ meV} & \text{for NH} \\ 99.75 \text{ meV} & \text{for IH} \end{cases}. \qquad (66)$$

There exist some different constraints for $\sum m_\nu$, for example, $\sum m_\nu < 0.152$ eV, $\sum m_\nu < 0.118$ eV by adding the high-l polarization data [16], in NPDDE model $\sum m_\nu < 0.101$ eV, in the NPDDE+r model $\sum m_\nu < 0.093$ eV, and in NPDDE+r with the R16 prior $\sum m_\nu < 0.078$ eV [16]. The result in Eq. (66) is in consistence with the updated bounds on $\sum_\nu m_\nu$ taken from Ref. [16].

Expressions (30), (34)-(52) imply that k_{12}, τ_{12}, r_{12}, $\sin\delta_{CP}$ and $\eta_{1,2}$ depend on three mixing angles s_{13}, s_{12}, s_{23} and two model parameters θ and ψ. Using the best-fit values s_{13}, s_{12} and s_{23} [4], $s^2_{13} = 2.2\times 10^{-2}$, $s^2_{12} = 0.318$, $s^2_{23} = 0.574$ for NH, and $s^2_{13} = 2.225\times 10^{-2}$, $s^2_{12} = 0.318$, $s^2_{23} = 0.578$ for IH, we can describe the dependence of $k_{1,2}$, $\tau_{1,2}$, $r_{1,2}$, $\sin\delta_{CP}$ and $\eta_{1,2}$ on $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for both NH and IH as shown in Figs. 10-15, respectively.

Figures 10-15 imply that:

$$\kappa_1 \in \begin{cases} (-0.708,\ -0.694) & \text{for NH} \\ (1.1872, 1.1882) & \text{for IH} \end{cases},$$

$$\kappa_2 \in \begin{cases} (1.548,\ 1.554) & \text{for NH} \\ (-0.118, -0.108) & \text{for IH} \end{cases},$$

$$\tau_1 \in \begin{cases} (-0.956, -0.948) & \text{for NH} \\ (-0.700, -0.688) & \text{for IH} \end{cases},$$

$$\tau_2 \in \begin{cases} (-1.078,\ -1.070) & \text{for NH} \\ (1.543, 1.550) & \text{for IH} \end{cases},$$

$$r_1 \in \begin{cases} (1.1774, 1.1784) & \text{for NH} \\ (-0.948, -0.940) & \text{for IH} \end{cases},$$

$$r_2 \in \begin{cases} (-0.118, -0.108) & \text{for NH} \\ (-1.074, -1.066) & \text{for IH} \end{cases}. \quad (67)$$

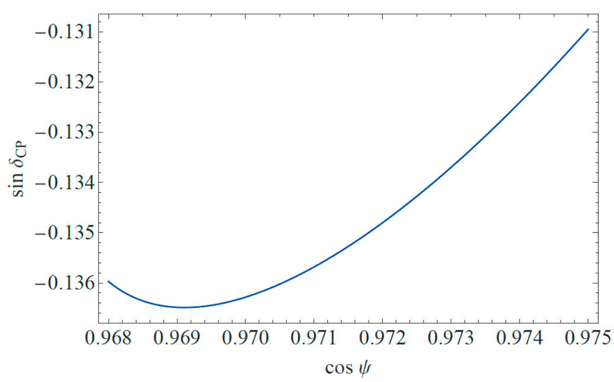

FIGURE 5. $\sin\delta_{CP}$ versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for both NH and IH.

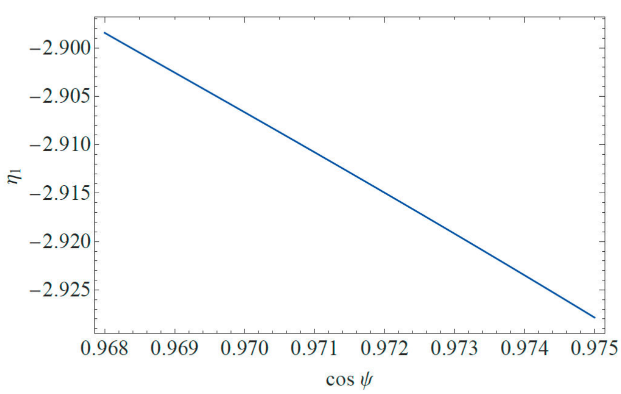

FIGURE 6. η_1 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for both NH and IH.

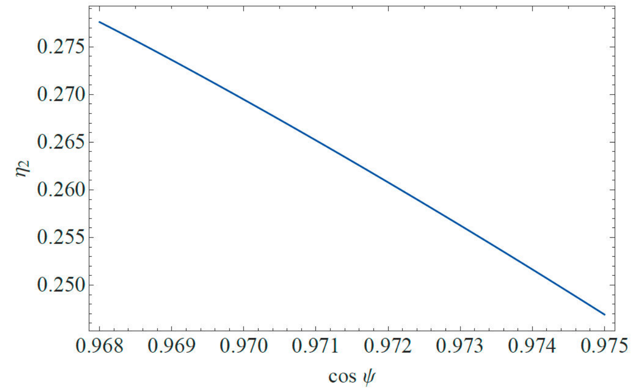

FIGURE 7. η_2 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for both NH and IH.

Furthermore, the dependence of $\sin\delta_{CP}$ and $\eta_{1,2}$ on $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for both NH and IH are plotted in Figs. 5-7, respectively.

Figures 5 tells us that $\sin\delta_{CP} \in (-0.145, -0.125)$, *i.e.*, $\delta_{CP} \in (351.7,\ 352.8)^\circ$ for both NH and IH which lies in the 3σ range of the best-fit value taken from Ref. [18] for both NH and IH and is in good agreement with the T2K data on Dirac CP violation [19]. In addition, Figs. 6 and 7 imply that $\eta_1 \in (-2.925,\ -2.900)\,\text{rad} \sim (192.40,\ 193.80)^\circ$ and $\eta_2 \in (0.250,\ 0.275)\,\text{rad} \sim (14.32,\ 15.76)^\circ$ for both NH

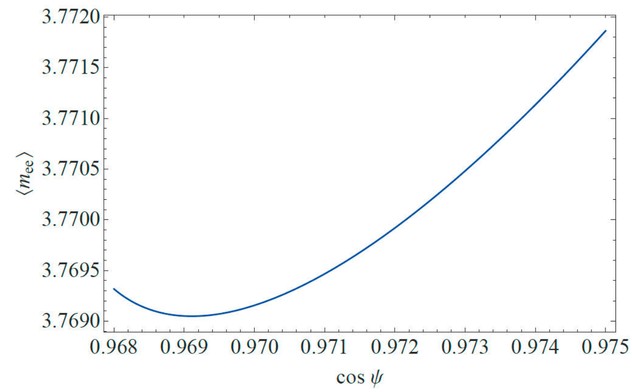

FIGURE 8. $\langle m_{ee}\rangle$ (in meV) versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH.

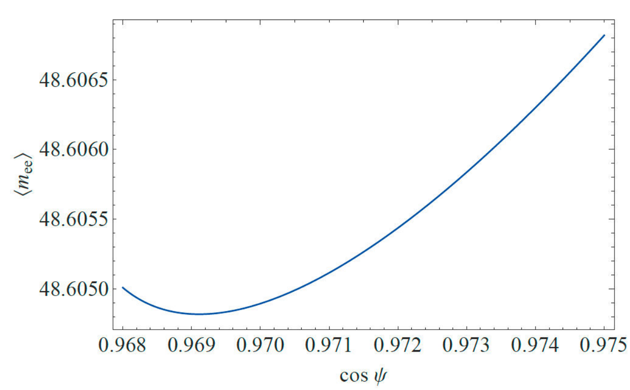

FIGURE 9. $\langle m_{ee}\rangle$ (in meV) versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for IH.

and IH which are acceptable since they are assumed to be in $[0,\,2\pi]$ [3].

The effective neutrino mass parameter governing $0\nu\beta\beta$, $\langle m_{ee}\rangle=\left|\sum_{i=1}^{3}U_{ei}^2 m_i\right|$, depends o n $\cos\psi$, which are plotted in Figs. 8 and 9 for NH and IH, respectively.

Figures 8 and 9 imply that:

$$\langle m_{ee}\rangle\in\left\{\begin{array}{ll}(3.764,\,3.776)\,\mathrm{meV} & \text{for NH}\\ (48.601,48.610)\,\mathrm{meV} & \text{for IH}\end{array}\right. . \tag{68}$$

The resulting effective neutrino mass in Eq. (68) are below all the constraints from current $0\nu\beta\beta$ decay experiments, for instance, MAJORANA [18] $\langle m_{ee}\rangle<0.24\div0.53\,\mathrm{eV}$, CUORE [19,20] $\langle m_{ee}\rangle<0.11\div0.5\,\mathrm{eV}$, EXO [21,23] $\langle m_{ee}\rangle<0.17\div0.49\,\mathrm{eV}$, GERDA [24] $\langle m_{ee}\rangle<0.12\div0.26\,\mathrm{eV}$, KamLAND-Zen [25] $\langle m_{ee}\rangle<0.05\div0.16\,\mathrm{eV}$.

5. Conclusions

We have suggested a gauge $B-L$ model with $D_5\times Z_4$ for explaining the lepton mass and mixing through the type-I seesaw mechanism. The model predicts the neutrino masses and mixing angles at their best-fit values while the Dirac CP phase lies in $(351.7,\,352.8)^\circ$ which being within 3σ range of the best-fit value for both NH and IH and is in good agreement with the T2K data on Dirac CP violation. The two Majorana phases are predicted to be $\eta_1\in(192.40,\,193.80)^\circ$ and $\eta_2\in(14.32,\,15.76)^\circ$ for both NH and IH. The model also predicts the effective neutrino parameters of $\langle m_{ee}\rangle\in(3.764,\,3.776)\,\mathrm{meV}$ for NH and $\langle m_{ee}\rangle\in(48.601,48.610)\,\mathrm{meV}$ which are highly consistent with the current constraints.

Non-Abelian discrete symmetry D_5 requires additional scalar fields which may be dark matter candidates. Howerver, a detailed study of this issue is beyond the scope of this study and it will be shown elsewhere.

Appendix

A. Tensor products of D_5

The D_5 group is the symmetry group of the regular pentagon which is generated by the $2\pi/5$ rotation a and the reflection b, and satisfying the relations, $a^5=e, b^2=e, bab=a^{-1}$. The D_5 group has ten elements which are divided into four conjugacy classes corresponding to four irreducible representations, denoted by $\underline{1}_1,\underline{1}_2$ and $\underline{2}_1$, $\underline{2}_2$ [10]. The Clebsch-Gordan coefficients for all the tensor products of D_5 are given by:

$$\underline{1}_1(\alpha)\otimes\underline{1}_1(y)=\underline{1}_2(\alpha)\otimes\underline{1}_2(\beta)=\underline{1}_1(\alpha\beta),\qquad \underline{1}_1(\alpha)\otimes\underline{1}_2(\beta)=\underline{1}_2(\alpha)\otimes\underline{1}_1(\beta)=\underline{1}_2(\alpha\beta), \tag{A.1}$$

$$\underline{1}_1(\alpha)\otimes\underline{2}_k(\beta_1,\,\beta_2)=\underline{2}_k(\alpha\beta_1,\,\alpha\beta_2),\qquad \underline{1}_2(\alpha)\otimes\underline{2}_k(\beta_1,\,\beta_2)=\underline{2}_k(\alpha\beta_1,\,-\alpha\beta_2),\quad (k=1,2) \tag{A.2}$$

$$\underline{2}_1(\alpha_1,\alpha_2)\otimes\underline{2}_1(\beta_1,\beta_2)=\underline{1}_1(\alpha_1\beta_2+\alpha_2\beta_1)\oplus\underline{1}_2(\alpha_1\beta_2-\alpha_2\beta_1)\oplus\underline{2}_2(\alpha_1\beta_1,\,\alpha_2\beta_2), \tag{A.3}$$

$$\underline{2}_2(\alpha_1,\alpha_2)\otimes\underline{2}_2(\beta_1,\beta_2)=\underline{1}_1(\alpha_1\beta_2+\alpha_2\beta_1)\oplus\underline{1}_2(\alpha_1\beta_2-\alpha_2\beta_1)\oplus\underline{2}_1(\alpha_2\beta_2,\,\alpha_1\beta_1), \tag{A.4}$$

$$\underline{2}_1(\alpha_1,\alpha_2)\otimes\underline{2}_2(\beta_1,\beta_2)=\underline{2}_1(\alpha_2\beta_1,\,\alpha_1\beta_2)\oplus\underline{2}_2(\alpha_2\beta_2,\,\alpha_1\beta_1), \tag{A.5}$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are the multiplet components of different representations.

B. Scalar potential

The scalar potential up to five dimension, which is invariant under $\mathbf{\Gamma}$ symmetry, reads^{vii}:

$$V_g=V(H)+V(\phi)+V(\varphi)+V(\eta)+V(\rho)+V(H,\phi)+V(H,\varphi)+V(H,\eta)+V(H,\rho)+V(\phi,\varphi)$$
$$+V(\phi,\eta)+V(\phi,\rho)+V(\varphi,\eta)+V(\varphi,\rho)+V(\eta,\rho)+V_{\mathrm{tri}}, \tag{B.1}$$

where

$$V(H)=\mu_H^2 H^\dagger H+\lambda^H(H^\dagger H)^2,\quad V(\phi)=\mu_\phi^2\phi^*\phi+\lambda^\phi(\phi^*\phi)^2, \tag{B.2}$$

$$V(\varphi)=\mu_\varphi^2\varphi^*\varphi+\lambda_1^\varphi(\varphi^*\varphi)_{\underline{1}_1}(\varphi^*\varphi)_{\underline{1}_1}+\lambda_2^\varphi(\varphi^*\varphi)_{\underline{1}_2}(\varphi^*\varphi)_{\underline{1}_2}+\lambda_3^\varphi(\varphi^*\varphi)_{\underline{2}_1}(\varphi^*\varphi)_{\underline{2}_1}, \tag{B.3}$$

$$V(\eta)=\mu_\eta^2\eta^*\eta+\lambda^\eta(\eta^*\eta)_{\underline{1}_1}(\eta^*\eta)_{\underline{1}_1}+\lambda_2^\eta(\eta^*\eta)_{\underline{1}_2}(\eta^*\eta)_{\underline{1}_2}+\lambda_3^\eta(\eta^*\eta)_{\underline{2}_2}(\eta^*\eta)_{\underline{2}_2}, \tag{B.4}$$

$$V(\rho)=\mu_\rho^2\rho^*\rho+\lambda^\rho(\rho^*\rho)^2,\quad V(H,\phi)=\lambda_1^{H\phi}(H^\dagger H)_{\underline{1}_1}(\phi^*\phi)_{\underline{1}_1}+\lambda_2^{H\phi}(H^\dagger\phi)_{\underline{1}_1}(\phi^*H)_{\underline{1}_1}, \tag{B.5}$$

$$V(H,\varphi)=\lambda_1^{H\varphi}(H^\dagger H)_{\underline{1}_1}(\varphi^*\varphi)_{\underline{1}_1}+\lambda_2^{H\varphi}(H^\dagger\varphi)_{\underline{2}_2}(\varphi^*H)_{\underline{2}_2},\quad V(H,\rho)=V(H,\phi\to\rho), \tag{B.6}$$

$$V(H,\eta)=\lambda_1^{H\eta}(H^\dagger H)_{\underline{1}_1}(\eta^*\eta)_{\underline{1}_1}+\lambda_2^{H\eta}(H^\dagger\eta)_{\underline{2}_1}(\eta^*H)_{\underline{2}_1},\quad V(\phi,\varphi)=V(H\to\phi,\varphi), \tag{B.7}$$

$$V(\phi,\eta)=V(H\to\phi,\eta),\quad V(\phi,\rho)=V(H\to\phi,\rho), \tag{B.8}$$

$$\begin{aligned}V(\varphi,\eta)&=\lambda_1^{\varphi\eta}(\varphi^*\varphi)_{\underline{1}_1}(\eta^*\eta)_{\underline{1}_1}+\lambda_2^{\varphi\eta}(\varphi^*\varphi)_{\underline{1}_2}(\eta^*\eta)_{\underline{1}_2}+\lambda_3^{\varphi\eta}(\varphi^*\eta)_{\underline{2}_1}(\eta^*\varphi)_{\underline{2}_1}\\&+\ \lambda_4^{\varphi\eta}(\varphi^*\eta)_{\underline{2}_2}(\eta^*\varphi)_{\underline{2}_2},\quad V(\varphi,\rho)=V(\varphi,H\to\rho),\quad V(\eta,\rho)=V(\eta,H\to\rho),\end{aligned} \tag{B.9}$$

$$V_{tri}=\lambda^{H\varphi\eta}(H^\dagger H)_{\underline{1}_1}(\varphi\varphi)_{\underline{2}_1}\eta+\left[\lambda^{\phi\varphi\eta}(\phi\varphi)_{\underline{2}_2}(\eta\eta)_{\underline{2}_2}+\lambda^{\varphi\eta\rho}(\varphi\varphi)_{\underline{2}_1}(\eta\rho^*\rho)_{\underline{2}_1}+h.c\right]. \tag{B.10}$$

All the other terms up to five dimensions that contain more than three different scalar fields (including quartic and quintic terms) are not included in the expression of V_g in Eq. (B.1) because they are not invariant under one (or some) of the model's symmetry $\mathbf{\Gamma}$. The existence of several couplings in the scalar potential V_g represents a characteristic of discrete symmetry models and it guarantees of freedom to choose a suitable scalar potential.

With the aim of showing that the scalar VEVs in Eq. (2) is a natural solution of the minimum condition of the scalar potential V_{tot}, we put $v^*_\lambda=v_\lambda$ ($v_\lambda=v,\ v_\phi, v_\varphi,\ v_{\eta_{1,2}},\ v_\rho$), the minimization condition of the scalar potential reads

$$2\lambda^H v^2+4\lambda^{H\varphi}v_{\varphi_1}v_{\varphi_2}+\lambda^{H\phi\eta}(v_{\varphi_1}^2v_{\eta_1}+v_{\varphi_2}^2v_{\eta_2})+4\lambda^{H\eta}v_{\eta_1}v_{\eta_2}+2\lambda^{H\phi}v_\phi^2+2\lambda^{H\rho}v_\rho^2+\mu_H^2=0, \tag{B.11}$$

$$v_\phi\left[\mu_\phi^2+2\left(\lambda^{H\phi}v^2+2\lambda^{\phi\varphi}v_{\varphi_1}v_{\varphi_2}+2\lambda^{\phi\eta}v_{\eta_1}v_{\eta_2}+\lambda^\phi v_\phi^2+\lambda^{\phi\rho}v_\rho^2\right)\right]+\lambda^{\phi\varphi\eta}(v_{\varphi_2}v_{\eta_1}^2+v_{\varphi_1}v_{\eta_2}^2)=0, \tag{B.12}$$

$$\begin{aligned}&\mu_\varphi^2v_{\varphi_2}+2\lambda^{H\varphi}v^2v_{\varphi_2}+6\lambda^\varphi v_{\varphi_1}v_{\varphi_2}^2+\lambda^{H\phi\eta}v^2v_{\varphi_1}v_{\eta_1}+4\lambda^{\varphi\eta}v_{\varphi_2}v_{\eta_1}v_{\eta_2}+\lambda^{\phi\varphi\eta}v_{\eta_2}^2v_\phi\\&+2\lambda^{\phi\varphi}v_{\varphi_2}v_\phi^2+2(\lambda^{\varphi\rho}v_{\varphi_2}+\lambda^{\phi\eta\rho}v_{\varphi_1}v_{\eta_1})v_\rho^2=0,\end{aligned} \tag{B.13}$$

$$\begin{aligned}&\mu_\varphi^2v_{\varphi_1}+2\lambda^{H\varphi}v^2v_{\varphi_1}+6\lambda^\varphi v_{\varphi_2}v_{\varphi_1}^2+\lambda^{H\phi\eta}v^2v_{\varphi_2}v_{\eta_2}+4\lambda^{\varphi\eta}v_{\varphi_1}v_{\eta_1}v_{\eta_2}+\lambda^{\phi\varphi\eta}v_{\eta_1}^2v_\phi+2\lambda^{\phi\varphi}v_{\varphi_1}v_\phi^2\\&+2(\lambda^{\varphi\rho}v_{\varphi_1}+\lambda^{\phi\eta\rho}v_{\varphi_2}v_{\eta_2})v_\rho^2=0,\end{aligned} \tag{B.14}$$

$$\begin{aligned}&4\left[v_{\eta_2}\left(\lambda^{H\eta}v^2+2\lambda^{\varphi\eta}v_{\varphi_1}v_{\varphi_2}+3\lambda^\eta v_{\eta_1}v_{\eta_2}\right)+\lambda^{\phi\varphi\eta}v_{\varphi_2}v_{\eta_1}v_\phi+\lambda^{\phi\eta}v_{\eta_2}v_\phi^2\right]+2\mu_\eta^2v_{\eta_2}^2+\lambda^{H\phi\eta}v^2v_{\varphi_1}^2\\&+2(\lambda^{\phi\eta\rho}v_{\varphi_1}^2+2\lambda^{\eta\rho}v_{\eta_2})v_\rho^2=0,\end{aligned} \tag{B.15}$$

$$\begin{aligned}&4\left[v_{\eta_1}\left(\lambda^{H\eta}v^2+2\lambda^{\varphi\eta}v_{\varphi_1}v_{\varphi_2}+3\lambda^\eta v_{\eta_1}v_{\eta_2}\right)+\lambda^{\phi\varphi\eta}v_{\varphi_1}v_{\eta_2}v_\phi+\lambda^{\phi\eta}v_{\eta_1}v_\phi^2\right]+2\mu_\eta^2v_{\eta_1}^2+\lambda^{H\phi\eta}v^2v_{\varphi_2}^2\\&+2(\lambda^{\phi\eta\rho}v_{\varphi_2}^2+2\lambda^{\eta\rho}v_{\eta_1})v_\rho^2=0,\end{aligned} \tag{B.16}$$

$$2\left[\lambda^{H\rho}v^2+2\lambda^{\varphi\rho}v_{\varphi_1}v_{\varphi_2}+\lambda^{\phi\eta\rho}(v_{\varphi_1}^2v_{\eta_1}+v_{\varphi_2}^2v_{\eta_2})+2\lambda^{\eta\rho}v_{\eta_1}v_{\eta_2}+\lambda^{\phi\rho}v_\phi^2\right]+\mu_\rho^2+2\lambda^\rho v_\rho^2=0, \tag{B.17}$$

$$6\lambda^Hv^2+4\lambda^{H\varphi}v_{\varphi_1}v_{\varphi_2}+\lambda^{H\phi\eta}v_{\varphi_1}^2v_{\eta_1}+\lambda^{H\phi\eta}v_{\varphi_2}^2v_{\eta_2}+4\lambda^{H\eta}v_{\eta_1}v_{\eta_2}+\mu_H^2+2\lambda^{H\phi}v_\phi^2+2\lambda^{H\rho}v_\rho^2>0, \tag{B.18}$$

$$\mu_\phi^2+2\lambda^{H\phi}v^2+4\lambda^{\phi\varphi}v_{\varphi_1}v_{\varphi_2}+4\lambda^{\phi\eta}v_{\eta_1}v_{\eta_2}+6\lambda^\phi v_\phi^2+2\lambda^{\phi\rho}vr^2>0, \tag{B.19}$$

$$6\lambda^\varphi v_{\varphi_2}^2+\lambda^{H\phi\eta}v^2v_{\eta_1}+2\lambda^{\phi\eta\rho}v_{\eta_1}vr^2>0,\quad 3\lambda^\eta v_{\eta_2}^2+\lambda^{\phi\varphi\eta}v_{\varphi_2}v_\phi>0, \tag{B.20}$$

$$6\lambda^\varphi v_{\varphi_1}^2+\lambda^{H\phi\eta}v^2v_{\eta_2}+2\lambda^{\phi\eta\rho}v_{\eta_2}vr^2>0,\quad 3\lambda^\eta v_{\eta_1}^2+\lambda^{\phi\varphi\eta}v_{\varphi_1}v_\phi>0, \tag{B.21}$$

$$2\left(\lambda^{H\rho}v^2+2\lambda^{\varphi\rho}v_{\varphi_1}v_{\varphi_2}+\lambda^{\phi\eta\rho}v_{\varphi_1}^2v_{\eta_1}+\lambda^{\phi\eta\rho}v_{\varphi_2}^2v_{\eta_2}+2\lambda^{\eta\rho}v_{\eta_1}v_{\eta_2}\right)+\mu_\rho^2+2\lambda^{\phi\rho}v_\phi^2+6\lambda^\rho v_\rho^2>0, \tag{B.22}$$

where the following benchmark points were used[viii]

$$\begin{aligned}
&\lambda_1^{\varphi}=\lambda_3^{\varphi}=\lambda^{\varphi},\lambda_1^{\phi\eta\rho}=\lambda_2^{\phi\eta\rho}=\lambda^{\phi\eta\rho},\\
&\lambda_1^{\phi\eta}=\lambda_2^{\phi\eta}=\lambda^{\phi\eta},\quad \lambda_1^{\phi\varphi}=\lambda_2^{\phi\varphi}=\lambda^{\phi\varphi},\\
&\lambda_1^{H\phi}=\lambda_2^{H\phi}=\lambda^{H\phi},\quad \lambda_1^{\phi\rho}=\lambda_2^{\phi\rho}=\lambda^{\phi\rho},\\
&\lambda_1^{H\eta}=\lambda_2^{H\eta}=\lambda^{H\eta},\quad \lambda_1^{H\varphi}=\lambda_2^{H\varphi}=\lambda^{H\varphi},\\
&\lambda_1^{H\rho}=\lambda_2^{H\rho}=\lambda^{H\rho},\quad \lambda_1^{\eta}=\lambda_3^{\eta}=\lambda^{\eta},\\
&\lambda_1^{\phi\varphi\eta}=\lambda_2^{\phi\varphi\eta}=\lambda^{\phi\varphi\eta},\quad \lambda_1^{\eta\rho}=\lambda_2^{\eta\rho}=\lambda^{\eta\rho},\\
&\lambda_1^{\varphi\eta}=\lambda_2^{\varphi\eta}=\lambda_3^{\varphi\eta}=\lambda_4^{\varphi\eta}=\lambda^{\varphi\eta},\\
&\lambda_1^{\varphi\rho}=\lambda_2^{\varphi\rho}=\lambda^{\varphi\rho}.
\end{aligned} \tag{B.23}$$

The system of Eqs. (B.11)-(B.17) yields the following solution:

$$\lambda^{H\phi\eta}=-\left[2\left(\lambda^H v^2+2\lambda^{H\eta}v_{\eta_1}v_{\eta_2}+\lambda^{H\phi}v_\phi^2+\lambda^{H\rho}v_\rho^2+2\lambda^{H\varphi}v_{\varphi_1}v_{\varphi_2}\right)+\mu_H^2\right]/\lambda_0, \tag{B.24}$$

$$\lambda^{\phi\eta}=-\frac{\mu_\phi^2+2(\lambda^{H\phi}v^2+2\lambda^{\phi\varphi}v_{\varphi_1}v_{\varphi_2}+\lambda^{\phi}v_\phi^2+\lambda^{\phi\rho}v_\rho^2)}{4v_{\eta_1}v_{\eta_2}}-\frac{\lambda^{\phi\varphi\eta}(v_{\varphi_2}v_{\eta_1}^2+v_{\varphi_1}v_{\eta_2}^2)}{4v_{\eta_1}v_{\eta_2}v_\phi}, \tag{B.25}$$

$$\begin{aligned}
\lambda^{\phi\eta\rho}=&-\Bigg[v_{\varphi_2}\left(\mu_\varphi^2+2\lambda^{H\varphi}v^2+6\lambda^{\varphi}v_{\varphi_1}v_{\varphi_2}+2\lambda^{\varphi\rho}v_\rho^2+4\lambda^{\varphi\eta}v_{\eta_1}v_{\eta_2}+2\lambda^{\phi\varphi}v_\phi^2\right)-\frac{v^2v_{\varphi_1}v_{\eta_1}}{\lambda_0}\\
&\times\left(\mu_H^2+2\lambda^H v^2+4\lambda^{H\varphi}v_{\varphi_1}v_{\varphi_2}+4\lambda^{H\eta}v_{\eta_1}v_{\eta_2}+2\lambda^{H\phi}v_\phi^2+2\lambda^{H\rho}v_\rho^2\right)+\ \lambda^{\phi\varphi\eta}v_{\eta_2}^2v_\phi\Bigg]\Bigg/(2v_{\varphi_1}v_{\eta_1}v_\rho^2),
\end{aligned} \tag{B.26}$$

$$\lambda^{H\varphi}=-\frac{1}{2v^2}\left[\mu_\varphi^2+6\lambda^{\varphi}v_{\varphi_1}v_{\varphi_2}+4\lambda^{\varphi\eta}v_{\eta_1}v_{\eta_2}+2\lambda^{\phi\varphi}v_\phi^2+2\lambda^{\varphi\rho}v_\rho^2+\lambda^{\phi\varphi\eta}v_\phi(v_{\varphi_2}v_{\eta_2}^3-v_{\varphi_1}v_{\eta_1}^3)/\delta^{\varphi\eta}\right], \tag{B.27}$$

$$\begin{aligned}
\lambda^{\varphi\eta}=&\left[\lambda^{\phi\varphi\eta}v_\phi\left(v_{\eta_1}^2v_{\varphi_2}-v_{\varphi_1}v_{\eta_2}^2\right)v_{\varphi_1}\right]/\left(8v_{\varphi_2}v_{\eta_2}\delta^{\varphi\eta}\right)+\left[2\lambda^{H\phi}v^2v_\phi^2+4\lambda^{\phi\varphi}v_\phi^2v_{\varphi_1}v_{\varphi_2}-12\lambda^{\eta}v_{\eta_1}^2v_{\eta_2}^2\right.\\
&-2\mu_\eta^2v_{\eta_1}v_{\eta_2}-4\lambda^{H\eta}v^2v_{\eta_1}v_{\eta_2}-3\lambda^{\phi\varphi\eta}v_\phi v_{\eta_1}^2v_{\varphi_2}-2(2\lambda^{\eta\rho}v_{\eta_1}v_{\eta_2}-\lambda^{\phi\rho}v_\phi^2)v_\rho^2+\lambda^{\phi\varphi\eta}v_{\varphi_1}v_{\eta_2}^2v_\phi\\
&\left.+2\lambda^{\phi}v_\phi^4+\mu_\phi^2v_\phi^2\right]/\left(8v_{\varphi_1}v_{\varphi_2}v_{\eta_1}v_{\eta_2}\right),
\end{aligned} \tag{B.28}$$

$$\begin{aligned}
\lambda^{\phi\varphi}=&\left\{\mu_H^2v^2v_{\eta_1}^4-\mu_\phi^2v_{\eta_1}^4v_\phi^2-2\mu_\varphi^2v_{\varphi_1}^2v_{\eta_1}^2v_{\eta_2}^2+2\mu_\eta^2v_{\eta_1}^5v_{\eta_2}+\mu_\rho^2v_{\eta_1}^4v_\rho^2+8\lambda^{H\eta}v^2v_{\eta_1}^5v_{\eta_2}+2\lambda^Hv^4v_{\eta_1}^4+12\lambda^{\eta}v_{\eta_1}^6v_{\eta_2}^2\right.\\
&\left.-12\lambda^{\varphi}v_{\varphi_1}^4v_{\eta_2}^4-2\lambda^{\phi}v_{\eta_1}^4v_\phi^4+4\lambda^{H\rho}v_{\eta_1}^4v^2v_\rho^2+2\lambda^{\rho}v_{\eta_1}^4v_\rho^4+8\lambda^{\eta\rho}v_{\eta_1}^4v_{\eta_1}v_{\eta_2}v_\rho^2\right\}/\left(8v_{\varphi_1}^2v_{\eta_1}^2v_{\eta_2}^2v_\phi^2\right),
\end{aligned} \tag{B.29}$$

$$v_{\varphi_2}=\epsilon^2v_{\varphi_1}, \tag{B.30}$$

where

$$\lambda_0=v_{\varphi_1}^2v_{\eta_1}+v_{\varphi_2}^2v_{\eta_2},\qquad \delta^{\varphi\eta}=v_{\varphi_2}^2v_{\eta_2}-v_{\varphi_1}^2v_{\eta_1}. \tag{B.31}$$

C. Explicit expressions for $\mathrm{A}_\nu, \mathrm{B}_\nu, \mathrm{C}_\nu, \mathrm{D}_\nu, \mathrm{G}_\nu$ and H_ν

$$\mathrm{A}_\nu = \frac{a_{12D}\left(a_{12D}a_{2R} - 2a_{22D}a_{4R}\right) + a_{1R}a_{22D}^2}{a_{4R}^2 - a_{1R}a_{2R}}, \tag{C.1}$$

$$\mathrm{B}_\nu = \frac{a_{11D}\left(a_{11D}a_{2R} - 2a_{21D}a_{4R}\right) + a_{1R}a_{21D}^2}{a_{4R}^2 - a_{1R}a_{2R}}, \tag{C.2}$$

$$\mathrm{C}_\nu = \frac{a_{4D}\left(a_{1R}a_{4D} - 2a_{3D}a_{4R}\right) + a_{2R}a_{3D}^2}{a_{4R}^2 - a_{1R}a_{2R}}, \tag{C.3}$$

$$\mathrm{D}_\nu = \frac{a_{11D}a_{12D}a_{2R} + a_{1R}a_{21D}a_{22D}}{a_{4R}^2 - a_{1R}a_{2R}} - \frac{a_{4R}\left(a_{11D}a_{22D} + a_{12D}a_{21D}\right)}{a_{4R}^2 - a_{1R}a_{2R}}, \tag{C.4}$$

$$\mathrm{G}_\nu = \frac{a_{12D}a_{2R}a_{3D} + a_{1R}a_{22D}a_{4D}}{a_{4R}^2 - a_{1R}a_{2R}} - \frac{a_{4R}\left(a_{12D}a_{4D} + a_{22D}a_{3D}\right)}{a_{4R}^2 - a_{1R}a_{2R}}, \tag{C.5}$$

$$\mathrm{H}_\nu = \frac{a_{11D}a_{2R}a_{3D} + a_{1R}a_{21D}a_{4D}}{a_{4R}^2 - a_{1R}a_{2R}} - \frac{a_{4R}\left(a_{11D}a_{4D} + a_{21D}a_{3D}\right)}{a_{4R}^2 - a_{1R}a_{2R}}. \tag{C.6}$$

D. Explicit expressions of $\Omega_{1,2}$, $\kappa_{1,2}$, $\tau_{1,2}$ and $r_{1,2}$

$$\Omega_1 = \frac{\Omega_{01}}{2\left(a_{1R}a_{2R} - a_{4R}^2\right)}, \qquad \Omega_2 = \frac{\sqrt{\Omega_{02}}}{2\left(a_{1R}a_{2R} - a_{4R}^2\right)}, \tag{D.1}$$

$$\Omega_{01} = a_{1R}\left(a_{21D}^2 + a_{22D}^2 + a_{4D}^2\right) + a_{2R}\left(a_{11D}^2 + a_{12D}^2 + a_{3D}^2\right) - 2a_{4R}\left(a_{11D}a_{21D} + a_{12D}a_{22D} + a_{3D}a_{4D}\right), \tag{D.2}$$

$$\Omega_{02} = -4\left(a_{1R}a_{2R} - a_{4R}^2\right)\left[a_{11D}^2\left(a_{22D}^2 + a_{4D}^2\right) + a_{3D}^2\left(a_{21D}^2 + a_{22D}^2\right) - 2a_{11D}a_{21D}a_{3D}a_{4D} - 2a_{12D}a_{22D}\left(a_{11D}a_{21D} + a_{3D}a_{4D}\right) + a_{12D}^2\left(a_{21D}^2 + a_{4D}^2\right)\right] + \Omega_{01}^2, \tag{D.3}$$

$$\kappa_1 = \frac{a_{21D}a_{3D} - a_{11D}a_{4D}}{a_{11D}a_{22D} - a_{12D}a_{21D}}, \tag{D.4}$$

$$\kappa_2 = \frac{a_{22D}a_{3D} - a_{12D}a_{4D}}{a_{12D}a_{21D} - a_{11D}a_{22D}}, \tag{D.5}$$

$$\tau_1 = \frac{\tau_{11}}{2\tau_0}, \qquad \tau_2 = \frac{\tau_{22}}{2\tau_0}, \qquad r_1 = \frac{r_{11}}{2\tau_0}, \qquad r_2 = \frac{r_{22}}{2\tau_0}, \tag{D.6}$$

where

$$\begin{aligned}
\tau_{11} &= -2a_{11D}a_{12D}a_{21D}a_{2R}a_{3D} + a_{22D}a_{2R}a_{3D}^3 - a_{12D}^2a_{22D}a_{2R}a_{3D} + a_{12D}^3a_{2R}a_{4D} - a_{12D}a_{2R}a_{3D}^2a_{4D} \\
&- a_{1R}\left[(a_{22D}^2 - a_{4D}^2)(a_{22D}a_{3D} - a_{12D}a_{4D}) - 2a_{11D}a_{21D}a_{22D}a_{4D} + a_{21D}^2(a_{22D}a_{3D} + a_{12D}a_{4D})\right] \\
&+ 2a_{12D}a_{21D}^2a_{3D}a_{4R} + 2a_{12D}a_{22D}^2a_{3D}a_{4R} - 2a_{12D}^2a_{22D}a_{4D}a_{4R} - 2a_{22D}a_{3D}^2a_{4D}a_{4R} \\
&+ 2a_{12D}a_{3D}a_{4D}^2a_{4R} + a_{11D}^2(a_{22D}a_{2R}a_{3D} + a_{12D}a_{2R}a_{4D} - 2a_{22D}a_{4D}a_{4R}) + (a_{22D}a_{3D} - a_{12D}a_{4D})\sqrt{\delta_0}, \qquad \text{(D.7)}\\
\tau_{22} &= a_{11D}^3a_{2R}a_{4D} - a_{11D}^2a_{21D}(a_{2R}a_{3D} + 2a_{4D}a_{4R}) + a_{11D}\left\{-a_{4D}\left[a_{2R}a_{3D}^2 + a_{1R}(a_{22D}^2 - a_{21D}^2 + a_{4D}^2)\right]\right. \\
&\left.+a_{12D}^2a_{2R}a_{4D} + 2a_{3D}(a_{21D}^2 + a_{22D}^2 + a_{4D}^2)a_{4R} - 2a_{12D}a_{22D}a_{2R}a_{3D}\right\} + a_{21D}\left\{a_{1R}\left[2a_{12D}a_{22D}a_{4D}\right.\right. \\
&\left.\left.-(a_{21D}^2 + a_{22D}^2)a_{3D} + a_{3D}a_{4D}^2\right] + (a_{12D}^2 + a_{3D}^2)(a_{2R}a_{3D} - 2a_{4D}a_{4R})\right\} + (a_{21D}a_{3D} - a_{11D}a_{4D})\sqrt{\delta_0}, \\
r_{11} &= -2a_{11D}a_{12D}a_{21D}a_{2R}a_{3D} - a_{12D}^2a_{22D}a_{2R}a_{3D} + a_{22D}a_{2R}a_{3D}^3 + a_{12D}^3a_{2R}a_{4D} - a_{12D}a_{2R}a_{3D}^2a_{4D} \\
&- a_{1R}\left[(a_{22D}^2 - a_{4D}^2)(a_{22D}a_{3D} - a_{12D}a_{4D}) - 2a_{11D}a_{21D}a_{22D}a_{4D} + a_{21D}^2(a_{22D}a_{3D} + a_{12D}a_{4D})\right] \\
&+ 2a_{12D}a_{21D}^2a_{3D}a_{4R} + 2a_{12D}a_{22D}^2a_{3D}a_{4R} - 2a_{12D}^2a_{22D}a_{4D}a_{4R} - 2a_{22D}a_{3D}^2a_{4D}a_{4R} \\
&+ 2a_{12D}a_{3D}a_{4D}^2a_{4R} + a_{11D}^2(a_{22D}a_{2R}a_{3D} + a_{12D}a_{2R}a_{4D} - 2a_{22D}a_{4D}a_{4R}) + (a_{12D}a_{4D} - a_{22D}a_{3D})\sqrt{\delta_0}, \qquad \text{(D.8)}\\
r_{22} &= a_{11D}^3a_{2R}a_{4D} - a_{11D}^2a_{21D}(a_{2R}a_{3D} + 2a_{4D}a_{4R}) + a_{11D}\left\{-a_{4D}\left[a_{2R}a_{3D}^2 + a_{1R}(a_{22D}^2 - a_{21D}^2 + a_{4D}^2)\right]\right. \\
&\left.+a_{12D}^2a_{2R}a_{4D} + 2a_{3D}(a_{21D}^2 + a_{22D}^2 + a_{4D}^2)a_{4R} - 2a_{12D}a_{22D}a_{2R}a_{3D}\right\} + a_{21D}\left\{a_{1R}\left[2a_{12D}a_{22D}a_{4D}\right.\right. \\
&\left.\left.-(a_{21D}^2 + a_{22D}^2)a_{3D} + a_{3D}a_{4D}^2\right] + (a_{12D}^2 + a_{3D}^2)(a_{2R}a_{3D} - 2a_{4D}a_{4R})\right\} - a_{21D}a_{3D}\sqrt{\delta_0} + a_{11D}a_{4D}\sqrt{\delta_0}, \qquad \text{(D.9)}\\
\tau_0 &= a_{11D}a_{21D}(a_{1R}a_{4D}^2 - a_{2R}a_{3D}^2) + a_{12D}a_{22D}(a_{1R}a_{4D}^2 - a_{2R}a_{3D}^2) + a_{3D}(a_{21D}^2 + a_{22D}^2)(a_{3D}a_{4R} - a_{1R}a_{4D}) \\
&+ a_{11D}^2a_{4D}(a_{2R}a_{3D} - a_{4D}a_{4R}) + a_{12D}^2a_{4D}(a_{2R}a_{3D} - a_{4D}a_{4R}), \\
\delta_0 &= a_{1R}^2(a_{21D}^2 + a_{22D}^2 + a_{4D}^2)^2 + 2a_{1R}a_{2R}\left[4a_{11D}a_{21D}(a_{12D}a_{22D} + a_{3D}a_{4D}) + 4a_{12D}a_{22D}a_{3D}a_{4D}\right. \\
&\left.+a_{11D}^2(a_{21D}^2 - a_{22D}^2 - a_{4D}^2) - a_{3D}^2(a_{21D}^2 + a_{22D}^2 - a_{4D}^2) - a_{12D}^2(a_{21D}^2 - a_{22D}^2 + a_{4D}^2)\right] \\
&+ (a_{11D}^2 + a_{12D}^2 + a_{3D}^2)\left[a_{2R}^2(a_{11D}^2 + a_{12D}^2 + a_{3D}^2) - 4a_{2R}(a_{11D}a_{21D} + a_{12D}a_{22D} + a_{3D}a_{4D})a_{4R}\right. \\
&\left.+4(a_{21D}^2 + a_{22D}^2 + a_{4D}^2)a_{4R}^2\right] - 4a_{1R}a_{4R}(a_{11D}a_{21D} + a_{12D}a_{22D} + a_{3D}a_{4D})(a_{21D}^2 + a_{22D}^2 + a_{4D^2}), \qquad \text{(D.10)}
\end{aligned}$$

with $a_{ijD},\ \ a_{kD},\ \ a_{nR}\ \ (i,j=1,2;\, k=3,4;\, n=1,2,3,4)$ are defined in Eq. (25).

E. Excluded interactions

TABLE III. Excluded interactions.

Couplings	Prevented by
$(\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\phi)_{\underline{1}_1}, (\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\varphi)_{\underline{2}_2}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}H, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}\widetilde{H},$ $(\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\varphi)_{\underline{2}_2}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\varphi)_{\underline{2}_2}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\phi^*)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\varphi^*)_{\underline{1}_1},$ $(\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1},$	D_5
$(\overline{\psi}_{iL}l_{iR})_{\underline{1}}H, (\overline{\psi}_{iL}l_{iR})_{\underline{1}}(H\phi^*)_{\underline{1}}, (\overline{\psi}_{iL}l_{iR})_{\underline{2}_2}(H\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\eta)_{\underline{2}_1},$ $(\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\eta)_{\underline{2}_1}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\phi)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\phi^*)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\eta)_{\underline{2}_1},$ $(\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}\widetilde{H}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}\widetilde{H}, (\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\phi^*)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\phi^*)_{\underline{1}_1},$ $(\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1},$	Z_4
$(\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\rho^*)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\rho^*)_{\underline{1}_1},$	$B-L, D_5$
$(\overline{\psi}_{iL}l_{iR})_{\underline{1}_1}(H\rho)_{\underline{1}_1}, (\overline{\psi}_{iL}l_{iR})_{\underline{1}_1}(H\rho^*)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\rho)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\rho^*)_{\underline{1}_1},$ $(\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\rho^*)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\rho^*)_{\underline{1}_1},$	$B-L, Z_4$
$(\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\rho)_{\underline{1}_1}, (\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\rho^*)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\rho)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\rho^*)_{\underline{1}_1},$ $(\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\rho^*)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\rho)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\rho^*)_{\underline{1}_1},$	$B-L, D_5, Z_4$
$(\overline{\psi}_{iL}l_{iR})(H\eta)_{\underline{2}_1}, (\overline{\psi}_{iL}l_{3R})H, (\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\phi^*)_{\underline{1}_1}, (\overline{\psi}_{iL}l_{3R})_{\underline{2}_1}(H\varphi^*)_{\underline{2}_2},$ $(\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\phi)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\phi^*)_{\underline{1}_1}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\varphi)_{\underline{2}_2}, (\overline{\psi}_{3L}l_{iR})_{\underline{2}_1}(H\varphi^*)_{\underline{2}_2},$ $(\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\varphi)_{\underline{2}_2}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{3L}l_{3R})_{\underline{1}_1}(H\eta)_{\underline{2}_1}, (\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}\widetilde{H},$ $(\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\phi)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\phi^*)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}\widetilde{H}, (\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\phi)_{\underline{1}_1},$ $(\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\phi^*)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\varphi)_{\underline{2}_2}, (\overline{\psi}_{iL}\nu_{1R})_{\underline{2}_1}(\widetilde{H}\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\varphi)_{\underline{2}_2},$ $(\overline{\psi}_{iL}\nu_{2R})_{\underline{2}_1}(\widetilde{H}\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\phi)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\phi^*)_{\underline{1}_1}, (\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\varphi)_{\underline{2}_2},$ $(\overline{\psi}_{iL}\nu_{3R})_{\underline{2}_1}(\widetilde{H}\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\varphi^*)_{\underline{2}_2}, (\overline{\psi}_{3L}\nu_{1R})_{\underline{1}_1}(\widetilde{H}\eta)_{\underline{2}_1},$ $(\overline{\psi}_{3L}\nu_{2R})_{\underline{1}_1}(\widetilde{H}\eta)_{\underline{2}_1}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}\widetilde{H}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\phi)_{\underline{1}_1}, (\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\varphi)_{\underline{1}_1},$ $(\overline{\psi}_{3L}\nu_{3R})_{\underline{1}_1}(\widetilde{H}\eta)_{\underline{2}_1}, (\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\varphi\rho)_{\underline{2}_2},$ $(\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{2}_2}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\varphi\rho)_{\underline{2}_2}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{2}_2}, (\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\varphi\rho)_{\underline{2}_2},$ $(\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{2}_2}, (\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\varphi\rho)_{\underline{2}_2}, (\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{2}_2}, (\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\varphi\rho)_{\underline{2}_2},$ $(\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{2}_2}, (\overline{\nu}^C_{1R}\nu_{1R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{1R}\nu_{2R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{2R}\nu_{1R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1},$ $(\overline{\nu}^C_{2R}\nu_{2R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{3R}\nu_{3R})_{\underline{1}_1}(\eta\rho)_{\underline{2}_1}, (\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}\rho, (\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}(\varphi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}\rho, (\overline{\nu}^C_{1R}\nu_{3R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}(\varphi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{2R}\nu_{3R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}\rho, (\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}(\varphi\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{3R}\nu_{1R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}\rho, (\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}(\phi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}(\phi^*\rho)_{\underline{1}_1},$ $(\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}(\varphi\rho)_{\underline{1}_1}, (\overline{\nu}^C_{3R}\nu_{2R})_{\underline{1}_1}(\varphi^*\rho)_{\underline{1}_1},$	D_5, Z_4

F. The dependence of $k_{12}, \tau_{12}, r_{12}, \sin\delta_{CP}$ and $\eta_{1,2}$ on $\cos\psi$

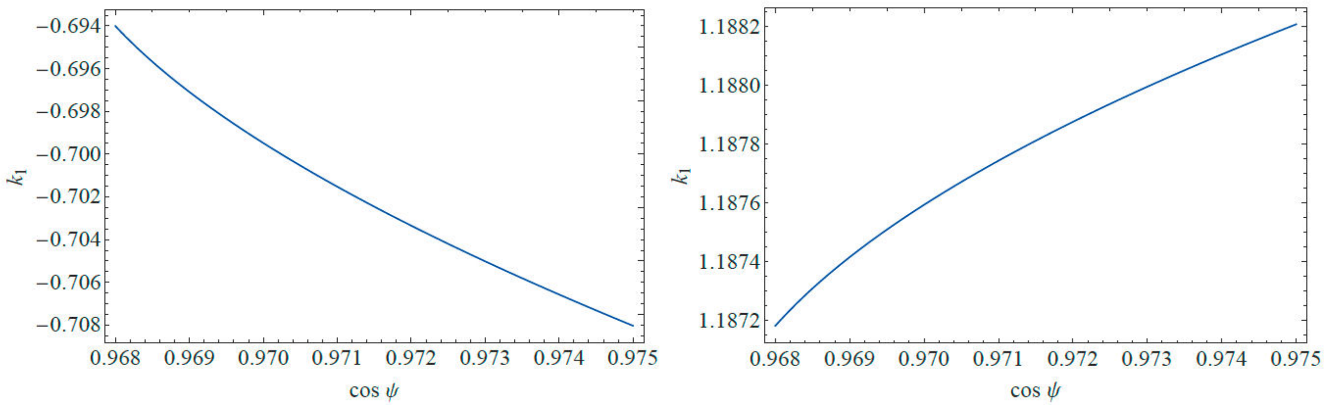

FIGURE 10. κ_1 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

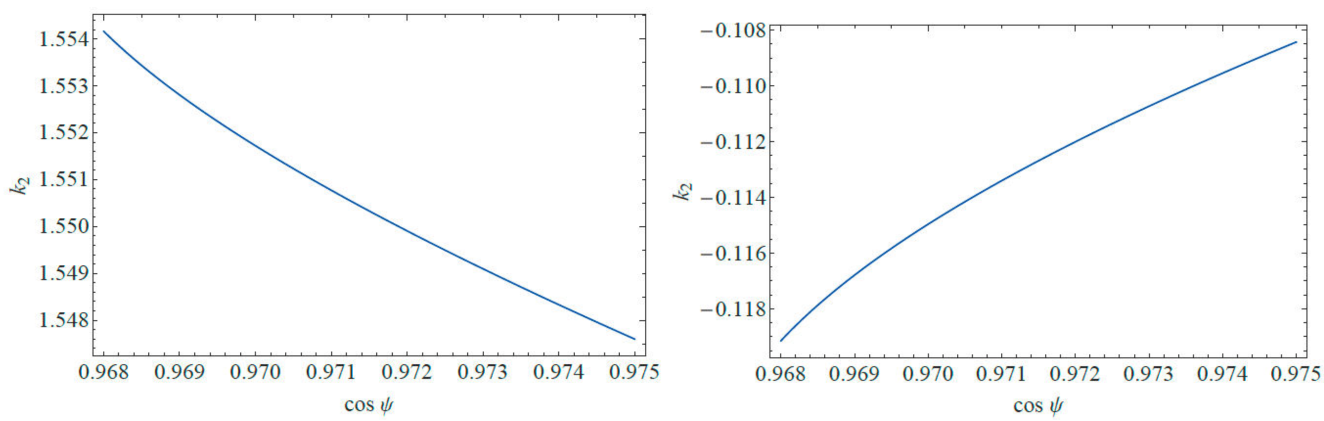

FIGURE 11. κ_2 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

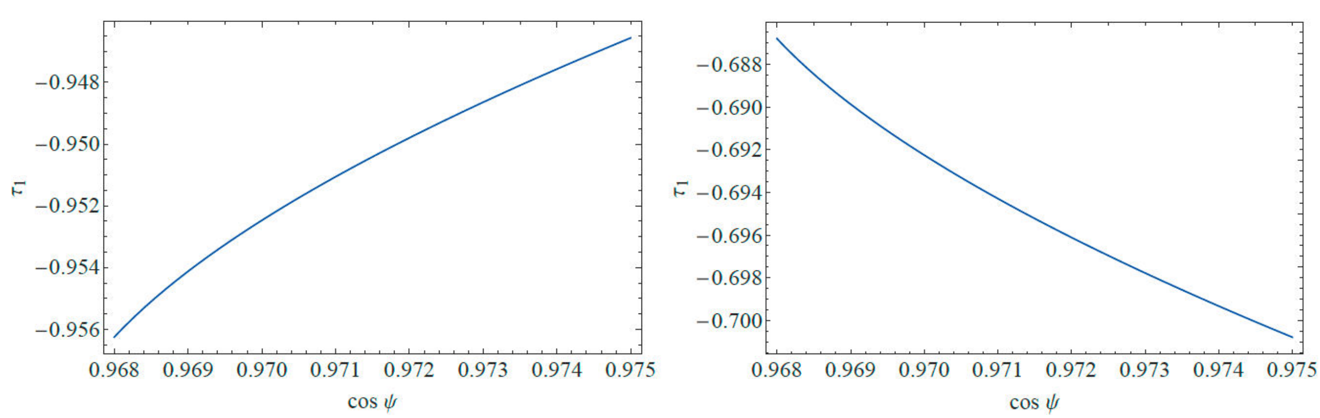

FIGURE 12. τ_1 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

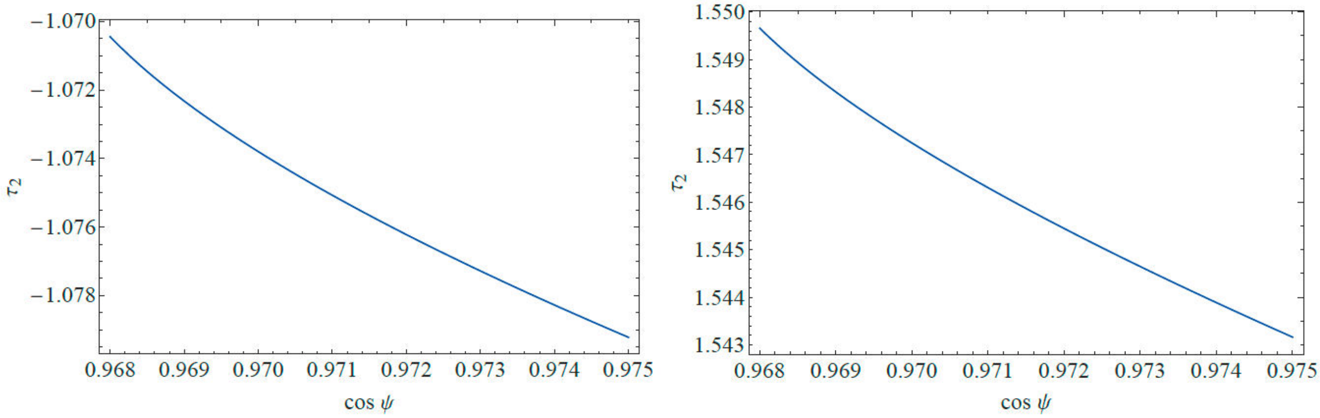

FIGURE 13. τ_2 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

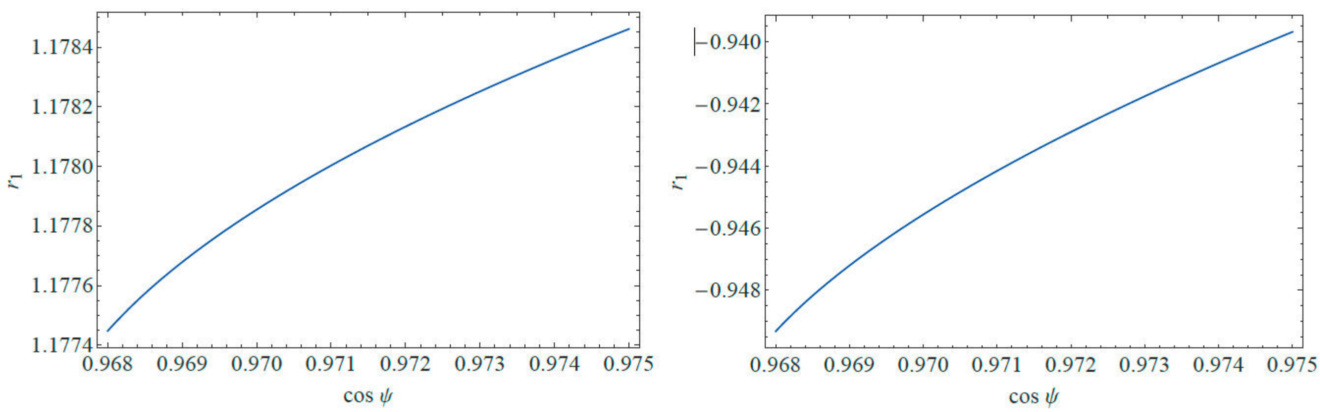

FIGURE 14. r_1 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

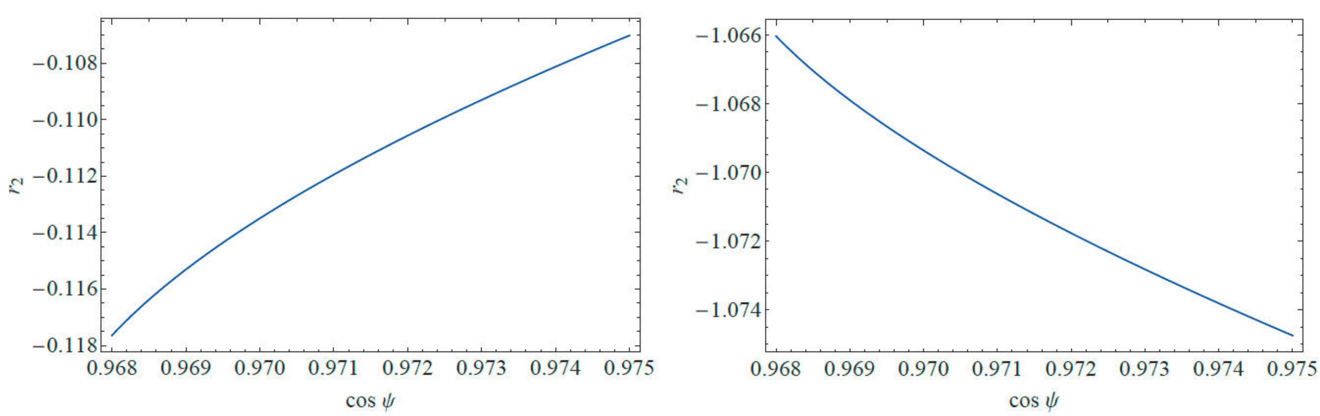

FIGURE 15. r_2 versus $\cos\psi$ with $\cos\psi \in (0.968, 0.975)$ for NH (left side) and IH (right side).

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i. As shown in Table I, the current model contains only one $SU(2)_L$ doublet and six singlets.

ii. For the convenience, we briefly present the Clebsch-Gordan coefficients of D_5 symmetry in Appendix A.

iii. Here, $a_0 = |a_l|$, $b_0 = |b_l|$ and $c_0 = |c_l|$.

iv. For simplicity, hereafter we use the notations $c_\theta = \cos\theta,\ s_\theta = \sin\theta,\ c_\psi = \cos\psi,\ s_\psi = \sin\psi$.

v. Here, $x_{0i} = |x_i|\ (i = 1, 2, 3)$.

vi. The range of v_{φ_2} and v_{η_2} are obtained from Eqs. (56) and (57).

vii. Here, the following notation is used: $V(\chi \to \chi', \zeta \to \zeta') = V(\chi, \zeta)|_{\chi=\chi', \zeta=\zeta'}$.

$viii$. The considered model has many free parameters in the scalar potential. For simplicity, we consider the case where the couplings in the same type of interaction are equal to each other which is similar to that of Ref. [25].

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