Operator associated to the index of degeneration of Landau's levels

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We determine the operators associated to the index of degeneration of Landau's levels for the Landau's gauge and symmetric gauge of the magnetic field, with the non separable eigenvalue solution obtained for a charged particle in a flat box under a magnetic field perpendicular to flat motion.

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1. Introduction

The problem of the quantum motion of a single charged particle q of mass m moving in a flat box under strong magnetic field, perpendicular to the flat motion and characterized by a vector potential $\mathbf{A}$, where $\mathbf{B} = \nabla \times \mathbf{A}$ [1], has received more attention due to the finding of non separable eigenfunction for the eigenvalue problem [2, 3]

$$\widehat{H} f_n^j = E_n f_n^j, \quad j, n \in \mathcal{Z}^+ \qquad \text{and}$$

$$\widehat{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2, \tag{1}$$

but having the same Landau's levels [5]

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right), \qquad \omega_c = \frac{qB}{mc}, \tag{2}$$

being ω_c the cyclotron frequency. These Landau's levels result to be numerable degenerate for each level E_n, $\{f_n^j\}_{j\in\mathcal{Z}}$, indicated by the index "j" in the eigenvalue functions, and this occurs with the Landau's gauge,

$$\mathbf{A} = (-By, 0, 0), \tag{3}$$

or with the symmetric gauge,

$$\mathbf{A} = \frac{B}{2}(y, -x, 0). \tag{4}$$

According to Quantum Mechanics Theory [6] , this index must correspond to an eigenvalue problem of some linear operator which commutes with the Hamiltonian. In this paper, we determine a possible operator with this characteristic which determines the index "j" of the eigenfunction f_n^j, and this will be done for the Landau's gauge and the symmetric gauge.

2. Analytical analysis

The Hamiltonian associated to the motion of charged particle of charge q and mass m in a transversal magnetic field determined by the vector potential $\mathbf{A} = (A_x, A_y, 0)$, where $A_i = A_i(x, y)$ $i = x, y$ and $\mathbf{B} = (\partial_x A_y - \partial_y A_x)\hat{\mathbf{k}}$, is given by (restricting the motion on the plane)

$$\widehat{H} = \frac{1}{2m}\left(p_x - \frac{q}{c}A_x\right)^2 + \frac{1}{2m}\left(p_y - \frac{q}{c}A_y\right)^2. \tag{5}$$

From the Heisenberg's evolution equation [7] of any linear operator $\widehat{\mathcal{O}}$,

$$\frac{d\widehat{\mathcal{O}}}{dt} = \frac{1}{i\hbar}[\widehat{\mathcal{O}}, \widehat{H}] + \frac{\partial\widehat{\mathcal{O}}}{\partial t}, \tag{6}$$

it follows that

$$m\dot{x} = \widehat{p}_x - \frac{q}{c}A_x, \qquad \text{and} \qquad m\dot{y} = \widehat{p}_y - \frac{q}{c}A_y, \tag{7}$$

which define the operators

$$\widehat{\pi}_x = \widehat{p}_x - \frac{q}{c}A_x, \qquad \text{and} \qquad \widehat{\pi}_y = \widehat{p}_y - \frac{q}{c}A_y. \tag{8}$$

(I): For the Landau's gauge (3), which represents the magnetic field in the $\hat{z} = \hat{\mathbf{k}}$ direction $\mathbf{B} = B\widehat{\mathbf{k}}$, we get the operators

$$\widehat{\pi}_x^{(+)} = \widehat{p}_x + m\omega_c y, \qquad \text{and} \quad \widehat{\pi}_y^{(+)} = \widehat{p}_y. \tag{9}$$

If we invert the direction of the magnetic field with the guage $\mathbf{A} = (0, -Bx, 0)$, $\mathbf{B} = -B\widehat{\mathbf{k}}$, the operators would be

$$\widehat{\pi}_x^{(-)} = \widehat{p}_x, \qquad \text{and} \quad \widehat{\pi}_y^{(-)} = \widehat{p}_y + m\omega_c x, \tag{10}$$

and the following commutation relations result

$$\begin{aligned}\left[\widehat{\pi}_x^{(+)}, \widehat{\pi}_x^{(-)}\right] &= \left[\widehat{\pi}_x^{(+)}, \widehat{\pi}_y^{(-)}\right] = \left[\widehat{\pi}_y^{(+)}, \widehat{\pi}_x^{(-)}\right] \\ &= \left[\widehat{\pi}_y^{(+)}, \widehat{\pi}_y^{(-)}\right] = 0,\end{aligned} \tag{11}$$

and

$$\left[\widehat{\pi}_x^{(-)}, \widehat{H}^{(+)}\right] = \left[\widehat{\pi}_y^{(-)}, \widehat{H}^{(+)}\right] = 0, \tag{12}$$

where $\widehat{H}^{(+)}$ is the Hamiltonian corresponding to the magnetic field in the $\widehat{\mathbf{k}}$ direction. From [3, 4], the eigenfunction $\{f_n^j\}_{j\in\mathcal{Z}}$ associated to the degeneration of the eigenvalue E_n are obtained just through the applications

$$f_n^j = (\widehat{\pi}_x^{(-)})^j f_n^0, \qquad \text{with}$$
$$f_n^0 = \frac{1}{\sqrt{L_y}}\left(\frac{m\omega_c}{\hbar}\right)^{1/4} e^{-i\frac{m\omega_c}{\hbar}xy}\varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\,x\right), \quad (13)$$

where φ_n is the harmonic oscillatior solution. In additon, we also note that

$$\widehat{\pi}_y^{(-)} f_n^j = 0. \quad (14)$$

It is not difficult to see that using this last expression, we get

$$\left[\widehat{\pi}_y^{(-)}, (\widehat{\pi}_x^{(-)})^{j+1}\right] = i\hbar m\omega_c (j+1)(\widehat{p}_x)^j. \quad (15)$$

So, applying this operator to f_n^0 and using (13), we get

$$\widehat{\pi}_y^{(-)}\widehat{\pi}_x^{(-)} f_n^j = i\hbar m\omega_c(j+1) f_n^j, \quad (16)$$

and the operator $\widehat{\pi}_y^{(-)}\widehat{\pi}_x^{(-)}$ is such that

$$\left[\widehat{\pi}_y^{(-)}\widehat{\pi}_x^{(-)}, \widehat{H}^{(+)}\right] = 0. \quad (17)$$

Thus, the non Hermitian operator $\widehat{\pi}_y^{(-)}\widehat{\pi}_x^{(-)}$ is a possible operator responsible of the numerable degeneration of Landau's levels, and this operator corresponds to the inverted direction of the magnetic field.

(II): For the symmetric gauge (4), representing the magnetic field in the negative direction $\mathbf{B} = -B\widehat{\mathbf{k}}$, the associated operators (8) are

$$\tilde{\pi}_x^{(-)} = \widehat{p}_x - \frac{m\omega_c}{2}y, \quad \text{and} \quad \tilde{\pi}_y^{(-)} = \widehat{p}_y + \frac{m\omega_c}{2}x. \quad (18)$$

If we select the magnetic field in the positive direction, $\mathbf{A} = B(-y, x, 0)/2$ and $\mathbf{B} = B\hat{\mathbf{k}}$, the resulting operators are

$$\tilde{\pi}_x^{(+)} = \widehat{p}_x + \frac{m\omega_c}{2}y, \quad \text{and} \quad \tilde{\pi}_y^{(+)} = \widehat{p}_y - \frac{m\omega_c}{2}x. \quad (19)$$

The following commutation relations result

$$\left[\tilde{\pi}_x^{(-)}, \tilde{\pi}_x^{(+)}\right] = \left[\tilde{\pi}_x^{(-)}, \tilde{\pi}_y^{(+)}\right] = \left[\tilde{\pi}_y^{(-)}, \tilde{\pi}_x^{(+)}\right] = \left[\tilde{\pi}_y^{(-)}, \tilde{\pi}_y^{(+)}\right] = 0, \quad (20)$$

and

$$\left[\tilde{\pi}_x^{(+)}, \widehat{H}^{(-)}\right] = \left[\tilde{\pi}_y^{(+)}, \widehat{H}^{(-)}\right] = 0, \quad (21)$$

where $\widehat{H}^{(-)}$ is the Hamiltonian with the magnetic field in the negative direction.

Previously [3, 4], the component $\hat{L}_z$ of the angular momentum was used as the generator of the degeneration of Landau's levels,

$$\tilde{f}_n^j = (\widehat{L}_z)^j \tilde{f}_n^0, \qquad \text{with}$$
$$\tilde{f}_n^0 = A_n e^{-\alpha(x^2+y^2)-\lambda(x-iy)}\big(2\alpha(x+iy)+\lambda\big)^n, \quad (22)$$

where $\alpha = qB/4\hbar c$ and $A_n = e^{-|\lambda|^2/4\alpha}/\sqrt{\pi(2\alpha)^{n-1}n!}$ is the constant of normalization. However, here we will use the operator $\tilde{\pi}_x^{(+)}$ to generate an equivalent set of eigenfunctions for the degenerated Landau's levels, and in fact, the operator $\widehat{L}_z$ can be written in terms of the operators $\tilde{\pi}_y^{(+)}$ and $\tilde{\pi}_x^{(+)}$ as $\widehat{L}_z = x\tilde{\pi}_y^{(+)} - y\tilde{\pi}_x^{(+)} + m\omega_c(x^2+y^2)/2$. In this way, we have

$$\tilde{\pi}_x^{(+)}\tilde{f}_n^0 = \left(\frac{m\omega_c}{2}(ix+y) + i\hbar\lambda\right)\tilde{f}_n^0 - i\hbar\sqrt{2\alpha n}\,\tilde{f}_{n-1}^0, \quad (23)$$

and

$$\tilde{\pi}_y^{(+)}\tilde{f}_n^0 = i\tilde{\pi}_x^{(+)}\tilde{f}_n^0 + 2\hbar\lambda\tilde{f}_n^0. \quad (24)$$

Then, the set $\{g_n^j\}_{j\in\mathcal{Z}}$ of degenerated functions associated to the eigenvalue E_n can be given by

$$g_n^j = (\tilde{\pi}_x^{(+)})^j\tilde{f}_n^0, \qquad \text{with} \qquad g_n^0 = \tilde{f}_n^0, \quad (25)$$

and it is not difficult to see that the following commutation relation results

$$\left[\left(\tilde{\pi}_x^{(+)}\right)^{j+1}, \tilde{\pi}_y^{(+)}\right] = i\hbar m\omega_c(j+1)\left(\tilde{\pi}_x^{(+)}\right)^j, \quad (26)$$

which brings about the following relation, using expression (25) and when is applied to g_n^0,

$$(\tilde{\pi}_x^{(+)})^j\tilde{\pi}_y^{(+)}g_n^0 - \tilde{\pi}_y^{(+)}\tilde{\pi}_x^{(+)}g_n^j = i\hbar m\omega_c(j+1)g_n^j, \quad (27)$$

or

$$\left[i(\tilde{\pi}_x^{(+)})^2 + 2\lambda\hbar\tilde{\pi}_x^{(+)} - \tilde{\pi}_y^{(+)}\tilde{\pi}_x^{(+)}\right]g_n^j = i\hbar m\omega_c(j+1)g_n^j. \quad (28)$$

Therefore, the operator $\widehat{\mathcal{O}}$ defined as

$$\widehat{\mathcal{O}} = i(\tilde{\pi}_x^{(+)})^2 + 2\lambda\hbar\tilde{\pi}_x^{(+)} - \tilde{\pi}_y^{(+)}\tilde{\pi}_x^{(+)}, \quad (29)$$

is an operator responsible of the index associated to the numerable degeneration of the Landau's levels,

$$\widehat{\mathcal{O}}g_n^j = i\hbar m\omega_c(j+1)g_n^j, \quad (30)$$

which commutes with the Hamiltonian associated to the magnetic field in the negative direction,

$$\left[\widehat{\mathcal{O}}, \widehat{H}^{(-)}\right] = 0. \quad (31)$$

3. Conclusions

We have constructed the operator responsible of the the index of the degeneration of Landau's levels for the problem of the quantum motion of a charged particle in a flat box (considering only the motion on the plane), and we have done this for the Landau's gauge and the symmetric gauges. These operators are intrinsic related with the inversion of the magnetic field on the system, and their commutation relations were determined, and it was verified that their commutation with the Hamiltonian was satisfied.

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