Black-Hole duality in four time and four space dimensions

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A black-hole solution in four time and four space dimensions ($(4+4)$-dimensions) is developed. It is emphasized that such a solution establishes a duality relation between the $(1+3)$ and the $(3+1)$ black-holes, which are part of the $(4+4)$-world. Moreover, it is found that a cosmological constant of the $(1+3)$-world is dual to the cosmological constant in the $(3+1)$-world.

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1. Introduction

Traditionally in relativity, in order to describe different phenomena in our universe, such as a trajectory of an object, one time and three space are the chosen number of real dimensions ($(1+3)$-dimensions). Yet the physical reasons why our world requires $(1+3)$-dimensions, remains as an open problem. It is evident that from the point of view of number theory the $(1+3)$-world the space and time are not symmetric. The natural question is: why three space and only one time dimension? Looking for the answer of this question one finds that the $(5+5)$-dimensional space-time (five time and five space dimensions) is a common signature to both type IIA strings and type IIB strings [1]. In fact, versions of M-theory [2-3] lead to type IIA and to type IIB string in space-time of signatures $(5+5)$. It turns out that by duality transformations string theories of signatures $(5+5)$ are related to other string signatures such as $(1+9)$ [3].

Of course, the $(5+5)$-dimensional world is more symmetrical in the number of space and time dimensions than the $(1+3)$-world. Thus, considering seriously the $(5+5)$-world, just as the $(1+3)$-dimensional signature can be considered as a reduced world of the de Sitter $(1+4)$-dimensional or anti-de Sitter $(2+3)$-dimensional signatures *via* the cosmological constants $\Lambda > 0$ and $\Lambda < 0$, respectively, here, one may assume that up to two cosmological constants, the $(4+4)$-world emerges from $(5+5)$-dimensional world. In fact, the $(4+4)$-dimensions can be considered as the transverse coordinates of the $(5+5)$-dimensions [4-5].

Fortunately, there are already a number of works with interesting results in the $(4+4)$-world that can be considered as additional motivation for increasing interest in such a scenario. First, the Dirac equation in $(4+4)$-dimensions is consistent with Majorana-Weyl spinors which give exactly the same number of components as the complex spinor of $1/2$-spin particles such as the electron or quarks [6,7]. Second, the most general Kruskal-Szekeres transformation of a black-hole coordinates in $(1+3)$-dimensions leads to 8-regions (instead of the usual 4-regions), which can be better described in $(4+4)$-dimensions [8]. Third, loop quantum gravity in $(4+4)$-dimensions [9-10] admits a self-duality curvature structure analogue to the traditional $(1+3)$-dimensions. It also has been shown [11] that duality

$$\sigma^2 \leftrightarrow \frac{1}{\sigma^2},$$

of a Gaussian distribution in terms of the standard deviation σ of 4-space coordinates associated with the de Sitter space (anti-de Sitter) and the vacuum zero-point energy yields to a Gaussian of 4-time coordinates of the same vacuum scenario. Finally, it has been suggested that the mathematical structures of matroid theory [12] (see also Refs. [13-20] and references therein) and surreal number theory [21-23] (see also Refs [24-25] and references therein) may provide interesting routes for a connection with the $(4+4)$-world.

2. The (4+4)-world black hole

Let us start considering the ansatz

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} -e^{f(r,\rho)} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{h(r,\rho)} & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2\tilde{g}_{ij}(\theta_{(+)}) & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{p(r,\rho)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{q(r,\rho)} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho^2\tilde{g}_{ab}(\theta_{(-)}) \end{pmatrix}. \tag{1}$$

Here, the indices $\hat{\mu}, \hat{\nu}, ...$run from 1 to 8 and the matrices $\tilde{g}_{ij}(\theta_{(+)})$ and $\tilde{g}_{ab}(\theta_{(-)})$ are defined as

$$\tilde{g}_{ij}(\theta_{(+)}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta_{(+)} \end{pmatrix}, \tag{2}$$

and

$$\tilde{g}_{ab}(\theta_{(-)}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta_{(-)} \end{pmatrix}, \tag{3}$$

respectively. Here, the notation $\theta_{(+)}$ and $\theta_{(-)}$ means that the angle $\theta_{(+)}$ refers to the $(1+3)$-world, while the angle $\theta_{(-)}$ corresponds to the $(3+1)$-world.

From the chosen form of $g_{\mu\nu}$ it is evident that one is dealing with a spheric symmetric static system in $(4+4)$-dimensions. The only unknown variables will be $f(r,\rho)$, $h(r,\rho)$, $p(r,\rho)$ and $q(r,\rho)$ which must be determined with the relativistic gravitational field equations in $(4+4)$-dimensions.

The non-vanishing Christoffel symbols associated with (1), involving the indices values of $\mu, \nu = 1$ to 4, are

$$\Gamma^1_{12} = \frac{f'}{2}, \qquad \Gamma^2_{22} = \frac{h'}{2}, \qquad \Gamma^2_{11} = \frac{e^{f-h}f'}{2}, \qquad \Gamma^2_{ij} = -re^{-h}\tilde{g}_{ij}, \qquad \Gamma^i_{2j} = \frac{\delta^i_j}{r}, \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \tag{4}$$

while for the values $\mu, \nu = 5$ to 8 one gets

$$\Gamma^5_{56} = \frac{\dot{p}}{2}, \qquad \Gamma^6_{66} = \frac{\dot{q}}{2}, \qquad \Gamma^6_{55} = \frac{e^{p-q}\dot{p}}{2}, \qquad \Gamma^6_{ab} = -\rho e^{-q}\tilde{g}_{ab}, \qquad \Gamma^a_{6b} = \frac{\delta^a_b}{\rho}, \qquad \Gamma^a_{bc} = \tilde{\Gamma}^a_{bc}, \tag{5}$$

where $A' = \partial A/\partial r$ and $\dot{B} = \partial B/\partial\rho$ for any arbitrary functions $A(r,\rho)$ and $B(r,\rho)$. One still must include the non-vanishing mixture Christoffel symbols

$$\Gamma^1_{16} = \frac{\dot{f}}{2}, \qquad \Gamma^6_{11} = -\frac{e^{f-q}\dot{f}}{2}, \qquad \Gamma^2_{26} = \frac{\dot{h}}{2}, \qquad \Gamma^6_{22} = \frac{e^{h-q}\dot{h}}{2}, \tag{6}$$

and

$$\Gamma^5_{52} = \frac{p'}{2}, \qquad \Gamma^2_{55} = -\frac{e^{p-h}p'}{2}, \qquad \Gamma^6_{62} = \frac{q'}{2}, \qquad \Gamma^2_{66} = \frac{e^{q-h}q'}{2}. \tag{7}$$

In vacuum the gravitational field equations simply establish that the Ricci tensor $R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}_{\hat{\mu}\hat{\alpha}\hat{\nu}}$ must vanish, that is one has

$$R_{\hat{\mu}\hat{\nu}} = 0. \tag{8}$$

Using the Christoffel symbols (4)-(7) one learns that (8) leads to

$$R_{11} = \frac{1}{2}e^{f-h}\left(f'' + \frac{1}{2}f'^2 - \frac{1}{2}f'h' + \frac{2}{r}f' + \frac{1}{2}f'p' + \frac{1}{2}f'q'\right) - \frac{1}{2}e^{f-q}\left(\ddot{f} + \frac{1}{2}\dot{f}^2 + \frac{1}{2}\dot{f}\dot{h} + \frac{2}{\rho}\dot{f} + \frac{1}{2}\dot{f}\dot{p} - \frac{1}{2}\dot{f}\dot{q}\right) = 0,$$

$$R_{22} = -\frac{1}{2}\left(f'' + \frac{1}{2}f'^2 - \frac{1}{2}f'h' - \frac{2}{r}h' + p'' + \frac{1}{2}p'^2 - \frac{1}{2}p'h' + q'' + \frac{1}{2}q'^2 - \frac{1}{2}q'h'\right)$$
$$+ \frac{1}{2}e^{h-q}\left(\ddot{h} + \frac{1}{2}\dot{h}^2 + \frac{1}{2}\dot{h}\dot{f} + \frac{1}{2}\dot{h}\dot{p} - \frac{1}{2}\dot{h}\dot{q} + \frac{2}{\rho}\dot{h}\right) = 0,$$

$$R_{ij} = e^{-h}\left(-\frac{1}{2}rf' + \frac{1}{2}rh' - \frac{1}{2}rp' - \frac{1}{2}rq' + e^h - 1\right)\tilde{g}_{ij} = 0, \tag{9}$$

(where due to (1) and (2) the indices $i, j, ...$run from 3 to 4), and also to

$$R_{55}=\frac{1}{2}e^{p-q}\left(\ddot{p}+\frac{1}{2}\dot{p}^2-\frac{1}{2}\dot{p}\dot{q}+\frac{2}{\rho}\dot{p}+\frac{1}{2}\dot{p}\dot{f}+\frac{1}{2}\dot{p}\dot{h}\right)-\frac{1}{2}e^{p-h}\left(p''+\frac{1}{2}p'^2+\frac{1}{2}p'q'+\frac{2}{r}p'+\frac{1}{2}p'f'-\frac{1}{2}p'h'\right)=0,$$

$$R_{66}=-\frac{1}{2}\left(\ddot{p}+\frac{1}{2}\dot{p}^2-\frac{1}{2}\dot{p}\dot{q}-\frac{2}{\rho}\dot{q}+\ddot{f}+\frac{1}{2}\dot{f}^2-\frac{1}{2}\dot{f}\dot{q}+\ddot{h}+\frac{1}{2}\dot{h}^2-\frac{1}{2}\dot{h}\dot{q}\right)$$
$$+\frac{1}{2}e^{q-h}\left(q''+\frac{1}{2}q'^2+\frac{1}{2}q'p'+\frac{1}{2}q'f'-\frac{1}{2}q'h'+\frac{2}{r}q'\right)=0,$$
$$R_{ab}=e^{-q}\left(-\frac{1}{2}\rho\dot{p}+\frac{1}{2}\rho\dot{q}-\frac{1}{2}\rho\dot{f}-\frac{1}{2}\rho\dot{h}+e^{q}-1\right)\tilde{g}_{ab}=0. \tag{10}$$

Here, according to the ansatz choice (1) and (3), the indices $a, b, ...$take the values 7 and 8.

3. Black-hole duality solution

Our next step is to look for a black-hole solution of (9) and (10). For this purpose, focusing in the last formula in (9) one observes that assuming the two equations

$$f'+h'=0,$$
$$p'+q'=0, \tag{11}$$

such a formula can be simplified in the form

$$rh'+e^{h}-1=0. \tag{12}$$

A general solution of this equation can be written as

$$e^{-h}=1-\frac{A(\rho)}{r}, \tag{13}$$

with $A(\rho)$ an arbitrary function of ρ. Following similar steps and assuming

$$\dot{p}+\dot{q}=0,$$
$$\dot{f}+\dot{h}=0, \tag{14}$$

the last equation in (10) leads to

$$\rho\dot{q}+e^{q}-1=0, \tag{15}$$

whose solution is

$$e^{-q}=1-\frac{B(r)}{\rho}, \tag{16}$$

with $B(r)$ an arbitrary function of r.

Our next step is to determine the functions $A(\rho)$ and $B(r)$. For this purpose, one may first focus in the first equation of (9). Considering (11) and (14) one see that such equation reduces to

$$\frac{1}{2}e^{f-h}\left(f''+f'^2+\frac{2}{r}f'\right)$$
$$-\frac{1}{2}e^{f-q}\left(\ddot{f}-\dot{f}\dot{q}+\frac{2}{\rho}\dot{f}\right)=0. \tag{17}$$

Since $e^{-h}=e^{f}$, with (13), one verifies that

$$f''+f'^2+\frac{2}{r}f'=0. \tag{18}$$

Thus, (17) is further reduced to

$$\ddot{f}-\dot{f}\dot{q}+\frac{2}{\rho}\dot{f}=0, \tag{19}$$

which can also be written as

$$\frac{\ddot{f}}{\dot{f}}-\dot{q}+\frac{2}{\rho}=0, \tag{20}$$

with $\dot{f}\neq 0$. This expression can be integrated yielding

$$\ln\dot{f}-q+\ln\rho^2=\ln a, \tag{21}$$

where $\ln a$ is a constant independent of ρ. This means that

$$\dot{f}e^{-q}=\frac{a}{\rho^2}. \tag{22}$$

From (16) one learns that (22) becomes

$$\dot{f}e^{-q}=-e^{-q}\dot{q}\frac{a}{B}. \tag{23}$$

Thus, if one sets $a=B$ one sees that

$$\dot{f}+\dot{q}=0, \tag{24}$$

in agreement with (14).

Hence, (19) can be written as

$$\ddot{f}+\dot{f}^2+\frac{2}{\rho}\dot{f}=0. \tag{25}$$

Substituting (13) into this equation one obtains

$$\ln\dot{A}+\ln\rho^2=-r_0^2,$$

with r_0 also a constant. Therefore,

$$\rho^2\dot{A}=-r_0^2, \tag{26}$$

and consequently one gets

$$A = \frac{r_0^2}{\rho}. \tag{27}$$

Substituting this result into (13) one discovers the surprising result

$$e^f = 1 - \frac{r_0^2}{\rho r}. \tag{28}$$

Following similar steps one shall obtain that the first equation in (10) leads to the solution

$$e^p = 1 - \frac{\rho_0^2}{\rho r}. \tag{29}$$

Summarizing, we have derived the black-hole solution in $(4+4)$-dimensions;

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{r_0^2}{\rho r}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1-\frac{r_0^2}{\rho r})} & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2\tilde{g}_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\frac{\rho_0^2}{\rho r}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{(1-\frac{\rho_0^2}{\rho r})} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho^2\tilde{g}_{ab} \end{pmatrix}. \tag{30}$$

In principle r_0 and ρ_0 are different constants but if one sets

$$r_0 = \rho_0 \equiv \xi. \tag{31}$$

the metric (30) becomes a totally dual black-hole solution. Let us explain in some detail this comment. First from (30) and (31) one sees that the line element in $(4+4)$-dimensions can be written as

$$ds^2 = ds_{(+)}^2 + ds_{(-)}^2, \tag{32}$$

where

$$ds_{(+)}^2 = -\left(1 - \frac{\xi^2}{\rho r}\right) c^2 dt_{(+)}^2 + \frac{dr^2}{(1-\frac{\xi^2}{\rho r})} + r^2\left(d\theta_{(+)}^2 + \sin^2\theta_{(+)} d\phi_{(+)}^2\right), \tag{33}$$

and

$$ds_{(-)}^2 = +\left(1 - \frac{\xi^2}{\rho r}\right) c^2 dt_{(-)}^2 - \frac{d\rho^2}{(1-\frac{\xi^2}{\rho r})} - \rho^2\left(d\theta_{(-)}^2 + \sin^2\theta_{(-)} d\phi_{(-)}^2\right). \tag{34}$$

In order for $ds_{(+)}^2$ to describe the usual black-hole element in $(1+3)$-world one must set

$$\frac{\xi^2}{\rho} = \frac{2GM^{(+)}}{c^2}, \tag{35}$$

with G the Newton gravitational constant and $M^{(+)}$ the mass source in the $(1+3)$-world. Similarly in order for $ds_{(-)}^2$ to describe the usual black-hole element in $(3+1)$-world one must set

$$\frac{\xi^2}{r} = \frac{2GM^{(-)}}{c^2}, \tag{36}$$

where $M^{(-)}$ is the source mass in the $(3+1)$-world. Thus, from the perspective of a $(1+3)$-world observer the combination ξ^2/ρ is just a related to the source mass $M^{(+)}$. This means that even if the parameter ρ associated with $(3+1)$-world appears in the line element $ds_{(+)}^2$ is in fact related to source mass $M^{(+)}$ according to (35). In dual form, the parameter r of the $(1+3)$-world is interpreted by a $(3+1)$-world observer as the mass source $M^{(-)}$ according to (36).

Even clearer dual properties of the line element (32)-(34) emerge when one considers the event horizon of (33) and (34). Suppose there are parameters r and ρ such that

$$\left(1 - \frac{\xi^2}{\rho_s r_s}\right) = 0. \tag{37}$$

Of course, in this case both the terms with dr^2 and $d\rho^2$ present an apparent singularity. From (37) one sees that this means that

$$\rho_s r_s = \xi^2, \tag{38}$$

which is clearly a dual relation: A large radius r_s of the event horizon of the $(1+3)$ black-hole corresponds to a small radius ρ_s of the $(3+1)$ black-hole and *vice versa*.

At least at the level of black-holes the above result establishes a dual link between the $(1+3)$-world and the $(3+1)$-world that needs to be consider when one looks for a solution of quantum gravity. In fact, thinking about the magnetic monopole g and the electric charge e duality, namely

$$e \longleftrightarrow \frac{n\hbar}{e}, \tag{39}$$

with $ge = n\hbar$ and $n = 1, 2, ...$, one is tempted to assume that (38) implies a quantum duality of the form

$$\rho_s r_s = n\xi^2, \tag{40}$$

Of course in order to fully understand the consequences of (38) or (40) one needs to clarify the meaning of the constant parameter ξ. At first sight one may propose that $\xi = l_P$, with l_P the Plank length. However, in this case (38) implies a smaller black-hole radius than the Planck-length. So, assuming that the Planck length l_P is the smallest possible length then one must expect that $\xi \sim 1$.

4. Cosmological constant duality

Let us now introduce two cosmological constants; Λ^+ for the $(1+3)$-world and Λ^- for the $(3+1)$-world. Thus, one assumes that the gravitation field equation (8) can be splitted as

$$R_{\mu\nu} = \Lambda^+ g_{\mu\nu}, \tag{41}$$

and

$$R_{AB} = \Lambda^- g_{ab}, \tag{42}$$

with the indice $\mu, \nu, ...$runing from 1 to 4 and the indices A, B runing from 5 to 8. Assuming again (11) and (14) we find that the relevant equation in the $(1+3)$-world will be

$$rh' + e^h - 1 = \Lambda^+ r^2, \tag{43}$$

with a general solution of the form

$$e^{-h} = 1 - \frac{A(\rho)}{r} + F(\rho)\Lambda^+ r^2, \tag{44}$$

with $A(\rho)$ and $F(\rho)$ arbitrary functions of ρ. For the corresponding equation for the $(3+1)$-world one shall have

$$\rho\dot{q} + e^q - 1 = \Lambda^- \rho^2, \tag{45}$$

whose a general solution becomes

$$e^{-q} = 1 - \frac{B(r)}{\rho} + G(r)\Lambda^- \rho^2, \tag{46}$$

with $B(r)$ and $G(r)$ arbitrary functions of r. Again, one may choose $A(\rho)$ and $B(r)$ as $A = r_0^2/\rho$ and $B = r_0^2/r$, respectively. While a dual solution for $F(\rho)$ and $G(r)$ is obtained by setting $F = l^{-2}\rho^2$ and $G = l^{-2}r^2$, with l a dimensional fundamental constant. An intesting aspect of this construction emerges if one chooses a black-hole horizon such that

$$\Lambda^+\Lambda^- = const. \tag{47}$$

Of course this formula leads to a cosmological constant duality of the form

$$\Lambda^- \leftrightarrow \frac{const.}{\Lambda^+}. \tag{48}$$

This means that a small cosmological constant Λ^+ in the $(1+3)$-world must lead to a large cosmological constant Λ^- in the $(3+1)$-world and *vice versa*, as predicted in Ref. [26].

5. Final remarks

The present work opens many possible physical routes for further work. First, it may be interesting to consider a generalized Kruskal-Szekeres transform of the line element (32). This must lead to a connection with the observation [8] that in the $(4+4)$-world such a transform implies 8-regions instead of the usual 4-regions. Second, since it has been shown that in $(4+4)$-dimensions there exist a kind of duality of the cosmological constant one wonders what is the relation of such a duality with dual black-hole solution developed in this work (see Ref. [11]). Finally the quantum relation (40) may motive to see the consequences of our dual black-hole solution with quantum gravity theory. At this respect, it is worth mentioning that oriented matroid theory [12] (see also Refs. [13-19] and references therein) and surreal number theory (see Ref. [20] and also Refs [21-24] and references therein) are two promising underlaying mathematical structures for dealing with the key dual concept in $(4+4)$-dimensions [18].

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