# Applications of the complex, double and dual numbers in Lagrangian mechanics

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It is shown that in some examples of classical mechanics, the complex, double and dual numbers are useful in the search of symmetries of the equations of motion. As a byproduct, we obtain non-standard Lagrangians for the systems under consideration.

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## 1. Introduction

Apart from their use in the calculation of some definite integrals, the complex numbers have numerous applications in physics, especially in electrodynamics, and they are essential in the standard formulation of quantum mechanics. By contrast, one finds very few examples of the application of the complex numbers in classical mechanics. On the other hand, there are two additional sets of numbers, somewhat analogous to the complex ones, called double and dual numbers here (though they receive other names in the literature), that are seldom employed in physics, or even in mathematics (see, however, Ref. [1]). Nevertheless, in recent works some examples of the application of the double and the dual numbers in the standard equations of mathematical physics have been given [2–5]. (The double numbers are also employed in the construction of alternative physical theories, see, *e.g.*, Refs. [6–10].)

In some of the applications of the complex numbers (*e.g.*, in general relativity), the number of equations can be reduced by half just because a complex equation is equivalent to two real equations. In a similar manner, an equation involving double or dual numbers is equivalent to two real equations that can be handled simultaneously [1, 5].

In classical mechanics, apart from the interest in solving the equations of motion, a related problem is that of finding the variational symmetries of a given Lagrangian because they are associated with conserved quantities (see, *e.g.*, Refs. [11–14]).

The aim of this paper is to show that in some problems of classical mechanics, the use of complex, double or dual numbers greatly simplifies the search of symmetries of the equations of motion, by means of an appropriate Lagrangian. (Any variational symmetry of a Lagrangian leaves invariant the form of the corresponding equations of motion, but the converse is not true (see, *e.g.*, Ref. [15]).) Here, again, we see that the fact that the double and the dual numbers are not fields in the algebraic sense, does not impede their use in various ways. Moreover, a great advantage is that these numbers obey most of the algebraic rules applicable to the real and complex ones and, therefore, we can perform the computations in exactly the same manner as if we were dealing with real variables.

All examples considered in this paper correspond to mechanical systems with a number of degrees of freedom equal to two and their standard Lagrangians are polynomials of degree two in $\dot{x}$ and $\dot{y}$; hence, the partial differential equation that determines their variational symmetries leads to a system of *ten* differential equations that only involve $x, y$ and $t$ (which are obtained by considering the coefficients of $\dot{x}^3, \dot{x}^2\dot{y}, \dot{x}\dot{y}^2, \dot{y}^3, \dot{x}^2, \dot{x}\dot{y}, \dot{y}^2, \dot{x}, \dot{y}$ and the terms that do not contain $\dot{x}$ or $\dot{y}$). By contrast, the use of a complex, double or dual variable, $z$, leads to *four* differential equations that only involve $z$ and $t$ (which are obtained by considering the coefficients of $\dot{z}^3, \dot{z}^2, \dot{z}$ and the terms that do not contain $\dot{z}$).

The examples considered here belong to a special class: their equations of motion, written in terms of the complex combination $z \equiv x + \mathrm{i}y$, amount to $\ddot{z} = f(z, \dot{z}, t)$, where $f$ is an analytic function of $z$ and $\dot{z}$, and any such equation can be expressed in the form

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z}, \tag{1}$$

where $L(z, \dot{z}, t)$ is a suitable function (again, $L$ is an analytic function of $z$ and $\dot{z}$). The proof of the existence of $L$ and the recipe to construct it is given, *e.g.*, in Refs. [14, 16] for the case of real variables but, under the present assumptions, they can be applied without change in the case of complex functions.

In Sec. 2 we give two examples making use of complex numbers and in Sec. 3 we give examples that can be conveniently treated using complex, double or dual numbers.

## 2. Examples with complex numbers

There are two nice examples, closely related to each other, where the complex numbers are very useful: the problem of a charged particle moving on a plane in a uniform magnetic field and the two-dimensional isotropic harmonic oscillator. In the first case, the equations of motion for a charged particle of mass $m$ and electric charge $e$ moving on a plane in a magnetic field $B_0$ perpendicular to this plane, expressed in Cartesian coordinates, are

$$m\ddot{x} = m\omega_{\rm c}\dot{y}, \qquad m\ddot{y} = -m\omega_{\rm c}\dot{x}, \tag{2}$$

where

$$\omega_{\rm c} \equiv \frac{eB_0}{mc},$$

is the so-called cyclotron frequency (in Gaussian units). Then, with the definition $z \equiv x + {\rm i}y$ these equations amount to the single complex equation

$$m\ddot{z} = -m{\rm i}\,\omega_{\rm c}\dot{z}. \tag{3}$$

Applying the standard procedure to find Lagrangians for a given second-order ordinary differential equation (see, *e.g.*, Refs. [13, 14, 16]), one readily finds the Lagrangian

$$L = \tfrac{1}{2}m{\rm e}^{{\rm i}\omega_{\rm c}t}\dot{z}^2, \tag{4}$$

corresponding to Eq. (3).

It may be noticed that the complex Lagrangian (4), written in terms of $\dot{x}$ and $\dot{y}$, is

$$\begin{aligned} L &= \tfrac{1}{2}m\big[(\dot{x}^2 - \dot{y}^2)\cos(\omega_{\rm c}t) - 2\dot{x}\dot{y}\sin(\omega_{\rm c}t)\big] \\ &\quad + {\rm i}\tfrac{1}{2}m\big[(\dot{x}^2 - \dot{y}^2)\sin(\omega_{\rm c}t) + 2\dot{x}\dot{y}\cos(\omega_{\rm c}t)\big] \end{aligned} \tag{5}$$

and one can verify that the real and imaginary parts of $L$ are two, essentially equivalent to each other, (real) Lagrangians leading to Eqs. (2), which depend explicitly on the time and *both coordinates*, $x$ and $y$, *are ignorable*. (It may be remarked that, by contrast, the usual Lagrangian for this problem is time-independent but the coordinates $x$ and $y$ cannot be both ignorable.)

In order to find more symmetries of the Lagrangian (4) and the corresponding constants of motion, we look for groups of variational symmetries of $L$ by copying the equation for (the infinitesimal generators of) these symmetries obtained in the usual case of real-valued coordinates (see, *e.g.*, Refs. [11–14])

$$\frac{\partial L}{\partial z}\eta + \frac{\partial L}{\partial \dot{z}}\left(\frac{{\rm d}\eta}{{\rm d}t} - \dot{z}\frac{{\rm d}\xi}{{\rm d}t}\right) + \frac{\partial L}{\partial t}\xi + L\frac{{\rm d}\xi}{{\rm d}t} = \frac{{\rm d}G}{{\rm d}t}, \tag{6}$$

where $\eta, \xi$ and $G$ are functions of $z$ and $t$ only and ${\rm d}f/{\rm d}t$ is an abbreviation for $\partial f/\partial t + \dot{z}\,\partial f/\partial z$; at this point the variables $z, \dot{z}$ and $t$ are independent (we are assuming that $\eta, \xi$ and $G$ are *analytic* functions of $z$, that is, there is no dependence on the complex conjugate of $z$). Equation (6) can also be expressed in the form

$$\frac{\partial L}{\partial z}\eta + \frac{\partial L}{\partial \dot{z}}\frac{{\rm d}\eta}{{\rm d}t} - \left(\dot{z}\frac{\partial L}{\partial \dot{z}} - L\right)\frac{{\rm d}\xi}{{\rm d}t} + \frac{\partial L}{\partial t}\xi = \frac{{\rm d}G}{{\rm d}t}. \tag{7}$$

As in the usual case of real-valued quantities one can show that, by virtue of the Lagrange equation (1), Eq. (7) is equivalent to the conservation of $\varphi$, defined by

$$\varphi = \eta\frac{\partial L}{\partial \dot{z}} - \xi\left(\dot{z}\frac{\partial L}{\partial \dot{z}} - L\right) - G. \tag{8}$$

Substituting (4) into Eq. (7) we get

$$\begin{aligned} &m{\rm e}^{{\rm i}\omega_{\rm c}t}\dot{z}\left(\frac{\partial\eta}{\partial t} + \dot{z}\frac{\partial\eta}{\partial z}\right) - \frac{m}{2}{\rm e}^{{\rm i}\omega_{\rm c}t}\dot{z}^2\left(\frac{\partial\xi}{\partial t} + \dot{z}\frac{\partial\xi}{\partial z}\right) \\ &\qquad + \frac{m}{2}{\rm i}\omega_{\rm c}{\rm e}^{{\rm i}\omega_{\rm c}t}\dot{z}^2\xi = \frac{\partial G}{\partial t} + \dot{z}\frac{\partial G}{\partial z}, \end{aligned} \tag{9}$$

which has to be satisfied for all values of $z, t$ and $\dot{z}$. Since $\eta, \xi$ and $G$ are functions of $z$ and $t$ only, the coefficients of the various powers of $\dot{z}$ on each side of this equation must coincide separately. By equating the coefficients of $\dot{z}^3$ we find that $\xi$ must be a function of $t$ only

$$\xi = A(t), \tag{10}$$

where $A$ is a real-valued function of a single variable (on the other hand, $\eta$ has complex values; this follows from the fact that $\xi = \partial t'/\partial s|_{s=0}$ and $\eta = \partial z'/\partial s|_{s=0}$, considering a one-parameter family of coordinate transformations $z' = z'(z,t,s)$, $t' = t'(z,t,s)$, such that, for $s = 0$, $z' = z$, $t' = t$ [11–14]). From the equality of the coefficients of $\dot{z}^2$ we obtain

$$\frac{\partial\eta}{\partial z} = \frac{1}{2}\frac{{\rm d}A}{{\rm d}t} - \frac{1}{2}{\rm i}\omega_{\rm c}A,$$

which implies that

$$\eta = \frac{z}{2}\left(\frac{{\rm d}A}{{\rm d}t} - {\rm i}\omega_{\rm c}A\right) + B(t), \tag{11}$$

where $B$ is a complex-valued function.

The equality of the terms proportional to $\dot{z}$ yields

$$\begin{aligned} \frac{1}{m}\frac{\partial G}{\partial z} &= {\rm e}^{{\rm i}\omega_{\rm c}t}\frac{\partial\eta}{\partial t} \\ &= {\rm e}^{{\rm i}\omega_{\rm c}t}\left[\frac{z}{2}\left(\frac{{\rm d}^2A}{{\rm d}t^2} - {\rm i}\omega_{\rm c}\frac{{\rm d}A}{{\rm d}t}\right) + \frac{{\rm d}B}{{\rm d}t}\right], \end{aligned} \tag{12}$$

where we have made use of (11), and from the equality of the terms independent of $\dot{z}$ we get

$$\frac{1}{m}\frac{\partial G}{\partial t} = 0. \tag{13}$$

Then, the equality of the mixed second partial derivatives of $G$ gives

$$\begin{aligned} &{\rm i}\omega_{\rm c}\left[\frac{z}{2}\left(\frac{{\rm d}^2A}{{\rm d}t^2} - {\rm i}\omega_{\rm c}\frac{{\rm d}A}{{\rm d}t}\right) + \frac{{\rm d}B}{{\rm d}t}\right] \\ &\qquad + \frac{z}{2}\left(\frac{{\rm d}^3A}{{\rm d}t^3} - {\rm i}\omega_{\rm c}\frac{{\rm d}^2A}{{\rm d}t^2}\right) + \frac{{\rm d}^2B}{{\rm d}t^2} = 0. \end{aligned} \tag{14}$$

The fulfillment of this condition for all values of $z$ (taking into account that $A$ and $B$ are functions of $t$ only) implies that

$$\frac{\mathrm{d}^3 A}{\mathrm{d}t^3} + {\omega_\mathrm{c}}^2 \frac{\mathrm{d}A}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2 B}{\mathrm{d}t^2} + \mathrm{i}\omega_\mathrm{c} \frac{\mathrm{d}B}{\mathrm{d}t} = 0, \tag{15}$$

and therefore

$$A(t) = c_1 \cos(\omega_\mathrm{c} t) + c_2 \sin(\omega_\mathrm{c} t) + c_3, \tag{16}$$

where $c_1, c_2$ and $c_3$ are arbitrary real constants, and

$$B(t) = c_4 + \mathrm{i}c_5 + (c_6 + \mathrm{i}c_7)\mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} t}, \tag{17}$$

where $c_4, \ldots, c_7$ are four additional arbitrary real constants.

Hence, from Eqs. (10)–(13), (16) and (17), we find that the most general solution of Eq. (9) is given by

$$\begin{aligned}
\xi &= c_1 \cos(\omega_\mathrm{c} t) + c_2 \sin(\omega_\mathrm{c} t) + c_3, \\
\eta &= \tfrac{1}{2}\omega_\mathrm{c} z\big( - c_1 \mathrm{i}\, \mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} t} + c_2 \mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} t} - c_3 \mathrm{i}\big) \\
&\quad + c_4 + \mathrm{i}c_5 + (c_6 + \mathrm{i}c_7)\mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} t}, \\
G &= m \left[ -(c_1 + \mathrm{i}c_2) \frac{{\omega_\mathrm{c}}^2 z^2}{4} - (c_6 + \mathrm{i}c_7)\mathrm{i}\omega_\mathrm{c} z \right],
\end{aligned} \tag{18}$$

where $c_1, \ldots, c_7$ are arbitrary real constants. Thus, the Lagrangian (4) possesses a seven-dimensional group of variational symmetries. Substituting (18) into Eq. (8), using the fact that $c_1, \ldots, c_7$ are arbitrary one obtains seven constants of motion, which cannot be functionally independent since for a regular system with a number of degrees of freedom equal to two there exist four functionally independent constants of motion.

It is interesting to note that the one-parameter group obtained from Eqs. (18) with $c_3 = 1$, and all the other constants $c_k$ equal to zero, is given by $t' = t + s$, $z' = z\, \mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} s/2}$, that is, translations in the time accompanied by specific rotations in the $xy$-plane. Such transformations are, separately, symmetries of the equations of motion (2), but the Lagrangian (4) does not possess these symmetries one by one. This is an example of the fact that the symmetries of the equations of motion, *may not be* variational symmetries of a Lagrangian leading to such equations (see, *e.g.*, Ref. [15]).

### 2.1. The two-dimensional isotropic harmonic oscillator

In the case of the two-dimensional isotropic harmonic oscillator, the equations of motion, in Cartesian coordinates, are given by

$$m\ddot{x} = -m\omega^2 x, \qquad m\ddot{y} = -m\omega^2 y, \tag{19}$$

where $\omega$ is a real constant. Also in this case the complex variable $z \equiv x + \mathrm{i}y$ is useful because Eqs. (19) are equivalent to

$$m\ddot{z} = -m\omega^2 z. \tag{20}$$

Equation (20) (which has the form of the equation of motion for a one-dimensional harmonic oscillator) can be obtained by means of Eq. (1) from the Lagrangian

$$L = \tfrac{1}{2} m\dot{z}^2 - \tfrac{1}{2} m\omega^2 z^2. \tag{21}$$

Substituting (21) into Eq. (7) we get

$$\begin{aligned}
-m\omega^2 z\, \eta + m\dot{z}\left(\frac{\partial\eta}{\partial t} + \dot{z}\frac{\partial\eta}{\partial z}\right) - \left(\frac{m}{2}\dot{z}^2 + \frac{m}{2}\omega^2 z^2\right) \\
\times \left(\frac{\partial\xi}{\partial t} + \dot{z}\frac{\partial\xi}{\partial z}\right) = \frac{\partial G}{\partial t} + \dot{z}\frac{\partial G}{\partial z},
\end{aligned} \tag{22}$$

and following the same steps as in the preceding case one finds that all the solutions of this last equation are given by

$$\xi = c_1 \cos(2\omega t) + c_2 \sin(2\omega t) + c_3, \tag{23}$$

$$\begin{aligned}
\eta &= -\omega z \big[c_1 \sin(2\omega t) - c_2 \cos(2\omega t)\big] \\
&\quad + (c_4 + \mathrm{i}c_5)\cos(\omega t) + (c_6 + \mathrm{i}c_7)\sin(\omega t),
\end{aligned} \tag{24}$$

$$\begin{aligned}
G &= m\Big\{ - \omega^2 z^2 \big[c_1 \cos(2\omega t) + c_2 \sin(2\omega t)\big] \\
&\quad - (c_4 + \mathrm{i}c_5)\, \omega z \sin(\omega t) + (c_6 + \mathrm{i}c_7)\, \omega z \cos(\omega t)\Big\},
\end{aligned} \tag{25}$$

where $c_1, \ldots, c_7$ are arbitrary real constants. Hence, the Lagrangian (21) also admits a seven-dimensional group of variational symmetries.

Another advantage of the use of the complex quantities introduced above is that with their help we can readily establish a connection between the sets of equations of motion (2) and (19). In fact, it is easy to see that if $z$ is a solution to Eq. (3) then $w = \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t/2} z$ satisfies Eq. (20), provided that $\omega = \omega_c/2$, and conversely. Furthermore, with this relation, the Lagrangian (4) becomes

$$L = \tfrac{1}{2} m\dot{w}^2 - \tfrac{1}{2} m\omega^2 w^2 + \dot{w}\frac{\partial}{\partial w}\left(-\mathrm{i}\tfrac{1}{2} m\omega w^2\right),$$

which is the Lagrangian (21) up to the "total derivative with respect to the time" of $-\mathrm{i}\frac{1}{2} m\omega w^2$ and, therefore, (4) and (21) lead to equivalent equations of motion. (Note that, at this point, $w$, $\dot{w}$, and $t$, *must* be regarded as independent variables and therefore it is an abuse to speak of a total derivative with respect to the time, though this is the standard usage.) In view of this connection, it is not strange that we end up with symmetry groups of the same dimension (and it is to be expected that these two groups are isomorphic, though we are not dealing with this question here).

Writing the Lagrangian (21) in terms of $(x, y, \dot{x}, \dot{y})$ one finds

$$L = \tfrac{1}{2} m(\dot{x}^2 - \dot{y}^2) - \tfrac{1}{2} m\omega^2 (x^2 - y^2) + \mathrm{i}(m\dot{x}\dot{y} - m\omega^2 xy).$$

The real and the imaginary parts of this function are acceptable real Lagrangians for the equations of motion (19), which differ from the standard one.

## 3. Examples with double and dual numbers

In a simplified manner, the double numbers can be defined as expressions of the form $a + \mathrm{j}b$, with $a, b \in \mathbb{R}$, where the unit j is such that $\mathrm{j}^2 = 1$, but $\mathrm{j} \neq \pm 1$, while the dual numbers are expressions of the form $a + \varepsilon b$, with $a, b \in \mathbb{R}$, where the unit $\varepsilon$ is such that $\varepsilon^2 = 0$, but $\varepsilon \neq 0$. The sum and product of these numbers are commutative and associative, and the multiplication is distributive over the sum (see also Ref. [1]).

It may be noticed that, by contrast with Eqs. (3) and (4), Eqs. (20) and (21) do not contain the imaginary unit i and, actually, Eqs. (20) and (21) remain valid if we employ the combination $z = x + \mathrm{h}y$, where h can be either, i, j or $\varepsilon$. Furthermore, the form of Eq. (22) is unchanged and its solution is given by

$$\xi = c_1 \cos(2\omega t) + c_2 \sin(2\omega t) + c_3, \tag{26}$$

$$\eta = -\omega z\big[c_1 \sin(2\omega t) - c_2 \cos(2\omega t)\big] + (c_4 + \mathrm{h}c_5)\cos(\omega t) + (c_6 + \mathrm{h}c_7)\sin(\omega t), \tag{27}$$

$$G = m\big\{ -\omega^2 z^2 \big[c_1 \cos(2\omega t) + c_2 \sin(2\omega t)\big] -(c_4+\mathrm{h}c_5)\,\omega z \sin(\omega t)+(c_6+\mathrm{h}c_7)\,\omega z\cos(\omega t)\big\}, \tag{28}$$

where $c_1, \ldots, c_7$ are arbitrary real constants. By decomposing Eqs. (26) and (27) into their real and "imaginary" parts one obtains the same transformations for the variables $x, y$ and $t$, regardless of the unit chosen (i, j or $\varepsilon$).

### 3.1. An example with dual numbers

The system of equations

$$m\ddot{x} = 0, \qquad m\ddot{y} = -mg, \tag{29}$$

corresponds to a particle of mass $m$ in a uniform gravitational field, where the constant $g$ is the acceleration of gravity. Making use of the variable $z \equiv x + \varepsilon y$, with values in the dual numbers, Eqs. (29) amount to the single equation

$$m\ddot{z} = -\varepsilon mg, \tag{30}$$

and one can readily verify that this equation can be obtained from the Lagrangian

$$L = \tfrac{1}{2}m\dot{z}^2 - \varepsilon mgz. \tag{31}$$

(Note that $L$ also has values in the dual numbers, and it is not constructed from the standard Lagrangian for the system of equations (29); in fact, the imaginary part of (31) is $m\dot{x}\dot{y} - mgx$, which is a real Lagrangian that leads to Eqs. (29), but this is not the standard Lagrangian for this problem.)

Since the Lagrangian (31) does not depend explicitly on the time, the Jacobi integral, $J \equiv \dot{z}\,(\partial L/\partial \dot{z}) - L$, is conserved. In fact, one finds that

$$\begin{aligned} J &= \tfrac{1}{2}m\dot{z}^2 + \varepsilon mgz \\ &= \tfrac{1}{2}m(\dot{x}^2 + 2\varepsilon\dot{x}\dot{y}) + \varepsilon mg(x + \varepsilon y) \\ &= \tfrac{1}{2}m\dot{x}^2 + \varepsilon(m\dot{x}\dot{y} + mgx). \end{aligned}$$

One can verify that the real and imaginary parts of $J$ are separately conserved as a consequence of Eqs. (29).

Substituting (31) into Eq. (7) we obtain the equation

$$-\varepsilon mg\eta + m\dot{z}\left(\frac{\partial\eta}{\partial t} + \dot{z}\frac{\partial\eta}{\partial z}\right) - \left(\frac{m}{2}\dot{z}^2 + \varepsilon mgz\right) \times \left(\frac{\partial\xi}{\partial t} + \dot{z}\frac{\partial\xi}{\partial z}\right) = \frac{\partial G}{\partial t} + \dot{z}\frac{\partial G}{\partial z}, \tag{32}$$

which has to be satisfied for all values of $z, t$ and $\dot{z}$. Hence, the coefficients of the powers of $\dot{z}$ on each side of this equation must coincide separately. By equating the coefficients of $\dot{z}^3$ we find that $\xi$ must be a function of $t$ only

$$\xi = A(t), \tag{33}$$

where $A$ is a real-valued function of a single variable. From the equality of the coefficients of $\dot{z}^2$ we obtain

$$\frac{\partial\eta}{\partial z} = \frac{1}{2}\frac{\mathrm{d}A}{\mathrm{d}t},$$

which implies that

$$\eta = \frac{z}{2}\frac{\mathrm{d}A}{\mathrm{d}t} + B(t), \tag{34}$$

where $B$ is a function with values in the dual numbers.

The equality of the terms proportional to $\dot{z}$ and the ones independent of $\dot{z}$ yields

$$\frac{1}{m}\frac{\partial G}{\partial z} = \frac{\partial\eta}{\partial t} = \frac{z}{2}\frac{\mathrm{d}^2A}{\mathrm{d}t^2} + \frac{\mathrm{d}B}{\mathrm{d}t}, \tag{35}$$

and

$$\frac{1}{m}\frac{\partial G}{\partial t} = -\varepsilon g\eta - \varepsilon gz\frac{\mathrm{d}A}{\mathrm{d}t}, \tag{36}$$

respectively. Then, the equality of the mixed second partial derivatives of $G$ gives

$$\frac{z}{2}\frac{\mathrm{d}^3A}{\mathrm{d}t^3} + \frac{\mathrm{d}^2B}{\mathrm{d}t^2} = -\frac{3}{2}\varepsilon g\frac{\mathrm{d}A}{\mathrm{d}t}. \tag{37}$$

The fulfillment of this condition for all values of $z$ (taking into account that $A$ and $B$ are functions of $t$ only) implies that $\mathrm{d}^3A/\mathrm{d}t^3 = 0$ and therefore

$$A(t) = c_8t^2 + c_7t + c_3, \tag{38}$$

where $c_3, c_7$ and $c_8$ are arbitrary real constants (the labeling of the constants is chosen to get agreement with that employed in Sec. 3.1 of Ref. [13]); then, Eq. (37) gives

$$\frac{\mathrm{d}^2B}{\mathrm{d}t^2} = -\varepsilon g(3c_8t + \tfrac{3}{2}c_7),$$

which leads to

$$B(t) = -\varepsilon g\big(\tfrac{1}{2}c_8t^3 + \tfrac{3}{4}c_7t^2\big) + (c_4+\varepsilon c_5)t + (c_1+\varepsilon c_2), \tag{39}$$

where $c_1, c_2, c_4$ and $c_5$ are four additional arbitrary real constants. Substituting (38) and (39) into Eqs. (35) and (36) one gets

$$G = m\big[ -c_1\varepsilon gt + c_4(z - \tfrac{1}{2}\varepsilon gt^2) + c_5\varepsilon z - \tfrac{3}{2}c_7\varepsilon gtz + c_8(\tfrac{1}{2}z^2 - \tfrac{3}{2}\varepsilon gt^2 z)\big]. \tag{40}$$

In this way we obtain a seven-dimensional group of variational symmetries of the Lagrangian (31), which is a symmetry group of the equations of motion (29). By contrast, the standard (real) Lagrangian for the equations (29) possesses an eight-dimensional group of variational symmetries [13], which does not represent a big difference taking into account the simplification achieved with the use of the dual numbers.

## 4. Concluding remarks

In the examples considered here, we started by expressing the equations of motion as a single second-order differential equation for a complex, double or dual variable, $z$, for which a suitable Lagrangian was constructed, from scratch, and, as a byproduct, we have found real Lagrangians for the systems under consideration, which differ from the standard ones.

As we can see, a successful use of the complex, double or dual numbers depends on the choice of the coordinates and on their convenient pairing.

It is important to stress the fact that a complex, double or dual number is not simply a pair of real numbers because, apart from having the usual algebraic operations of $\mathbb{R}^2$, these sets of numbers are closed under the multiplication.

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