The influence of noncommutativity on the energy spectra of bosonic particles in the framework of the DKGE with improved spatially-dependent mass including mixed scalar-vector Coulomb potentials in the ERQM symmetries

A. Maireche

Department of Physics, M'sila University, Laboratory of Physics and Material Chemistry, M'sila University, BP 239, Algeria.
e-mail: abdelmadjid.maireche@univ-msila.dz

Received 4 July 2022; accepted 29 September 2022

The bound state solutions of the deformed Klien-Gordon equation (DKGE) have been determined in the extended relativistic quantum mechanics ERQM symmetries using the improved spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb potentials (ISDM-SVCPs) model. The spatially-dependent mass Coulomb potential, as well as a combination of ($1/r^3$ and $1/r^4$), are included in the ISDM-SVCPs model, which is coupled with the coupling $\mathbf{L\Theta}$, which explains the interaction of the physical features of the system with the topological deformations of space-time. The new relativistic energy eigenvalues for the ISDM-CP have been derived using the parametric Bopp's shift method and standard perturbation theory. Quantum numbers (j, l, s, m), mixed potential depths ($q/s_c, m_0, m_1$), and noncommutativity parameters (Θ,τ,χ) seemed to affect the new values we obtained. Within the framework of relativistic extended quantum mechanics, we have addressed certain significant particular instances that we hope will be valuable to the specialized researcher. In DKGE symmetries, we've also looked at the improved pure scalar Coulomb-like potential. The formulation of total energy was also discovered in the context of extended symmetries, which unified the energies of bosonic particles and antiparticles into a single mathematical formula. When the three simultaneous limits (Θ, τ, χ) were applied, we recovered the normal results of relativistic in the literature $(0, 0, 0)$.

Keywords: Klien-Gordon equation; spatially-dependent mass; mixed scalar-vector Coulomb potentials; noncommutative quantum mechanics; star product.

DOI: https://doi.org/10.31349/RevMexFis.69.030801

1. Introduction

It is important to obtain the bound state energy spectrum of the Klein-Gordon (KG), Dirac or Duffin-Kemmer-Petiau (DKP) equations in order to successfully examine relativistic effects in many area of physics, such as nuclear physics, high energy physics, and so on. Recently, many authors have focused on solving these equations with physical potentials the scalar potential is not equal to but greater than the vector potential. By inserting the scalar potential as a modification in the mass component of the KG equation, Bakke and Furtado explored the effect of a Coulomb type potential on the KG oscillator [1]. Vitória *et al.* studied the relativistic quantum dynamics of an electrically charged particle in the presence of the KG oscillator and the Coulomb potential, as well as the behavior of a relativistic position-dependent mass particle in the presence of the KG oscillator and the Coulomb potential [2]. The influence of a spatially variable mass on the solution of the KG equation in (1+1) dimensions for inversely linear scalar potentials was studied by Dutra and Jia [3].The energy shift due to noncommutativity was obtained by Motavalli and Akbarieh using the stationary KG equation for the Coulomb potential in noncommutative space using the perturbation theory, and showed that the degeneracy of the initial spectral line is broken in the transition from commutative to noncommutative space [4]. Darroodi *et al.* investigated the KG equation for the Kratzer potential in the spherical polar coordinate in noncommutative space and obtained the energy shift due to noncommutativity via the perturbation theory [5]. On the personal level, we have had recent contributions regarding the study of the Coulomb potential combined with other potentials within the framework of the KG [6–10], Schrödinger [11–13] and Dirac [14] equations, but so far we have not studied this potential singlet within the framework of noncommutative quantum mechanics symmetries With the help of the Bopp's shift method and standard perturbation theory. Many works in the framework of usual-symmetric quantum mechanics have investigated the bound states of the KG and Dirac equations in arbitrary dimensions with Coulomb-like scalar plus vector potentials of constant mass. Gu *et al.* found exact solutions for DE using a Coulomb potential and showed the energy levels and fine structure in the generalized (D+1) space-time [15]. Dong used the Tricomi equation approach to study the (D+1)-dimensional DE with the Coulomb potential and expressed the eigenfunctions using the confluent hypergeometric function [16]. In higher-dimensional field theory, Ma examined the D-dimensional Klein–Gordon equation (D-KGE) with a Coulomb plus scalar potential and found the eigenfunctions that appear as a function to the confluent hypergeometric functions [17]. The eigenfunctions of the D-KGE with a Coulomb potential were derived and described analytically by the confluent hypergeometric function by Dong *et al.* [18]. In the context of spatially-dependent mass within the framework of relativistic quantum mechanics, we have three studies of interest related to Coulomb potentials. Hamzavi *et al.* (2010) solved the Dirac equation (DE) for spatially-dependent mass Coulomb potentials, including a Coulomb-like tensor potential $U(r) = -H/r$, using an asymptotic iteration method with an arbitrary spin-orbit coupling number k, and obtained the energy eigenvalues and corresponding eigenfunctions in the pseudospin symmetry limit [19]. Ikhdair and Ramazan investigated the effect of a spatially-dependent mass function on the solution of the DE with the Coulomb potential in the (3+1)-dimensions, and found the analytic bound state energy eigenvalues and corresponding upper and lower two-component spinors of the two Dirac particles in closed form using the Nikiforov-Uvarov (NU) approach in the context of spin and pseudospin symmetry for any arbitrary spin-orbit j state [20]. Ikhdair found the exact bound-state energy eigenvalues by analyzing the effect of spatially dependent mass functions on the solution of the KG equation in the (3+1) dimensions for spinless bosonic particles with mixed scalar-vector Coulomb-like field potentials and masses that are directly proportional and inversely proportional to the distance $m(r) = m_0 + m_1/r$ from the force center. The NU approach is also used to obtain the KG's related wave functions for mixed scalar-vector and pure scalar Coulomb-like field potentials [21]. The study of quantum theories in deformed spaces with noncommutative coordinates has recently been revived as a topic of interest [22–25]. In addition to the postulates that we know within the framework of quantum mechanics known in the literature, the non-commutative formula depends on two new postulates $\widehat{x}_{\mu}^{(s,h,i)} * \widehat{x}_{\nu}^{(s,h,i)} \neq \widehat{x}_{\nu}^{(s,h,i)} * \widehat{x}_{\mu}^{(s,h,i)}$ and $\widehat{p}_{\mu}^{(s,h,i)} * \widehat{p}_{\nu}^{(s,h,i)} \neq \widehat{p}_{\nu}^{(s,h,i)} * \widehat{p}_{\mu}^{(s,h,i)}$, here $(*)$ stands for the Weyl-Moyal star product. Despite quantum mechanics' brilliant successes in treating physical and chemical systems in various research fields, significant physical problems have arisen, such as the standard model's divergence problem, gravity quantization, and the problem of unifying it with the rest of the fundamental interactions, and so on [26–37]. It should be noted that Heisenberg in 1930 suggested the idea of extended noncommutativity to the coordinates as a possible solution for removing the infinite quantities of field theories before the renormalization method was developed and had gained attention. In an effort to standardize QFT, Snyder published the first paper on its history in 1947 [38], and Connes introduced its geometric analysis in 1991 and 1994 [39, 40]. Seiberg and Witten obtain a new version of gauge fields in noncommutative gauge theory [41] by extending earlier ideas on the advent of NC geometry in string theory with a nonzero B-field. One of the potential goals of NC deformation of space-space and phase-phase [42] is to eliminate the observed undesired divergences or infinities that appear to cause short-range in field theories such as gravitational theory by generating new quantum fluctuations. In addition, the emergence of NC-QFT in string theory gives more credibility to their work. I believe that this research will contribute to further subatomic scale investigations and scientific knowledge of elementary particles. The improved spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb potentials (ISDM-SVCPs) in the DKGT symmetries was motivated by the fact that it had not been reported in the literature for bosonic particles and antiparticles. The following are the vector and scalar ISDM-SVCPs models that will be used in this study ($V_{sc}(\widehat{r})$ and $S_{sc}(\widehat{r})$) as follows:

$$\begin{cases} V_{sc}(\widehat{r}) = V_{sc}(r) - \frac{1}{2r}\frac{\partial V_{sc}(r)}{\partial r}\mathbf{L}\Theta + O(\Theta^2) \\ S_{sc}(\widehat{r}) = S_{sc}(r) - \frac{1}{2r}\frac{\partial S_{sc}(r)}{\partial r}\mathbf{L}\Theta + O(\Theta^2) \end{cases}, \quad (1)$$

In addition to the new spatially-dependent bosonic mass $m_{sc}(\widehat{r})$, in DKGT, which is expressed as:

$$m_{sc}(\widehat{r}) = m(r) - \frac{1}{2r}\frac{\partial m_{sc}(r)}{\partial r}\mathbf{L}\Theta + O(\Theta^2), \quad (2)$$

where $(V_{sc}(r), S_{sc}(r), m(r))$ are the vector and scalar potentials according to the view of RQM known in the literature [21]:

$$\begin{cases} V_{sc}(r) = V_0 + \beta S_{sc}(r) \text{ and } S_{sc}(r) = -\frac{\hbar c q_s}{r} \\ m(r) = m_0\left(1 + \frac{\lambda_0 b}{r}\right) \text{ with } r \neq 0 \text{ and } \lambda_0 = \frac{\hbar}{m_0 c} \end{cases}, \quad (3)$$

where m_0 is the integration constant (rest mass of the fermionic particle), m_1 is the perturbed mass and, λ_0 is the Compton-like wavelength in fm units. The constant mass b is a dimensionless real constant that should be set to zero. ($\widehat{r}$ and r) is the distance between the two particles in the DKGT symmetries and QM symmetries, respectively. The coupling ($\mathbf{L\Theta} \equiv \mathbf{L}.\mathbf{\Theta}$) is the scalar product of the usual components of the angular momentum operators $\mathbf{L}(L_x, L_y, L_z)$ and the modified noncommutativity vector $\mathbf{\Theta}(\theta_{12}, \theta_{23}, \theta_{13})/2$ which present as is the noncommutativity elements parameter. In the case of G_{NC}, the noncentral generators can be suitably realized as self-adjoint differential operators ($\widehat{x}_{\mu}^{(s,h,i)}$, $\widehat{p}_{\nu}^{(s,h,i)}$) appear in n three varieties. The first one is the canonical structure (CS), the second is the Lie structure (LS), while the last corresponds to the quantum plane (QP) in the representations of Schrödinger, Heisenberg, and interactions pictures, satisfying a deformed algebra of the form (For simplicity, we have used the natural units $\hbar = c = 1$): [43–51]:

$$\left[x_{\mu}^{(s,h,i)}, p_{\nu}^{(s,h,i)}\right] = i\hbar\delta_{\mu\nu} \Longrightarrow$$
$$\left[\widehat{x}_{\mu}^{(s,h,i)} \stackrel{*}{,} \widehat{p}_{\nu}^{(s,h,i)}\right] = i\hbar_{eff}\delta_{\mu\nu}, \quad (4)$$

and

$$\left[x_{\mu}^{(s,h,i)}, x_{\nu}^{(s,h,i)}\right] = 0 \Longrightarrow \left[\widehat{x}_{\mu}^{(s,h,i)} \stackrel{*}{,} \widehat{x}_{\nu}^{(s,h,i)}\right] =$$
$$\begin{cases} i\theta_{\mu\nu} : \theta_{\mu\nu} \in IC \text{ For CS,} \\ if_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{(s,h,i)} : f_{\mu\nu}^{\alpha} \in IC \text{ For LS,} \\ iC_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{(s,h,i)}\widehat{x}_{\beta}^{(s,h,i)} : C_{\mu\nu}^{\alpha\beta} \in IC \text{ For QP} \end{cases}. \quad (5)$$

In the DKGT symmetries, the generalized coordinates $\widehat{x}_{\mu}^{(s,h,i)}$and the generalizing momentums $\widehat{p}_{\mu}^{(s,h,i)}$ are equal $\left(\widehat{x}_{\mu}^{s}, \widehat{x}_{\mu}^{h}, \widehat{x}_{nc\mu}^{i}\right)$ and $\left(\widehat{p}_{\mu}^{s}, \widehat{p}_{\mu}^{h}, \widehat{p}_{\mu}^{i}\right)$ while the corresponding coordinates $x_{\mu}^{(s,h,i)}$and $p_{\mu}^{(s,h,i)}$ are equal $\left(x_{\mu}^{s}, x_{\mu}^{h}, x_{\mu}^{i}\right)$ and (p_{μ}^{s}, p_{μ}^{h}, p_{μ}^{i}) in the RQM symmetries are, respectively. Here IC denotes the complex number field. Furthermore, the usual uncertainty relation corresponding to the LHS of Eq. (5) will be extended to become two uncertainties the following formula in the new form symmetries is as follows:

$$\left|\Delta x_{\mu}^{(s,h,i)}\Delta p_{\nu}^{(s,h,i)}\right| \geqslant \hbar\delta_{\mu\nu}/2 \Longrightarrow$$
$$\left|\Delta \widehat{x}_{\mu}^{(s,h,i)}\Delta \widehat{p}_{\nu}^{(s,h,i)}\right| \geqslant \hbar_{eff}\delta_{\mu\nu}/2\,, \tag{6}$$

and

$$\left|\Delta \widehat{x}_{\mu}^{(s,h,i)}\Delta \widehat{x}_{\nu}^{(s,h,i)}\right| \geqslant \begin{cases} \left|\theta_{\mu\nu}\right|/2 \text{ For CS,} \\ F_{\mu\nu}/2 \text{ For LS,} \\ G_{\mu\nu}/2 \text{ For QP,} \end{cases} \tag{7}$$

with $F_{\mu\nu}$ and $G_{\mu\nu}$ are equal to the average values $\left|\left\langle f_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{(s,h,i)}\right\rangle\right|$ and $\left|\left\langle C_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{(s,h,i)}\widehat{x}_{\beta}^{(s,h,i)}\right\rangle\right|$, respectively. The second uncertainty relation in Eq. (7) is the consequence of the deformation of space-space that arises from the RHS of Eq. (5) that is divided into three varieties, while the first uncertainty relation in Eq. (6) is the result of the generalization of LHS Eq. (4) to RHS form. There is no equivalent in the literature for the novel incertitude relation in Eq. (6) in the framework of quantum mechanics. Under the Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame, Eqs. (4) and (5) are covariant equations (have the same behavior as $\widehat{x}_{\mu}^{(s,h,i)}$). The MASCCCRs were extended in DKGT to include Heisenberg and interaction pictures. When compared to the energy values and elements of antisymmetric (3×3) real matrices, $\hbar_{eff}\cong\hbar$ is the effective Planck constant, $\theta_{\mu\nu}=\epsilon_{\mu\nu}\theta$ (θ is the noncommutative parameter, and is just an antisymmetric number ($\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}=1$ for $\mu\neq\nu$ and $\epsilon_{\epsilon\epsilon}=0$) which is an infinitesimal parameter, and is the Kronecker symbol. The Weyl-Moyal $*$-product is generalized to define the new deformed scalar product $h(x)*g(x)$ in three varieties as [52–57, 59, 78]:

$$h(x)*g(x)=$$
$$\begin{cases} \exp\left(i\epsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hg)(x) \text{ For CS,} \\ \exp\left(\frac{i}{2}x_{nc\mu}^{(s,h,i)}g_k\left(i\partial_{\mu}^{x}, i\partial_{\nu}^{x}\right)\right)(hg)(x) \text{ For LS,} \\ iq^{G\left(u,v,\partial_{\mu}^{u},\partial_{\nu}^{v}\right)}h(u,v)\,g(u',v')\rfloor_{u'\to u}^{v'\to v} \text{ For QP,} \end{cases} \tag{8}$$

with

$$g_H(k,p)=-k_{\mu}p_{\nu}f_k^{\nu\nu}$$
$$+\frac{1}{6}k_{\mu}p_{\nu}\left(p_H-k_H\right)f_l^{\nu\nu}f_m^{lH}+\dots. \tag{9}$$

The first variety is used in this research, allowing us to rewrite $(h*g)(x)$ at the first order of the noncommutativity parameter $\epsilon^{\mu\nu}\theta$ as follows [60–68]:

$$(h*g)(x)=\exp\left(i\epsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hg)(x)$$
$$\approx(hg)(x)-\frac{i\epsilon^{\mu\nu}\theta}{2}\partial_{\mu}^{x}h\partial_{\nu}^{x}g\rfloor_{x^{\mu}=x^{\nu}}+O\left(\theta^{2}\right). \tag{10}$$

Every sum indices (μ or ν) can be equal to 1,2,3 in $D=3$. The effects of space-space noncommutativity are represented physically by the second term in Eq. (10). The following is a summary of the current paper's structure. The scope and objective of our investigation are presented in the first section, and the remainder of the paper is organized as follows: Section 2 presents an overview of the KGE under the ISDM-SVCPs model. Section 3 is devoted to investigating the DKGE using the well-known Bopp's shift method to obtain the ISDM-SVCPs model's effective potential. Furthermore, using standard perturbation theory, we find the expectation values of the radial terms ($1/r^3$ and $1/r^4$) to calculate the corrected relativistic energy generated by the effect of the perturbed effective potential $\Sigma_{\text{pert}}^{sc}(r)$ of the ISDM-SVCPs model, and we derive the global corrected energies for bosonic particles and antiparticles whose spin quantum number has an integer value $(0,1,2...)$ and satisfies the Bose-Einstein statistics under the ISDM-SVCPs model. Section 4 is reserved to study important relativistic particular cases in DKGT. The improved pure scalar Coulomb-like potential in DKG symmetries will be studied in the next section. The sixth section is devoted to the conclusions.

2. An overview of KGE under SDM-SVCPs in RQM symmetry

In order to construct a physical model describing a physical system that interacted with the spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb potentials (SDM-SVCPs) model in the DKGE, it is useful to recall the eigenvalues and the corresponding eigenfunctions under the influence of this system within the framework of relativistic quantum mechanics, RQM, known in the literature. In this case, the system is governed by the following Klien-Gordon equation:

$$\Big(\nabla^{2}+\Big[\left(E_{nl}-V_{sc}(r)\right)^{2}$$
$$-\left(m(r)-S_{sc}(r)\right)^{2}\Big]\Big)\Psi_{nl}(r,\theta,\varphi)=0, \tag{11}$$

here $\mathbf{p}=-\mathbf{i}\hbar\nabla$ is the momentum. The repulsive vector potential $V_{sc}(r)$ and space-time attractive scalar potential $S_{sc}(r)$ are produced from the four-vector linear momentum operator A^{μ} ($V_{sc}(r)$, $\mathbf{A}=\mathbf{0}$) and the mass $m(r)$, respectively while E_{nl} is the relativistic eigenvalues, (n,l) represent the principal and spin-orbit coupling terms. It should be noted that the scalar potential $S_{sc}(r)$ describes a situation in which the difference in potential energies of an object in two different positions is determined solely by the positions and not by the particle path taken in displacement from one point to another, whereas the vector potential $V_{sc}(r)$ is a vector field whose curl is a given vector field. Since the spatially-dependent mass Coulomb including a Coulomb-like tensor interaction has spherical symmetry, allowing the wave function solution $\Psi_{nl}(r,\theta,\varphi)$ of the known form $(u_{nl}(r)/r)Y_m^l(\theta,\varphi)$ while $Y_m^l(\theta,\varphi)$ is spherical harmonics and m is the projections on the z-axis. The radial component $u_{nl}(r)$ satisfies the differential equation as below:

$$\left\{\frac{d^2}{dr^2}+\left(E_{nl}^2+V_{sc}(r)^2-2E_{nl}V_{sc}(r)-m(r)^2c^4-S_{sc}(r)^2-2c^2m(r)S_{sc}(r)-\frac{l(l+1)\hbar^2c^2}{r^2}\right)\right\}u_{nl}(r)=0. \quad (12)$$

Ikhdair in Ref. [21] rewrite the above equation as follows:

$$\left(\frac{d^2}{dr^2}-\frac{\gamma_2}{r^2}-\frac{\gamma_1}{r}-\epsilon_{nl}^2\right)u_{nl}(r)=0 \quad (13)$$

with

$$\begin{cases}\epsilon_{nl}^2=\sqrt{m_0^2-E_{nl}^2},\gamma_1=2(b-q)m_0c^2-2q\beta E_{nl},\\ \gamma_2=b(b-2q)+q^2\left(1-\beta^2\right)+l(l+1).\end{cases} \quad (14)$$

The author of Ref. [21] used the NU method to obtain the expression of $u_{nl}(r)$ as a function of generalized Laguerre polynomial $L_n^{2L+1}(2\epsilon_{nl}r)$ in usual RQM symmetries as,

$$u_{nl}(r)=\sqrt{\frac{n!\,(2\epsilon_{nl})^{2L+3}}{2(n+L+1)\Gamma(n+2L+2)}}r^{\left(1+\sqrt{1+4\gamma_2}\right)/2}\exp(-\epsilon_{nl}r)L_n^{2L+1}(2\epsilon_{nl}r), \quad (15)$$

here

$$L=\sqrt{\left(l+\frac{1}{2}\right)^2+b(b-2q)+q^2(1-\beta^2)+l(l+1)}-\frac{1}{2}.$$

Allowing the spinor solution $\Psi_{nl}(r,\theta,\varphi)$ as follows:

$$\Psi_{nl}(r,\theta,\varphi)=\sqrt{\frac{n!\,(2\epsilon_{nl})^{2L+3}}{2(n+L+1)\Gamma(n+2L+2)}}r^{\left(1+\sqrt{1+4\gamma_2}\right)/2-1}\exp(-\epsilon_{nl}r)L_n^{2L+1}(2\epsilon_{nl}r)Y_m^l(\theta,\varphi). \quad (16)$$

The corresponding equation of energy for a bosonic particles E_{nl}^+ and antiparticles E_{nl}^- are given by [21]:

$$E_{nl}^{\pm}=-V_0+\frac{q(b-q)\beta\pm(n+1+L)\sqrt{(n+1+L)^2-q^2(1-\beta^2)-b(b-2q)}}{q^2\beta^2+(n+1+L)^2}. \quad (17)$$

3. The new solutions of DKGE under the ISDM-SVCPs in the DKGT symmetries:

3.1. Review of BS method

Let us begin in this subsection by finding the relativistic DKGE in the symmetries of extended relativistic quantum mechanics ERQM or noncommutative quantum mechanics NCQM under ISDM-SVCPs. Our objective is achieved by applying the new principles which we have seen in the introduction, Eqs. (5), (6) and (10), summarized in new relationships MASCCCRs and the notion of the Weyl-Moyal star product. These data allow us to rewrite the usual radial KG equations in Eq. (11) in the DKGT symmetries as follows:

$$\left(\frac{d^2}{dr^2}-\frac{\gamma_2}{r^2}-\frac{\gamma_1}{r}-\epsilon_{nl}^2\right)*u_{nl}(r)=0. \quad (18)$$

Among the possible paths to finding the solution to Eq. (18) is the application of the Connes method [39, 40], or the Seiberg and Witten map [41]. It is known to specialists that the star product can be translated into the ordinary product known in the literature using what is called Bopp's shift method. F. Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules:

$$(x,p)\rightarrow\left(\widehat{x}=x-\frac{i}{2}\partial_p,\widehat{p}=p+\frac{i}{2}\partial_x\right),$$

instead of ordinary correspondence:

$$(x,p)\rightarrow\left(\widehat{x}=x,\widehat{p}=p+\frac{i}{2}\partial_{x}\right),$$

respectively. This procedure is known as Bopp's shifts (BS) method and this quantization procedure is known as Bopp quantization [69–72]. This method has been widely successful in the last two decades. At the nonrelativistic level, within the framework of solving the deformed Schrödinger equation DSE, the exact and approximate solutions for many typical potentials that are applied to many fields of physics and chemistry have been successfully found (See for example [73, 74]). This success is not only limited to the DSE but also goes beyond that to the relativistic case in the framework of the three equations. Within the framework of the deformed Klein-Gordon equation DKGE, approximate or exact solutions were found for several central potentials of wide application(See typical refs. [75–80]). As for the deformed Dirac equation DDE, many typical potentials were successfully processed, despite the complexity of the calculations (See some refs. [81–84]) while the relativistic deformed Duffin-Kemmer-Petiau equation DDKPE for particles with spin-(1,2,...) [85, 86]. Thus, Bopp's shift method BS method is based on reducing second-order linear differential equations of the DSE, DKGE, DDE, and DDKPE with Weyl-Moyal star product to second-order linear differential equations of SE, KGE, DE, and DKPE without Weyl-Moyal star product with simultaneous translation in the space-space. It is worth motioning that the BS method permutes us to reduce the Eq. (18) to the simplest form:

$$\left(\frac{d^{2}}{dr^{2}}-\frac{\gamma_{2}}{\widehat{r}^{2}}-\frac{\gamma_{1}}{\widehat{r}}-\epsilon_{nl}^{2}\right)u_{nl}\left(r\right)=0. \tag{19}$$

The modified algebraic structure of covariant canonical commutation relations with the notion of Weyl-Moyal star product in Eqs. (5) and (6) become new MASCCCRs with ordinary known products in literature as follows (see, *e.g.*, [70–73]):

$$\left[\widehat{x}_{\mu}^{(s,h,i)},\widehat{p}_{\nu}^{(s,h,i)}\right]=i\hbar_{eff}\delta_{\mu\nu}\quad\text{and}\quad\left[\widehat{x}_{\mu}^{(s,h,i)},\widehat{x}_{\nu}^{(s,h,i)}\right]=i\theta_{\mu\nu}. \tag{20}$$

The generalized positions and momentum coordinates ($\widehat{x}_{\mu}^{(s,h,i)}$ and $\widehat{p}_{\mu}^{(s,h,i)}$), in the symmetries of DKGT, are defined as [70–73]:

$$\widehat{x}_{\mu}^{(s,h,i)}=x_{\mu}^{(s,h,i)}-\sum_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_{\nu}^{(s,h,i)}\text{ and }\widehat{p}_{\mu}^{(s,h,i)}=p_{\mu}^{(s,h,i)}. \tag{21}$$

This allows us to find the operator $\widehat{r}^{2}$, in the DKGT symmetries, equal [76–78, 78–81]:

$$\widehat{r}^{2}=r^{2}-\mathbf{L}\Theta+O\left(\Theta^{2}\right) \tag{22}$$

Thus, after straightforward calculations, we obtain the new operators $-\gamma_{2}/\widehat{r}^{2}$ and $-\gamma_{1}/\widehat{r}$ in the DKGT symmetries, as:

$$\begin{cases}-\frac{\gamma_{2}}{\widehat{r}^{2}}=-\frac{\gamma_{2}}{r^{2}}-\gamma_{2}\frac{\mathbf{L}\Theta}{r^{4}}+O\left(\Theta^{2}\right)\\ -\frac{\gamma_{1}}{\widehat{r}}=-\frac{\gamma_{1}}{r}-\gamma_{1}\frac{\mathbf{L}\Theta}{2r^{3}}+O\left(\Theta^{2}\right)\end{cases}. \tag{23}$$

Substituting Eqs. (23) into Eq. (19), we obtain the following like Schrödinger equations:

$$\left(\frac{d^{2}}{dr^{2}}-\frac{\gamma_{2}}{r^{2}}-\frac{\gamma_{1}}{r}-\epsilon_{nl}^{2}-\Sigma_{sc}^{pert}\left(r\right)\right)u_{nl}\left(r\right)=0, \tag{24}$$

with

$$\Sigma_{sc}^{pert}\left(r\right)=\left(\frac{\gamma_{2}}{r^{4}}+\frac{\gamma_{1}}{2r^{3}}\right)\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{25}$$

By comparing Eq. (24) and Eq. (13) , we observe an additive potential $\Sigma_{sc}^{\text{pert}}\left(r\right)$ dependent on two radial terms ($1/r^{3}$ and $1/r^{4}$) which is coupled with the coupling $\mathbf{L\Theta}$ that explains the interaction of the physical features of the system with the topological deformations of space-time. From a physical point of view, this means that the spontaneously generated term $\Sigma_{sc}^{\text{pert}}\left(r\right)$ as a result of the topological properties of deformation space-space can be considered very small compared to the fundamental term

$$\Sigma_{sc}\left(r\right)=\epsilon_{nl}^{2}+\frac{\gamma_{2}}{r^{2}}+\frac{\gamma_{1}}{r}. \tag{26}$$

Furthermore, using the unit step function (also known as the Heaviside step function $\theta(x)$ or simply the theta function) to rewrite the global induced two potentials $\Sigma_{t\_sc}^{\text{pert}}(r)$ for bosonic particles and bosonic antiparticles in DKG symmetries as:

$$\Sigma_{t\_sc}^{\text{pert}}(r) = \Sigma_{sc}^{\text{pert}}(r)\,\theta(|E_{nc}^{sc}|) - \Sigma_{sc}^{\text{pert}}(r)\,\theta(-|E_{nc}^{sc}|) = \begin{cases} \Sigma_{sc}^{\text{pert}}(r) \text{ for bosonic particles} \\ -\Sigma_{sc}^{\text{pert}}(r) \text{ for bosonic antiparticles} \end{cases}, \tag{27}$$

where the step function $\theta(y)$ is given by:

$$\theta(y) = \begin{cases} 1 \text{ for } y > 0 \\ 0 \text{ for } y < 0 \end{cases}. \tag{28}$$

The spatially-dependent mass Coulomb potential including a Coulomb-like tensor interaction is extended by including new additive potential $\Sigma_{sc}^{\text{pert}}(r)$ expressed to the radial terms $1/r^3$ and $1/r^4$ to become the improved spatially-dependent mass Coulomb potential in DKGT symmetries. The global induced two potentials $\Sigma_{t\_sc}^{\text{pert}}(r)$ represent the physical interaction between the system's physical properties that correspond to bosonic particles and bosonic antiparticles in DKG symmetries with topological deformations of space-space characterized by noncommutativity vector $\boldsymbol{\Theta}$. The generated new effective potential $\Sigma_{sc}^{\text{pert}}(r)$ is also proportional to the infinitesimal coupling $\mathbf{L}\Theta$. This allows us to consider the new additive parts of the effective potential $\Sigma_{sc}^{\text{pert}}(r)$ as perturbation potential compared with the main potential $\Sigma_{sc}(r)$ which is also known as the parent potential operator in the symmetries of DKGT, that is, the inequality $\Sigma_{sc}^{\text{pert}}(r) << \Sigma_{sc}(r)$ has become achieved. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. If we consider $\Sigma_{sc}^{nc}(r)$, the global potential in DKG symmetries that equals $\left([\gamma_2/r^2] + [\gamma_1/r] + \epsilon_{nl}^2\right)$ which presents the corresponding potential in KG theory, and the new additive potential $\Sigma_{sc}^{\text{pert}}(r)$. Looking at the previous data, we find that the physical inequality $\Sigma_{sc}^{\text{pert}}(r) << \left([\gamma_2/r^2] + [\gamma_1/r] + \epsilon_{nl}^2\right)$ is fully satisfied with its conditions. This allows us to give a complete prescription for determining the energy level of the generalized $(n,l,m)^{th}$ excited states.

3.2. The expectation values under the ISDM-SVCPs in the DKGT for spin symmetry

In this subsection, we want to apply the perturbative theory, in the case of DKGT symmetries, we find the expectation values:

$$M_{1(nlm)}^{sc} \equiv \left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{sc} \quad \text{and} \quad M_{2(nlm)}^{sc} \equiv \left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{sc},$$

for bosonic particles taking into account the unperturbed $\Psi_{nl}(r,\theta,\varphi)$ which we have seen previously in Eq. (15). After straightforward calculations, we obtain the two expectation values $M_{1(nlm)}^{sc}$ and $M_{2(nlm)}^{sc}$ by applying the standard perturbation theory in first-order as follows:

$$M_{1(nlm)}^{sc} = \frac{n!\,(2\epsilon_{nl})^{2L+3}}{2(n+L+1)\,\Gamma(n+2L+2)} \int_0^{+\infty} r^{\left(1+\sqrt{1+4\gamma_2}\right)-2-1} \exp(-2\epsilon_{nl} r) \left[L_n^{2L+1}(2\epsilon_{nl} r)\right]^2 dr, \tag{29.1}$$

and

$$M_{2(nlm)}^{sc} = \frac{n!\,(2\epsilon_{nl})^{2L+3}}{2(n+L+1)\,\Gamma(n+2L+2)} \int_0^{+\infty} r^{\left(1+\sqrt{1+4\gamma_2}\right)-3-1} \exp(-2\epsilon_{nl} r) \left[L_n^{2L+1}(2\epsilon_{nl} r)\right]^2 dr. \tag{29.2}$$

We have used useful abbreviations $\langle R\rangle_{(nlm)}^{sp-sc} = \langle n,l,m\,|R|\,n,l,m\rangle$ to avoid the extra burden of writing , with $R = \left(\frac{1}{r^3} \text{ or } \frac{1}{r^4}\right)$. Furthermore, we have applied the property of the spherical harmonics, which has the form:

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta',\varphi')\,Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)\,d\theta d\varphi = \delta_{ll'}\delta_{mm'}. \tag{30}$$

Comparing Eq. (29. 1,2) with the integral of the form [87]:

$$\int_0^{+\infty} t^{\eta-1}\exp(-\alpha t)\,L_m^{\lambda}(\alpha t)\,L_n^{\beta}(\alpha t)\,dt = \frac{\alpha^{-\eta}\Gamma(n-\eta+\beta+1)\,\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\eta+\beta)\,\Gamma(\lambda+1)} \times {}_3F_2(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1), \tag{31}$$

with *Rel* $(\eta)\rangle 0$ and ${}_3F_2\left(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1\right)$ is obtained from the generalized hypergeometric function ${}_pF_q\big(\alpha_1,...,\alpha_p;\beta^1,...,\beta^q,1\big)$ for $p=3$ and $q=2$ while $\Gamma\left(x\right)$ denoting the usual Gamma function. After straightforward calculations we find:

$$M^{sc}_{1(nlm)}=\frac{\left(2\epsilon_{nl}\right)^{2L-\sqrt{1+4\gamma_2}+4}}{2\left(n+L+1\right)}\frac{\Gamma\left(n+2L+3-\sqrt{1+4\gamma_2}\right)}{n!\Gamma\left(2L+3-\sqrt{1+4\gamma_2}\right)\Gamma\left(2L+2\right)}$$

$$\times\,{}_3F_2\left(-n,\sqrt{1+4\gamma_2}-1,\sqrt{1+4\gamma_2}-2L-2;-n+\sqrt{1+4\gamma_2}-1,2L+2,1\right), \tag{32.1}$$

$$M^{sc}_{2(nlm)}=\frac{\left(2\epsilon_{nl}\right)^{2L-\sqrt{1+4\gamma_2}+5}}{2\left(n+L+1\right)}\frac{\Gamma\left(n+2L+4-\sqrt{1+4\gamma_2}\right)}{n!\Gamma\left(2L+4-\sqrt{1+4\gamma_2}\right)\Gamma\left(2L+2\right)}$$

$$\times\,{}_3F_2\left(-n,\sqrt{1+4\gamma_2}-2,\sqrt{1+4\gamma_2}-2L-3;-n+\sqrt{1+4\gamma_2}-2,2L+2,1\right). \tag{32.2}$$

3.3. The corrected energy for the ISDM-SVCPs in DKGT symmetries

The main objective underlined in this subsection is to find the contribution resulting from topological properties based on our strategy that we have successfully applied in previous works and which we try to develop in every new work. We can say that the global relativistic energy in the perspective of deformation KG theory produced with ISDM-SVCPs model as a result of a major contribution to relativistic energy known in the literature under DM-SVCPs model in usual KG theory and which we paved for through a quick look for the bosonic particles and antiparticles in Eqs. (17), while the new contribution is produced from the topological properties under space-space deformation, which can be evaluated through several contributions, we will address three of them. The first one is generated from the effect of the perturbed spin-orbit effective potentials $\Sigma^{\text{pert}}_{sc}\left(r\right)$ corresponding to the bosonic particles and antiparticles with spin-s. This perturbed effective potential is obtained by replacing the coupling of the angular momentum $\mathbf{L}$ operator and the NC vector Θ with the new equivalent coupling $\Theta\mathbf{L}S$ (with $\Theta^2=\Theta^2_{12}+\Theta^2_{23}+\Theta^2_{13}$). This degree of freedom comes considering that the infinitesimal NC vector $\boldsymbol{\Theta}$ is arbitrary. We have oriented the spin-$\mathbf{S}$ of the bosonic particles (or antiparticles) to become parallels to the vector Θ which interacted with the ISDM-SVCPs model. Moreover, we replace the new spin-orbit coupling $\Theta\mathbf{L}S$ with the corresponding new physical form $\left(\Theta/2\right)\mathbf{G}^2$, with $\mathbf{G}^2=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2$ for the bosonic particles (or antiparticles). Furthermore, in RQM, the operators ($\widehat{\mathbf{H}}^{sc}_{rnc}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, and the eigenvalues of the operator $\mathbf{G}^2$ are equal to the values:

$$2F\left(j,l,s\right)=j(j+1)-l(l+1)-s(s+1),$$

with $|l-s|\leq j\leq\ |l+s|$ for the bosonic particles (or antiparticles) in DKGT symmetry. As a direct consequence, the square partially corrected energies $\Delta E^{so2}_{sc}\left(n,q_s,m_0,m_1,\Theta,j,l,s\right)\equiv\Delta E^{so2}_{sc}$ due to the perturbed effective potential $\Sigma^{\text{pert}}_{sc}\left(r\right)$ produced for the $\left(n,l,m\right)^{th}$ excited state, in deformation Klien-Gordon theory symmetries as follows:

$$\Delta E^{so2}_{sc}=\Theta F\left(j,l,s\right)\ \left\langle Z\right\rangle^{sc}_{(nlm)}\left(n,q,m_0,m_1\right), \tag{33}$$

The global expectation values $\left\langle Z\right\rangle^{sc}_{(nlm)}\left(n,q,m_0,m_1\right)$ for the bosonic particles (or antiparticles), which were created from the effect of the ISDM-SVCPs model, are determined from the following expressions:

$$\left\langle Z\right\rangle^{sc}_{(nlm)}\left(n,q,m_0,m_1\right)=\gamma_2\left\langle\tfrac{1}{r^4}\right\rangle^{sc}_{(nlm)}+\gamma_1\left\langle\tfrac{1}{r^3}\right\rangle^{sc}_{(nlm)}\ . \tag{34}$$

The second main part is obtained from the magnetic effect of the perturbative effective potentials $\Sigma^{\text{pert}}_{sc}\left(r\right)$ under the ISDM-SVCPs model in the DKGT symmetries. These effective potentials are achieved when we replace both $\mathbf{L}\Theta$ a by $\tau\aleph L_z$, and Θ_{12} by $\tau\aleph$, here ($\aleph$ and τ) are present the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12}(length)^2 is the same unit of $\tau\aleph$, we have also need to apply:

$$\left\langle n',l',m'\left|L_z\right|n,l,m\right\rangle=m\delta_{m'm}\delta_{l'l}\delta_{n'n}\text{ with: }-|l|\leq m\leq+|l|\,.$$

for the bosonic particles (or antiparticles). All of these data allow for the discovery of the new square energy shift $\Delta E^{mg2}_{sc}(n,q,m_0,m_1,\ \tau,\ m)$ due to the perturbed Zeeman effect created by the influence of the ISDM-SVCPs model for the $\left(n,l,m\right)^{th}$ excited state in deformation Klien-Gordon theory symmetries as follows:

$$\Delta E^{mg2}_{sc}(n,q_s,m_0,m_1,\tau,m)=\tau\aleph\left\langle Z\right\rangle^{sc}_{(nlm)}\left(n,q,m_0,m_1\right)m. \tag{35}$$

After we have completed the first and second stages of self-production of energy, we are heading to another very important case under the ISDM-SVCPs model in DKGT symmetries. This physical phenomenon is produced automatically from the influence of perturbed effective potential $\Sigma_{sc}^{\text{pert}}(r)$ which we have seen in Eq. (25). We consider the bosonic particles (or antiparticles) under going rotation with angular velocity Ω. The features of this subjective phenomenon are determined through the replace the arbitrary vector $\boldsymbol{\Theta}$ with $\chi\Omega$. Allowing us to replace the coupling $\mathbf{L}\Theta$ with $\chi\mathbf{L}\Omega$, as following:

$$\mathbf{L}\Theta \ \rightarrow \chi\mathbf{L}\boldsymbol{\Omega}. \tag{36}$$

Here χ is just an infinitesimal real proportional constant. The effective potentials $\Sigma_{\text{pert}}^{sc-rot}(s)$, which induced the rotational movements of the bosonic particles, can be expressed as follows:

$$\Sigma_{sc}^{\text{pert}}(r) \longrightarrow \Sigma_{\text{pert}}^{sc-rot}(r) = \chi \langle Z\rangle_{(nlm)}^{sc}(n,q,m_0,m_1)\,\mathbf{L}\boldsymbol{\Omega}. \tag{37}$$

We chose a rotational velocity Ω parallel to the (Oz) axis ($\Omega = \boldsymbol{\Omega}\mathbf{e}_z$) to simplify the calculations; this, of course, does not change the physical characteristics of the examined problem as much as it simplifies the calculations. The spin-orbit coupling is then transformed into new physical phenomena as follows:

$$\Sigma_{\text{pert}}^{sc-rot}(r)\,\mathbf{L}\boldsymbol{\Omega} = \chi\Omega\Sigma_{\text{pert}}^{sc-rot}(r)\,L_z. \tag{38}$$

All of this data allows for the discovery of the new corrected square energy $\Delta E_{sc}^{rot2}(n,q,m_0,m_1,\chi,m)$ due to the perturbed effective potential $\Sigma_{\text{pert}}^{sc-rot}(r)$ which is generated automatically by the influence of the improved spatially-dependent mass Coulomb potential for the $(n,l,m)^{th}$ excited state in DKGT symmetries as follows:

$$\Delta E_{sc}^{rot2} = \chi\Omega\,\langle Z\rangle_{(nlm)}^{sc}(n,q,m_0,m_1)\,m. \tag{39}$$

It's worth noting that the authors of Ref. [88] investigated rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and three-dimensional space at zero temperature, but in this case, the rotational term was a DKGE to the Hamiltonian operator, whereas in our case, the rotation operator $\Sigma_{\text{pert}}^{sc-rot}(r)\,\mathbf{L}\Omega$ appears automatically due to them from the deformation of space-space under the improved spatially-dependent mass Coulomb potential model. The eigenvalues of the operations $\mathbf{G}^2$ for a bosonic particles and antiparticles (negative energy) with spin $s = (1,2..)$ are equal to the following values:

$$\digamma(j,l,s) = [j(j+1) - l(l+1) - s(s+1)]/2,$$

the possible values of j are:

$$j = \{|l-s|, |l-s|+1, ..., |l+s|\}.$$

In the symmetries of the DKGT symmetries, the total relativistic energy $E_{nc}^{sp}(n, q_s, m_0, m_1, \Theta, \tau, \chi, j, l, s, m)$ for the case of the bosonic particles (or antiparticles) with spin quantum number has an integer value $(0,1,2...)$ and satisfies the Bose-Einstein statistics such as ($\pi^{\pm}$ and π^0) with improved spatially-dependent mass Coulomb potential model, corresponding to the generalized $(n,l,m)^{th}$ excited states are expressed as:

$$E_{nc}^{sp}(n,q_s,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = E_{nl}^{\pm} \pm \left[\langle Z\rangle_{(nlm)}^{sc}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)\,m + \Theta\digamma(j,l,s)\right)\right]^{1/2}. \tag{40}$$

Where $E_{nl}^{\pm}$ are usual relativistic energies under spatially-dependent mass Coulomb potential model obtained from equations of energy in Eq.(17). It should be noted that the positive sign $(+)$ in the principal (first) and corrector (second) terms denotes the energy of the bosonic particles which corresponds to the positive energy, while the negative sign $(-)$ in the principal and corrector terms denotes the energy of the bosonic antiparticles which correspond the negative energy. We can now generalize our obtained energies E_{g-nc}^{sc-s}, in a unified formula, under the improved spatially-dependent mass Coulomb potential model which was produced with the global induced potential $\Sigma_{t\_sc}^{\text{pert}}(r)$ for bosonic particles (or antiparticles) as:

$$E_{g-nc}^{sc-s} = E_{nc}^{sc}\theta\left(|E_{nc}^{sc}|\right) - E_{nc}^{sc-s}\theta\left(-|E_{nc}^{sc}|\right) = \begin{cases} E_{nc}^{sc-s} \text{ for bosonic particles} \\ -E_{nc}^{sc-s} \text{ for bosonic antiparticles} \end{cases}. \tag{41}$$

It is important to note that applying perturbation theory to find corrections of the second order is not useful because we have only adopted corrections of the first order of infinitesimal parameters (Θ,τ,χ) .

4. Study of important relativistic particular cases in DKGT

We will look at some specific examples involving the new bound state energy eigenvalues in Eq. (40) in this section. By adjusting relevant parameters of the ISDM-SVCPs model in the deformation of the KG theory symmetries, we could derive some specific potentials useful for other physical systems for much concern the specialist reach**.** It should be noted that these special cases were treated within the framework of relativistic quantum mechanics known in the literature in Ref. [21], and we are now in the process of generalizing them to include extended relativistic quantum mechanics symmetries.

(1) When the scalar potential is equal to the vector potential in magnitude, $V_{sc}(r) = S_{sc}(r)$, and sign, *i.e.*, $V_0 = 0$ and $\beta = 1$, Eq. (40) can be reduced to the following forms:

$$E^{p}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = \frac{q(b-q) + B^{s}_{nl}\sqrt{B^{s2}_{nl} - b(b-2q)}}{q^2 + B^{s2}_{nl}} m_0 + \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)m + \Theta F(j,l,s)\right)\right]^{1/2}, \tag{42}$$

and

$$E^{ap}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = \frac{q(b-q) - B_{nl}\sqrt{B^{2}_{nl} - b(b-2q)}}{q^2 + B^{s2}_{nl}} m_0 - \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)m + \Theta F(j,l,s)\right)\right]^{1/2}, \tag{43}$$

with

$$B^{s}_{nl} = n + 1/2 + \sqrt{(l+1/2)^2 + b(b-2q)}.$$

The first two parts in RHS of Eqs. (42) and (43) describe the relativistic energy of a bosonic particles and anti-bosonic particles within the framework of relativistic quantum mechanics known in the literature while the rest terms are present the topological effect of the deformation space-space on the theses main energies.

(2) When the scalar potential is equal to the vector potential in magnitude, $V_{sc}(r) = S_{sc}(r)$, and sign, i.e., $V_0 = 0$, $m_1 = 0$ and $\beta = 1$, Eq. (40) can be reduced to the following forms:

$$E^{p}_{nc}(n,q,m_0,\Theta,\tau,\chi,j,l,s,m) = \frac{(n+l+1)^2 - q^2}{(n+l+1)^2 - q^2} m_0 + \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\boldsymbol{\Omega})m + \Theta F(j,l,s)\right)\right]^{1/2}, \tag{44}$$

and

$$E^{ap}_{nc}(n,q,m_0,\Theta,\tau,\chi,j,l,s,m) = -m_0 - \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)m + \Theta F(j,l,s)\right)\right]^{1/2}. \tag{45}$$

The first two parts in RHS of Eqs. (44) and (45) describe the relativistic energy of a bosonic particles and its antiparticles within the framework of relativistic quantum mechanics known in the literature while the rest terms are present the topological effect of the deformation space-space on the theses main energies. For the s-wave which corresponds $l = 0$ to or $m = 0$, Eqs. (44) reduce to:

$$E^{p}_{nc}(n,q,m_0,\Theta,\tau,\chi,j,l,s,m) = \frac{(n+1)^2 - q^2}{(n+1)^2 - q^2} m_0 + \sqrt{\Theta}\left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1) F(j,l,s)\right]^{1/2}. \tag{46}$$

The first part in RHS of Eq. (69) describes the relativistic energy of particle for the s-wave within the framework of relativistic quantum mechanics known in the literature while the rest term presents the topological effect of the deformation space-space on the main energy.

(3) When the scalar potential is equal to the vector potential in magnitude, $V_{sc}(r) = -S_{sc}(r)$, and sign, *i.e.*, $V_0 = 0$ and $\beta = 1$, Eq. (40) can be reduced to the following forms:

$$E^{p}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = \frac{-q\,(b-q) + B^{p}_{nl}\sqrt{B^{p2}_{nl} - b\,(b-2q)}}{q^2 + B^{p2}_{nl}} + \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)\,m + \Theta F\,(j,l,s)\right)\right]^{1/2}, \quad (47)$$

and

$$E^{ap}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = \frac{-q\,(b-q) - \sqrt{B^{p2}_{nl} - b\,(b-2q)}}{q^2 + B^{p2}_{nl}} - \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)\,m + \Theta F\,(j,l,s)\right)\right]^{1/2}. \quad (48)$$

The first two parts in RHS of Eqs. (47) and (48) describe the relativistic energy of a bosonic particles and its antiparticles within the framework of relativistic quantum mechanics known in the literature while the rest terms are present the topological effect of the deformation space-space on the theses main energies.

(4) If we consider the case when $V_{sc}(r) = -S_{sc}(r)$, *i.e.*, $q_s = -q_v$ and sign, *i.e.*, (V_0, m_1)= $(0,0)$, then Eq. (40) can be reduced to the following forms:

$$E^{p}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = \frac{(n+l)^2 - q^2}{(n+l)^2 - q^2}m_0 + \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)\,m + \Theta F\,(j,l,s)\right)\right]^{1/2}, \quad (49)$$

and

$$E^{ap}_{nc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m) = -m_0 - \left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\left((\tau\aleph + \chi\Omega)\,m + \Theta F\,(j,l,s)\right)\right]^{1/2}. \quad (50)$$

The first two parts in RHS of Eqs. (49) and (50) describe the relativistic energy of a bosonic particles and its antiparticles within the framework of relativistic quantum mechanics known in the literature while the rest terms are present the topological effect of deformation space-space on the theses main energies. For the s-wave which corresponds $l = 0$ to $m = 0$, Eq. (49) reduced to:

$$E^{p}_{nc}(n,q,m_0,\Theta,\tau,\chi,j,l,s,m) = -\frac{(n+1)^2 - q^2}{(n+1)^2 - q^2}m_0 + \sqrt{\Theta}\left[\langle Z\rangle^{sc}_{(nlm)}(n,q,m_0,m_1)\,F\,(j,l,s)\right]^{1/2}. \quad (51)$$

The first part in RHS of Eq. (51) describes the relativistic energy of a bosonic particles for the *s*-wave within the framework of relativistic quantum mechanics known in the literature while the rest term is present the topological effect of deformation space-space on the main energy.

5. The improved pure scalar Coulomb-like potential in DKG symmetries

Ikhdair in Ref. [21] used a pure scalar repulsive Coulomb-like field potential and the spatially dependent mass function having a linear form in the context of usual relativistic KG symmetries:

$$\begin{cases} S_{sc}(r) = \frac{s_c}{r} \quad \text{and} \quad V_{sc}(r) = 0 \\ m(r) = Ar \quad \text{with} \quad A = \frac{m_0}{L} \end{cases}, \quad (52)$$

where m_0 is the rest mass and L is a constant with space dimension. Ikhdair in Ref. [21] inserting Eqs. (52) into the radial part $u_{nl}(r)$, and obtained:

$$\left(\frac{d^2}{dr^2}+\epsilon_{nl}^2-\alpha_1 r^2-\frac{\alpha_2}{r^2}\right)u_{nl}(r)=0, \tag{53}$$

with ϵ_{nl}, α_1 and α_2 are equals $\sqrt{(2m_0 s_c/L)-E_{nl}^2}$, $\alpha_1=m_0/L$ and $s_c^2+l(l+1)$, respectively. The author of Ref. [21] used the NU method to obtain the expression for the radial part $u_{nl}(r)$ of the wave function $\Psi_{nl}(r,\theta,\varphi)$ as a function of the generalized Laguerre polynomial $L_n^{(2\Lambda+1)/2}\left(\alpha_1 r^2\right)$ in RQM symmetries. Allowing the wave function solution as follows:

$$\Psi_{nl}(r,\theta,\varphi)=Nr^{(\Lambda+1)/2}\exp\left(-\alpha_1 r^2/2\right)L_n^{(2\Lambda+1)/2}\left(\alpha_1 r^2\right)Y_m^l(\theta,\varphi), \tag{54}$$

the parameter Λ and the normalization constant N are given by:

$$\Lambda=\frac{1}{2}\left(\sqrt{(2l+1)^2+4s_c^2}-1\right)\text{ and }N=\sqrt{\frac{2A^{\frac{1}{2}\sqrt{(2l+1)^2+4s_c^2}+1}}{\begin{pmatrix}n-1\\n\end{pmatrix}\Gamma\left(\frac{1}{2}\sqrt{(2l+1)^2+4s_c^2}+1\right)}}.$$

while

$$\begin{pmatrix}n-1\\n\end{pmatrix}=\frac{(n-1)!}{n!}$$

is a generalized binomial coefficient. The corresponding equation of energy of a bosonic particles E_{nl}^{+} and antiparticles E_{nl}^{-} is given by:

$$E_{nl}^{\pm}=\pm\left[m_0\frac{2s_c}{L}+\frac{m_0}{L}\left(2n+1+\sqrt{(2l+1)^2+4s_c^2}\right)\right]^{1/2}. \tag{55}$$

Now, we apply the Weyl-Moyal star product to Eq. (53), in the context of DKGT symmetries, we obtain:

$$\left(\frac{d^2}{dr^2}+\epsilon_{nl}^2-\alpha_1 r^2-\frac{\alpha_2}{r^2}\right)*u_{nl}(r)=0. \tag{56}$$

It is worth motioning that the BS method permutes us to reduce the above equation to the simplest form without a star product as follows:

$$\left(\frac{d^2}{dr^2}+\epsilon_{nl}^2-\alpha_1^2\hat{r}^2-\frac{\alpha_2}{\hat{r}^2}\right)u_{nl}(r)=0. \tag{57}$$

Thus, after straightforward calculations using Eq. (22), the new operators $\left(-\alpha_1^2\hat{r}^2\right)$ and $\left(-\frac{\alpha_2}{\hat{r}^2}\right)$ that appear in the above equation, in the DKGT symmetries, are expressed as:

$$\begin{cases}-\alpha_1^2\hat{r}^2=-\alpha_1^2 r^2+\alpha_1^2\mathbf{L}\Theta+O\left(\Theta^2\right)\\ -\frac{\alpha_2}{\hat{r}^2}=-\frac{\alpha_2}{r^2}-\alpha_2\frac{\mathbf{L}\Theta}{r^4}+O\left(\Theta^2\right)\end{cases}. \tag{58}$$

Substituting Eqs. (58) into Eq. (57), we obtain the following like the Shrodinger equation:

$$\left(\frac{d^2}{dr^2}+\epsilon_{nl}^2-\alpha_1 r^2-\frac{\alpha_2}{r^2}-U_{sc}^{\text{pert}}(r)\right)u_{nl}(r)=0. \tag{59}$$

The generated effective potential $U_{sc}^{\text{pert}}(r)$ that appears in Eq. (59) is expressed as:

$$U_{sc}^{\text{pert}}(r)=\left(\frac{\alpha_2}{r^4}-\alpha_1^2\right)\mathbf{L}\Theta+O\left(\Theta^2\right). \tag{60}$$

The inequality $U_{sc}^{\text{pert}}(r)<<U_{sc}(r)$ has become achieved (here $U_{sc}(r)$ equal $\epsilon_{nl}^2-\alpha_1 r^2-[\alpha_2/r^2]$). Thus, we need to calculate the expectation value $X_{2(nlm)}^{psc}\equiv\left\langle 1/r^4\right\rangle_{(nlm)}^{psc}$ taking into account the unperturbed wave function $\Psi_{nl}(r,\theta,\varphi)$ which we have seen previously in Eq. (54). Thus after straightforward calculations, we obtain the following integral:

$$X_{2(nlm)}^{psc}=N^2\int_0^{+\infty}r^{(\Lambda+1)-4}\exp\left(-\alpha_1 r^2\right)\left[L_n^{(2\Lambda+1)/2}\left(\alpha_1 r^2\right)\right]^2 dr. \tag{61}$$

Introducing the variable change $z = r^2 \in [0, +\infty]$, then the above equation takes a new form:

$$X_{2(nlm)}^{psc} = \frac{N^2}{2}\int_0^{+\infty} z^{(\Lambda+1)/2-3/2-1}\exp\left(-\alpha_1 z\right)\left[L_n^{(2\Lambda+1)/2}\left(\alpha_1 z\right)\right]^2 dz. \tag{62}$$

Comparing Eq. (62) with the integral in Eq. (31), thus after straightforward calculations we find:

$$X_{2(nlm)}^{psc} = \frac{N^2}{2}\frac{\alpha_1^{1-\Lambda/2}\Gamma\left(n+\Lambda/2+5/2\right)\Gamma\left(n+\Lambda+3/2\right)}{n!^2\Gamma\left(\Lambda/2+5/2\right)\Gamma\left(\Lambda+3/2\right)} \times {}_3F_2\left(-n, \Lambda/2-1, -1/2-\Lambda/2; -n+\Lambda/2-1, \Lambda+3/2, 1\right). \tag{63}$$

To avoid repeating the calculations, we will follow the same physical strategy that we saw in the previous section to find,the total energy $E_{nc}^{psc}(n, s_c, m_0, \Theta, \tau, \chi, j, l, s, m)$ for the case of the bosonic particles (or antiparticles) for bosonic particles with spin-s sunder improved pure scalar Coulomb-like potential model, corresponding to the generalized $(n, l, m)^{th}$excited states:

$$E_{nc}^{psc}(n, s_c, m_0, \Theta, \tau, \chi, j, l, s, m) = \begin{cases} \left[m_0\frac{2s_c}{L}+\frac{m_0}{L}\left(2n+1+\sqrt{(2l+1)^2+4s_c^2}\right)\right]^{1/2}+\left[\langle Z\rangle_{(nlm)}^{psc}\left(n, s_c, m_0\right)\left(\left(\tau\aleph+\chi\Omega\right)m+\Theta F\left(j, l, s\right)\right)\right]^{1/2} \\ \text{for bosonic particles with spin-s} \\ -\left[m_0\frac{2s_c}{L}+\frac{m_0}{L}\left(2n+1+\sqrt{(2l+1)^2+4s_c^2}\right)\right]^{1/2}-\left[\langle Z\rangle_{(nlm)}^{psc}\left(n, s_c, m_0\right)\left(\left(\tau\aleph+\chi\Omega\right)m+\Theta F\left(j, l, s\right)\right)\right]^{1/2} \\ \text{for bosonic antiparticles with spin-s} \end{cases}. \tag{64}$$

The global expectation values $\langle Z\rangle_{(nlm)}^{psc}\left(n, s_c, m_0\right)$ for the bosonic particles (or antiparticles), which were created from the effect of the ISDM-SVCPs model, are determined from the following expressions:

$$\langle Z\rangle_{(nlm)}^{psc}\left(n, s_c, m_0\right) = \left(s_c^2+l\left(l+1\right)\right)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{psc} - \left(\frac{m_0}{L}\right)^2. \tag{65}$$

At the end of this section, we apply the physical two limits achieved for the validity of the results for the improved spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb potentials and the improved pure scalar Coulomb-like potential in DKG symmetries:

$$\lim_{(\Theta,\tau,\chi)\to(0,0,0)} E_{nc}^{p}(n, q, m_0, \Theta, \tau, \chi, j, l, s, m) = E_{nl}^{p}(n, q, m_0) = -\frac{(n+1)^2-q^2}{(n+1)^2-q^2}m_0, \tag{66}$$

and

$$\lim_{(\Theta,\tau,\chi)\to(0,0,0)} E_{nc}^{psc}(n, s_c, m_0, \Theta, \tau, \chi, j, l, s, m) = E_{nl}^{psc}(n, s_c, m_0) = \left[m_0\frac{2s_c}{L}+\frac{m_0}{L}\left(2n+1+\sqrt{(2l+1)^2+4s_c^2}\right)\right]^{1/2}. \tag{67}$$

As it is known in the literature that the KG equation describes particles with zero spins, but in our case in which we studied both the improved spatially-dependent mass Coulomb potential and the improved pure scalar Coulomb-like potential, in the DKGT symmetries, we found that the DKGE equation can play another role, which is to describe the bosonic particles and antiparticles whose spin quantum number has an integer value (0,1,2...) and satisfies the Bose-Einstein statistics, which in this case is similar to the Duffin-Kemmer-Petiau equation. Furthermore, we have found the expression of total energy in the framework of extended symmetries that unified the energies of the bosonic particles and antiparticles in a single mathematical formula instead of two separate expressions for the two energy values. Worthwhile it is better to mention that for the three-simultaneous limits $(\Theta, \tau, \chi) \to (0, 0, 0)$, we recover the equations of energy for the spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb potentials and the pure scalar Coulomb-like potential in the KG symmetries in Ref. [21].

6. Summary and Conclusions

In summary, this work presents an approximate analytical solution of the 3-dimensional deformed Klein-Gordon equation with the improved spatially dependent mass Coulomb potential and the improved pure scalar Coulomb-like potential. We have obtained the new approximate bound-state energies that appeared sensitive to the quantum numbers $(j, l, , s, m)$, the potential depths $(n, q/s_c, m_0, m_1)$ of the studied potentials, and noncommutativity parameters (Θ, τ, χ). As we know, we derived some specific potentials useful for other physical systems. We also ended our research with this treatment of the nonrelativistic limit of the spatially-dependent mass Coulomb potential in ENRQM symmetries. Among the most important results of our research is the unification of the energy equation of the boson particles and antiparticles within the framework of extended relativistic symmetries, where we have one formula describing the two states together instead of two separate equations. It is worth mentioning that, for all cases, to make the three simultaneous limits $(\Theta, \tau, \chi) \to (0, 0, 0)$, the ordinary physical quantities are recovered in Ref. [21]. Finally, a feature of a noncommutative geometry on the 3-dimensional deformed Klien-Gordon equation with the improved spatially-dependent mass Coulomb potential with mixed scalar-vector Coulomb-like field potentials would be the presence of many physics phonemes which usually appear automatically such as spin-orbit and pseudospin-orbit, modified Zeeman effect and others and cause the behavior of topological properties of deformed space-space.

Acknowledgement

This work was supported by the Research Fund of the Laboratory of Physics and Material Chemistry, University of M'sila, and DGRSDT in Algeria with the project number B00L02UN280120180001. We are grateful to the anonymous referee for the careful reading of the manuscript and useful remarks.

---

1. K. Bakke and C. Furtado, *On the Klein-Gordon oscillator subject to a Coulomb-type potential*, Annals of Physics **355** (2015) 48, `https://doi.org/10.1016/j.aop.2015.01.028`.
2. R.L.L. Vitória, C. Furtado and K. Bakke, *On a relativistic particle and a relativistic position-dependent mass particle subject to the Klein-Gordon oscillator and the Coulomb potential*, Annals of Physics **370** (2016) 128, `https://doi.org/10.1016/j.aop.2016.03.016`.
3. A. De Souza Dutra and C.S. Jia, *Classes of exact Klein–Gordon equations with spatially dependent masses: Regularizing the one-dimensional inversely linear potential*, Phys. Lett. A **352**(6) (2006) 484, `https://doi.org/10.1016/j.physleta.2005.12.0`.
4. H. Motavalli and A.R. Akbarieh, *Klien-Gordon equation for Coulomb potential in noncommutative space*, Mod. Phys. Lett. A **25**(29) (2010) 2523, `https://doi.org/10.1142/s0217732310033529`.
5. M. Darroodi, H. Mehraban and S. Hassanabadi, *The Klein–Gordon equation with the Kratzer potential in the noncommutative space*, Mod. Phys. Lett. A **33** (2018) 1850203, `https://doi.org/10.1142/s0217732318502036`.
6. A. Maireche, *The Klein-Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space*, Mod. Phys. Lett. A **35** (2020) 052050015, `https://doi.org/10.1142/s0217732320500157`.
7. A. Maireche, *Heavy quarkonium systems for the deformed unequal scalar and vector Coulomb-Hulthén potential within the deformed effective mass Klein-Gordon equation using the improved approximation of the centrifugal term and Bopp's shift method in RNCQM symmetries*, Int. J. Geo. Met. Mod. Phys. **18** (2021) 2150214, `https://doi.org/10.1142/S0219887821502145`.
8. A. Maireche, *The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics*, Jordan J. Phys. **14** (2021) 59, `https://doi.org/10.47011/14.1.6`.
9. A. Maireche, *A Theoretical Model of Deformed Klein-Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries*, Few-Body Syst. **62** (2021) 12, `https://doi.org/10.1007/s00601-021-01596-2`.
10. A. Maireche, *The Investigation of Approximate Solutions of Deformed Klein-Gordon and Schrödinger Equations Under Modified More General Exponential Screened Coulomb Potential Plus Yukawa Potential in NCQM Symmetries*, Few-Body Syst. **62** (2021) 66, `https://doi.org/10.1007/s00601-021-01639-8`.
11. A. Maireche, *Heavy light mesons in the symmetries of extended nonrelativistic quark model*, Yanbu J. Eng. Sc. **17** (2019) 51, `https://doi.org/10.53370/001c.23732`.
12. A. Maireche, *A new study of energy levels of hydrogenic atoms and some molecules for new more general exponential screened Coulomb potential*, Open Acc J Math Theor Phy. **1** (2018) 232, `https://doi.org/10.15406/oajmtp.2018.01.00040`.
13. A. Maireche, *A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, Sri Lankan J. of Phys. **21**(2020) 11, `https://doi.org/10.4038/sljp.v21i1.8069`.
14. A. Maireche, *New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales*, J. Nano- Electron. Phys. **8** (2016) 01020, `https://doi.org/10.21272/jnep.8(1).01020`.

15. X.Y. Gu, Z.Q. Ma and S.H. Dong, *Exact -solutions to the Dirac equation for a Coulomb potential in D+1 dimensions,* Int. J. Mod. Phys. *E* **11** (2002) 335, https://doi.org/10.1142/s0218301302000879.

16. S.H. Dong, *The Dirac equation with a Coulomb potential in D dimensions,* J. Phys. A: Math. Gen. **36** (2003) 4977, https://doi.org/10.1088/0305-4470/36/18/303.

17. Z.Q. Ma, S.H. Dong, X.Y. Gu, J. Yu and M. Lozada-cassou, *The Klien-Gordon equation with a Coulomb plus scalar potential in D dimensions,* Int. J. Mod. Phys. E **13** (2004) 597, https://doi.org/10.1142/s0218301304002338.

18. S. H. Dong, X. Y. Gu, Z.Q. Ma and J. Yu, *The Klein–Gordon Equation with a Coulomb Potential in D Dimensions,* Int. J. Mod. Phys. E **12** (2003) 555, https://doi.org/10.1142/s0218301303001387.

19. M. Hamzavi, A.A. Rajabi and H. Hassanabadi, *Exact pseudospin symmetry solution of the Dirac equation for spatially-dependent mass Coulomb potential including a Coulomb-like tensor interaction via asymptotic iteration method,* Phys. Lett. A **374** (2010) 4303, https://doi.org/10.1016/j.physleta.2010.08.065.

20. S.M. Ikhdair and R. Sever, *Solutions of the spatially-dependent mass Dirac equation with the spin and pseudospin symmetry for the Coulomb-like potential,* Applied Mathematics and Computation **216** (2010) 545, https://doi.org/10.1016/j.amc.2010.01.072.

21. S. M. Ikhdair, *Exact Klein-Gordon equation with spatially dependent masses for unequal scalar-vector Coulomb-like potentials,* Eur. Phys. J. A **40** (2009)143, https://doi.org/10.1140/epja/i2009-10758-9.

22. T.C. Adorno, M.C. Baldiotti, M. Chaichian, D.M. Gitman and A. Tureanu, *Dirac equation in noncommutative space for hydrogen atom,* Phys. Lett. B **682** (2009) 235, https://doi.org/10.1016/j.physletb.2009.11.003.

23. V.G. Kupriyanov, *A hydrogen atom on curved noncommutative space,* J. Phys. A: Math. Theor. **46** (2013) 245303, https://doi.org/10.1088/1751-8113/46/24/245303.

24. N. Chair and M.A. Dalabeeh, *The noncommutative quadratic Stark effect for the H-atom,* J. Phys. A: Math. Gen. **38** (2005) 1553, https://doi.org/10.1088/0305-4470/38/7/010.

25. M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, *Noncommutativity of space-time and the hydrogen atom spectrum,* Eur. Phys. J. C **36** (2004) 251, https://doi.org/10.1140/epjc/s2004-01886-1.

26. M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, *Hydrogen atom spectrum and the Lamb Shift in noncommutative QED,* Phys. Rev. Lett. **86** (2001) 2716, https://doi.org/10.1103/physrevlett.86.2716.

27. M. Solimanian, J. Najia and Kh. Ghasemian, *The noncommutative parameter for $c\bar{c}$ in nonrelativistic limit*, Eur. Phys. J. Plus **137** (2022) 331, https://doi.org/10.1140/epjp/s13360-022-02546-5.

28. O. Bertolami, J.G. Rosa, C.M.L. De Aragao, P. Castorina and D. Zappala, *Scaling of variables and the relation between noncommutative parameters in noncommutative quantum mechanics,* Mod. Phys. Lett. A **21** (2006) 795, https://doi.org/10.1142/s0217732306019840.

29. A. Mirjalili and M. Taki, *Noncommutative correction to the Cornell potential in heavy-quarkonium atoms,* Theor. Math. Phys. **186** (2016) 280, https://doi.org/10.1134/s0040577916020112.

30. A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and Matrix theory,* JHEP **02** (1998) 003, https://doi.org/10.1088/1126-6708/1998/02/003.

31. S. Capozziello, G. Lambiase and G. Scarpetta, *Generalized uncertainty principle from quantum geometry,* Int. J. Theor. Phys. **39** (2000) 15, https://doi.org/10.1023/A:1003634814685.

32. S. Doplicher, K. Fredenhagen and J.E. Roberts, *Space-time quantization induced by classical gravity,* Phys. Lett. B **331** (1994) 39, https://doi.org/10.1016/0370-2693(94)90940-7.

33. E. Witten, *Refection on the fate spacetime,* Phys. Today **49** (1996) 24, https://doi.org/10.1063/1.881493.

34. A. Kempf, G. Mangano and R.B. Mann, *Hilbert space representation of the minimal length uncertainty relation,* Phys. Rev. D **52** (1995) 1108, https://doi.org/10.1103/physrevd.52.1108.

35. R.J. Adler and D.I. Santigo, *On gravity and the uncertainty principal*, Mod. Phys. Lett. A **14** (1999) 1371, https://doi.org/10.1142/s0217732399001462.

36. T. Kanazawa, G. Lambiase, G. Vilasi and A. Yoshioka, *Noncommutative Schwarzschild geometry and generalized uncertainty principle,* Eur. Phys. J. C. **79** (2019) 95, https://doi.org/10.1140/epjc/s10052-019-6610-1.

37. F. Scardigli, *Generalized uncertainty principle in quantum gravity from micro-black hole Gedanken experiment,* Phys. Lett. B **452** (1999) 39, https://doi.org/10.1016/s0370-2693(99)00167-7.

38. H.S. Snyder, *Quantized Space-Time,* Phys. Rev. **71** (1947) 38, https://doi.org/10.1103/PhysRev.71.38.

39. A. Connes, Noncommutative Geometry (ISBN-9780121858605) (1994).

40. A. Connes and J. Lott, *Particle models and noncommutative geometry (expanded version),* Nucl. Phys. Proc. Suppl. **18B** (1991) 29,

41. N. Seiberg, and E. Witten, String theory and noncommutative geometry, *JHEP* **1999**(09) (1999) 32, https://doi.org/10.1088/1126-6708/1999/09/032.

42. P. Nicolini, *Noncommutative black holes, the final appeal to quantum gravity: a review,* Int. J Mod. Phys. A **24** (2009) 1229, https://doi.org/10.1142/s0217751x09043353.

43. A. Maireche, *A Novel Exactly Theoretical Solvable of Bound States of the Dirac-Kratzer-Fues Problem with Spin and Pseudo-Spin Symmetry,* Int. Front. Sci. Lett. **10** (2016) 8, https://doi.org/10.18052/www.scipress.com/IFSL.10.8.

44. A. Maireche, *New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms,* Afr. Rev. Phys. **12** (2017) 130.

45. K.P. Gnatenko, *Kinematic variables in noncommutative phase space and parameters of noncommutativity,* Mod. Phys. Lett. A **32** (2017) 1750166, https://doi.org/10.1142/s0217732317501668.

46. A. Maireche, *New bound-state solutions of the deformed Klien-Gordon and Shrodinger equations for arbitrary l-state with the modified equal vector and scalar Manning-Rosen plus a class of Yukawa potentials in RNCQM and NRNCQM symmetries,* J. Phys. Stud. **25** (2021) 4301, https://doi.org/10.30970/jps.25.4301.

47. A. Maireche, *Nonrelativistic Atomic Spectrum for Companied Harmonic Oscillator Potential and its Inverse in both NC-2D: RSP,* Int. Lett. Chem. Phys. Astr. **56** (2015) 1, https://doi.org/10.18052/www.scipress.com/ILCPA.56.1.

48. P.M. Ho and H.C. Kao, *Noncommutative quantum mechanics from noncommutative quantum field Theory,* Phys. Rev. Lett. **88** (2002) 151602-1, https://doi.org/10.1103/physrevlett.88.151602.

49. K.P. Gnatenko, *Composite system in noncommutative space and the equivalence principle,* Phys. Lett. A **377** (2013) 3061, https://doi.org/10.1016/j.physleta.2013.09.036.

50. A. Maireche, *Modified unequal mixture scalar vector Hulthén-Yukawa potentials model as a quark-antiquark interaction and neutral atoms via relativistic treatment using the improved approximation of the centrifugal term and Bopp's shift method,* Few-Body Syst. **61** (2020) 30, https://doi.org/10.1007/s00601-020-01559-z.

51. E.F. Djemaï and H. Smail, *On quantum mechanics on noncommutative quantum phase space,* Commun. Theor. Phys. **41** (2004) 837, https://doi.org/10.1088/0253-6102/41/6/837.

52. A. Maireche, *Bound state solutions of Klein-Gordon and Schrödinger equations with linear combination of Hulthén and Kratzer potentials,* Afr. Rev Phys. **15** (2020) 19.

53. O. Bertolami, J. G. Rosa, C.M. L. de Aragão, P. Castorina and D. Zappalà, *Noncommutative gravitational quantum well,* Phys. Rev. D **72** (2005) 025010-1, https://doi.org/10.1103/physrevd.72.025010.

54. S. I. Vacaru, *Exact solutions with noncommutative symmetries in Einstein and gauge gravity,* J. Math. Phys. **46** (2005) 042503, https://doi.org/10.1063/1.1869538.

55. A. Maireche, *A New Approach to the approximate analytic solution of the three-dimensional Schrödinger equation for Hydrogenic and neutral atoms in the generalized Hellmann potential model,* Ukr. J. Phys. **65** (2020) 987, https://doi.org/10.15407/ujpe65.11.987.

56. J. Zhang, *Fractional angular momentum in non-commutative spaces,* Phys. Lett. B **584** (2004) 204, https://doi.org/10.1016/j.physletb.2004.01.049.

57. A. Maireche, *A new Theoretical Investigations of the Modified Equal Scalar and Vector Manning-Rosen plus quadratic Yukawa Potential within the Deformed Klein-Gordon and Schrodinger Equations using the Improved Approximation of the Centrifugal term and Bopp's shift Method in RNCQM and NRNCQM Symmetries,* SPIN J. **11** (2021) 2150029, https://doi.org/10.1142/S2010324721500296.

58. A. Maireche, *The investigation of approximate solutions of Deformed Klein-Fock-Gordon and Schrödinger Equations under Modified Equal Scalar and Vector Manning-Rosen and Yukawa Potentials by using the Improved Approximation of the Centrifugal term and Bopp's shift Method in NCQM Symmetries,* Lat. Am. J. Phys. Educ. **15** (2021) 2310-1.

59. A. Maireche, *Bound-state solutions of the modified Klein-Gordon and Schrödinger equations for arbitrary l-state with the modified Morse potential in the symmetries of noncommutative quantum mechanics,* J. Phys. Stud. **25** (2021) 1002, https://doi.org/10.30970/jps.25.1002.

60. A. Maireche, *Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics*, Int. J. Geo. Met. Mod. Phys. **17** (2020) 2050067, https://doi.org/10.1142/S021988782050067X.

61. S. Aghababaei and G. Rezaei, *Energy level splitting of a 2D hydrogen atom with Rashba coupling in non-commutative space, Commun. Theor. Phys.* **72** (2020) 125101, https://doi.org/10.1088/1572-9494/abb7cc.

62. J. Wang and K. Li, *The HMW effect in noncommutative quantum mechanics,* J. Phys. A Math. Theor. **40** (2007) 2197, https://doi.org/10.1088/1751-8113/40/9/021.

63. A. Maireche, *A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in RNCQM and NRNCQM symmetries, Rev. Mex. Fis.* **67** (2021) 050702, https://doi.org/10.31349/RevMexFis.67.050702.

64. E. M.C. Abreu, J.A. Neto, A.C.R. Mendes C. Neves, W. Oliveira and M.V. Marcial, *Lagrangian formulation for noncommutative nonlinear systems,* Int. J. Mod. Phys. A **27** (2012) 1250053, https://doi.org/10.1142/s0217751x12500534.

65. A. Maireche, *A model of modified Klein-Gordon equation with modified scalar-vector Yukawa potential,* Afr. Rev Phys. **15** (2020) 1.

66. A. Maireche, *Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles,* Int. Front. Sci. Lett. **11** (2017) 29, https://doi.org/10.18052/www.scipress.com/IFSL.11.29.

67. Y. Yi, K. Kang, W. Jian-Hua and C. Chi-Yi, *Spin-1/2 relativistic particle in a magnetic field in NC phase space,* Chin. Phys. C **34** (2010) 543, https://doi.org/10.1088/1674-1137/34/5/005.

68. A. Maireche, *A New Relativistic Study for Interactions in One-electron atoms (Spin Particles) with Modified Mie-type Potential*, J. Nano- Electron. Phys. **8** (2016) 04027, https://doi.org/10.21272/jnep.8(4(1)).04027.

69. L. Mezincescu, *Star Operation in Quantum Mechanics* (2000). https://arxiv.org/abs/hep-th/0007046.

70. L. Gouba, *A comparative review of four formulations of noncommutative quantum mechanics,* Int. J. Mod. Phys. A **31** (2016) 1630025, https://doi.org/10.1142/s0217751x16300258.

71. F. Bopp, *La mécanique quantique est-elle une mécanique statistique classique particulière*, Ann. Inst. Henri Poincaré **15** (1956) 81, https://www.numdam.org/item?id=AIHP_1956__15_2_81_0.

72. J. Gamboa, M. Loewe and J.C. Rojas, *Noncommutative quantum mechanics,* Phys. Rev. D. **64** (2001) 067901, https://doi.org/10.1103/PhysRevD.64.067901.

73. A. Maireche, *Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics,* J. Nano- Electron. Phys. **10** (2018) 06015-1, https://doi.org/10.21272/jnep.10(6).06015.

74. A. Maireche, *A Recent Study of Excited Energy Levels Diatomics for Modified more General Exponential Screened Coulomb Potential: Extended Quantum Mechanics,* J. Nano-Electron. Phys. **9** (2017) 03021, https://doi.org/10.21272/jnep.9(3).03021.

75. A. Maireche, *Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics,* Sri Lankan J. of Phys. **22** (2021) 1, https://doi.org/10.4038/sljp.v22i1.8079.

76. A. Maireche, *Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Nonrelativistic treatment in the NCQM symmetries,* Lat. Am. J. Phys. Educ. **14** (2020) 3310-1.

77. H. Motavalli and A.R. Akbarieh, *Klein-Gordon equation for the Coulomb potential in noncommutative space,* Mod. Phys. Lett. A **25** (2010) 2523, https://doi.org/10.1142/s0217732310033529.

78. B. Mirza and M. Mohadesi, *The Klein-Gordon and the Dirac Oscillators in a Noncommutative Space,* Commun. Theor. Phys. (Beijing, China) **42** (2004) 664, https://doi.org/10.1088/0253-6102/42/5/664.

79. A. Maireche, *A new theoretical study of the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials within the deformed Klein-Gordon equation in RNCQM symmetries,* Mod. Phys. Lett. A **36** (2021) 2150232, https://doi.org/10.1142/S0217732321502321.

80. A. Maireche, *Diatomic Molecules with the Improved Deformed Generalized Deng–Fan Potential Plus Deformed Eckart Potential Model through the Solutions of the Modified Klein-Gordon and Schrödinger Equations within NCQM Symmetries,*Ukr. J. Phys. **67** (2022) 183, https://doi.org/10.15407/ujpe67.3.183.

81. A. Maireche, *New relativistic and nonrelativistic model of diatomic molecules and fermionic particles interacting with improved modified Mobius potential in the framework of noncommutative quantum mechanics symmetries,* Yanbu J. Eng. Sc. **18** (2021) 10, https://doi.org/10.53370/001c.28090.

82. A. Maireche, *Approximate k-state solutions of the deformed Dirac equation in spatially dependent mass for the improved Eckart potential including the improved Yukawa tensor interaction in ERQM symmetries,* Int. J. Geo. Met. Mod. Phys. **19** (2022) 2250085, https://doi.org/10.1142/S0219887822500852.

83. A. Maireche, *Diatomic Molecules and Fermionic Particles With Improved Hellmann-Generalized Morse Potential through the Solutions of the Deformed Klein-Gordon, Dirac and Schrödinger Equations in Extended Relativistic Quantum Mechanics and Extended Nonrelativistic Quantum Mechanics Symmetries,* Rev. Mex. Fis. **68** (2022) 020801, https://doi.org/10.31349/RevMexFis.68.020801.

84. A. Maireche, *Approximate Arbitrary k State Solutions of Dirac Equation with Improved Inversely Quadratic Yukawa Potential within Improved Coulomb-like Tensor Interaction in Deformation Quantum Mechanics Symmetries, Few-Body Syst.* **63** (2022) 54, https://doi.org/10.1007/s00601-022-01755-z.

85. A. Saidi and M. B. Sedra, *Spin-one (1+3)-dimensional DKP equation with modified Kratzer potential in the noncommutative space*, Mod. Phys. Lett. A **35** (2020) 2050014, https://doi.org/10.1142/s0217732320500145.

86. A. Houcine and B. Abdelmalek, *Solutions of the Duffin-Kemmer Equation in Non-Commutative Space of Cosmic String and Magnetic Monopole with Allowance for the Aharonov-Bohm and Coulomb Potentials*, Phys. Part. Nuc. Lett. **16** (2019) 195,

87. Wolfram Research: https://functions.wolfram.com/.

88. K. Bencheikh, S. Medjedel and G. Vignale, *Current reversals in rapidly rotating ultracold Fermi gases*, Phys. Lett. A **89** (2014) 063620-1, https://doi.org/10.1103/physreva.89.063620.