Spherical circles and constant angle surfaces

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Received 2 July 2022; accepted 10 January 2023

In this present paper, we obtain a general version of constant angle surfaces constructed concerning any direction in three dimensional Euclidean space. This constant angle surface is the special case of developable ruled surfaces whose direction is a spherical circle. Here, we obtain the constant angle surfaces by taking the circles (small circles) whose radius is less than the radius of the sphere, as the base curve. Also, the relationship between the isophote curve and this surface and its physical interpretation is mentioned. When we beam from a light source in a constant direction, the intensity of the light will be the same at every point on this constant angle surface. This study is very important in terms of associating optics, a branch of physics, with geometry, a branch of mathematics. Finally, we classify the singular points of these constant angle surfaces.

Keywords: Constant angle surface; spherical circle; isophote curve; optic; singularity.

DOI: https://doi.org/10.31349/RevMexFis.69.041301

1. Introduction

A constant angle surface is a surface whose tangent planes make a constant angle with a fixed vector field of space. In another saying, constant angle surfaces whose unit normal form a constant angle with an assigned direction field in the Euclidean $3-$space. This surface is the generalization of a helical curve. An interesting motivation to study helix surfaces or constant angle surfaces arises from physics. The most basic known application areas of constant angle surfaces are for light such as crystal, liquid, shape from shading problems. In recent years, many authors have studied these special surfaces to take advantage of their applications in mathematics and physics. Cermelli and Scala discuss some properties of constant angel surfaces in terms of the Hamilton-Jacobi equation. They investigate the properties of a constant angle surface when the direction field is singular along a line or a point, [1]. Munteanu and Nistor obtain a classification for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to R in Euclidean $3-$space, [2]. Many studies have been done on constant angle surfaces and developable surfaces [3, 4]. In [5], the author investigates the constant angle ruled surfaces generated by Frenet frame vectors. Recently the theory of constant angle surfaces are extended to other ambient spaces. For example; in [6, 7], they study these surfaces in Minkowski $3-$space. In [8], the authors extend the concept of constant angle surfaces to a Lorentzian ambient space. Also, in product spaces $\mathbb{S}^2 \times \mathbb{R}$ [9, 10], in $\mathbb{H}^2 \times \mathbb{R}$ [11] and in Heisenberg group [12, 13].

On the other hand, an isophote curve is defined as the locus of surface points whose normal vectors make a constant angle with a given constant vector as seen in Fig. 1. So, we can say that the curves on the constant angle surface are isophote curves. The isophote curve is a nice corollary to Lambert's law of cosines in the optics branch of physics. This law states that the illuminance intensity on a diffused surface is proportional to the cosine of the angle formed between the normal vector of the surface and the light vector. So, we can say the geometric description of isophote curves on surfaces which are the surface normal vectors in points of the curve make a constant angle with a fixed light direction, [14]. In recent years, there have been many applications of these curves in different branches. In [15], the authors are developed a novel technique to detect caries lesions using isophote concepts. Also, in [16], they present the implementation of a real-time eye detection method that uses the properties of isophotes, to achieve robustness against changes in illumination, eye rotation and pupil size.

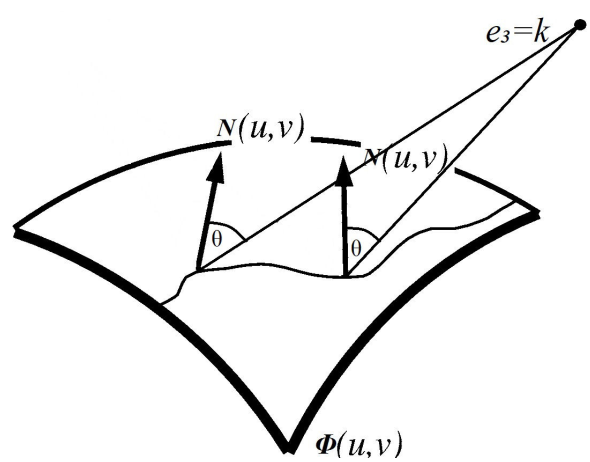


FIGURE 1. An isophote on a surface.

In this present paper, we investigate the spherical circles and constant angle surfaces in $\mathbb{E}^3$. This study has emerged by considering the study of Munteanu and Nistor in [2] from a different perspective. The difference of the present paper is that a constant angle surface is obtained with respect to any direction and some characterizations are given in three dimensional Euclidean space. This constant angle surface is the developable ruled surface whose direction is the spherical circle whose radius is less than the radius of the sphere. Also, by the definition of isophote curves, the curves on this surface are isophote curves. These curves have applications in many fields. At the beginning of these is optics, which is its application in physics. There are many studies that bring together the optics branch of physics and the geometry branch of mathematics [17–21]. This study is one of them. Based on that, we can say that when we beam from a light source in a constant direction, the intensity of the light will be the same at every point on this constant angle surface. On the other hand, the singularity of the ruled surfaces has been studied by many authors. We also investigate the singularity types of this special surface. Finally, as an application, we give some illustrated examples which support the theory of the paper.

2. Preliminaries

Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3$ be an arbitrary curve in $\mathbb{E}^3$. The curve α is said to be a unit speed if $\langle \alpha^{'}(s), \alpha^{'}(s)\rangle = 1$ for any $s \in I$. Assume that $\{t(s), n(s), b(s)\}$ be the moving frame of the curve α which satisfies the Frenet equations

$$\frac{d}{ds}\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \quad (1)$$

where $t(s), n(s), b(s), \kappa(s)$ and $\tau(s)$ are the tangent, the principal normal and the binormal vector fields, curvature and torsion of $\alpha(s)$, respectively, [22].

Let the position vector of the surface M in the standard form of Euclidean space $\mathbb{E}^3$ is

$$\Phi(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)).$$

Then the standard unit normal vector field N on the surface can be defined by

$$N = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}, \quad (2)$$

where $\Phi_u = (\partial\Phi(u, v)/\partial u)$ and $\Phi_v = (\partial\Phi(u, v)/\partial v)$. Also, the first and second fundamental forms of the surface are as follows

$$I = Edu^2 + 2Fdudv + Gdv^2,$$
$$II = edu^2 + 2fdudv + gdv^2 \quad (3)$$

where the E, F and G components are called the coefficients of the first fundemental form of the surface, and the e, f and g components are called the coefficients of the second fundemental form, respectively. The following equations are given for the first and second fundamental form coefficients of the surface

$$E = \langle \Phi_u, \Phi_u\rangle, \ F = \langle \Phi_u, \Phi_v\rangle, \ G = \langle \Phi_v, \Phi_v\rangle, \quad (4)$$

and

$$e = \langle \Phi_{uu}, N\rangle, \ f = \langle \Phi_{uv}, N\rangle, \ g = \langle \Phi_{vv}, N\rangle. \quad (5)$$

On the other hand, the Gaussian curvature K and the mean curvature H of the surface are as follows

$$K = \frac{eg - f^2}{EG - F^2}, \quad (6)$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \quad (7)$$

Definition 1. *Let α, γ be curves and M be a surface in Euclidean $3-$space. Surfaces formed by the movement of a line along a curve in space are called ruled surfaces. The parameterization of the ruled surface for any two differentiable curves α and γ is as follows*

$$\Phi(u, v) = \alpha(v) + u\gamma(v),$$

where $\alpha(v)$ is called base curve of the ruled surface and $\gamma(v)$ is a unit direction vector of an oriented line in $\mathbb{E}^3$. In addition, if the direction curve is not constant, that is, $\gamma'(v) \neq 0$, the surface is called a non-cylindrical ruled surface, and the surfaces with a constant direction curve are called the generalized cylindrical surface, [23].

Theorem 1. *Let M be a regular ruled surface with the parameterization $\Phi(u, v) = \alpha(v) + u\gamma(v)$. If the Gaussian curvature of the surface is zero, the surface M is called the developable surface. Also, another characterization for developable ruled surfaces is that $\det(\alpha'(v), \gamma(v), \gamma'(v)) = 0$,* [24, 25].

Theorem 2. *Let M be a surface in Euclidean $3-$space. For the surface $M = \Phi(u, v) = \alpha(v) + u\gamma(v)$, line of striction is calculated as,* [26]

$$\bar{\alpha}(v) = \alpha(v) - \frac{\langle \gamma(v) \times \gamma'(v), \gamma(v) \times \alpha'(v)\rangle}{\|\gamma(v) \times \gamma'(v)\|^2}\gamma(v).$$

Definition 2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in E^3. Points with $\alpha'(t) = 0$ on the curve $\alpha(t)$ are called singular points,* [27].

The study of ruled surfaces is a main subject in differential geometry in Euclidean space. Generally, ruled surfaces have singularities. Briefly speaking, the cuspidal edge $C \times \mathbb{R}$, the cuspidal cross-cap CCR or the swallowtail appear SW as singularities of developable surfaces in general. The figures of these types of singularity are given below in Fig. 2, [28].

Theorem 3. *Let $\Phi_{(\alpha,\gamma)} : I \times J \to \mathbb{R}^3$ be a noncylindrical developable surface and $\mu, \lambda : I \to \mathbb{R}$ be smooth functions with $\alpha^{'}(t) = \mu(t)\gamma(t) + \lambda(t)\gamma'(t)$. Let $(t_0, u_0) \in I \times J$ be a singular point of $\Phi_{(\alpha,\gamma)}$ and put $x_0 = \alpha(t_0) + u_0\gamma(t_0) = \Phi_{(\alpha,\gamma)}(t_0, u_0)$.*

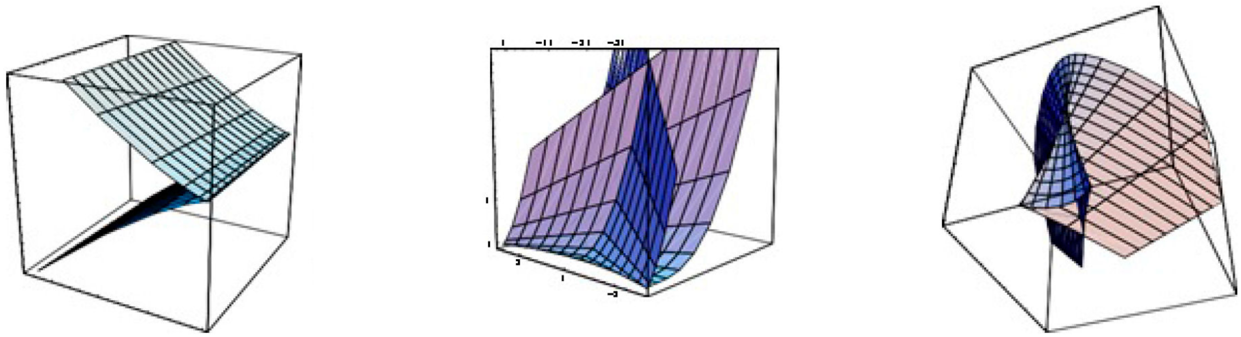


FIGURE 2. Types of singularity ($C \times \mathbb{R}$, CCR and SW, respectively).

1. *Suppose that* $\det(\gamma(t_0), \gamma'(t_0), \gamma''(t_0)) \neq 0$. *Then*

- **a.** *The germ of* $\Phi_{(\alpha,\gamma)}(I \times J)$ *at* x_0 *is locally diffeomorphic to* $C \times \mathbb{R}$ if $u_0 = \lambda(t_0)$ *and* $\mu(t_0) \neq \lambda'(t_0)$.
- **b.** *The germ of* $\Phi_{(\alpha,\gamma)}(I \times J)$ at x_0 *is locally diffeomorphic to* SW *if* $u_0 = \lambda(t_0)$, $\mu(t_0) = \lambda'(t_0)$ *and* $\mu'(t_0) = \lambda''(t_0)$.

2. *Suppose that* $\det(\gamma(t_0), \gamma'(t_0), \gamma''(t_0)) = 0$. *Then the germ of* $\Phi_{(\alpha,\gamma)}(I \times J)$ *at* x_0 *is locally diffeomorphic to* CCR *if* $u_0 = \lambda(t_0)$, $\mu(t_0) \neq \lambda'(t_0)$ *and* $\det(\gamma(t_0), \gamma'(t_0), \gamma^{(3)}(t_0)) \neq 0$, [29].

Definition 3. *A curve lying on a sphere is called a spherical curve*, [27].

Definition 4. *Let* $\alpha : I \to S^2$ *be a unit speed spherical curve. We denote* s *as the arc-length parameter of* α. *Let us denote by*

$$\alpha(s) = \alpha(s), \quad T(s) = \alpha'(s), \quad S(s) = \alpha(s) \times T(s)$$

where $T(s)$ *is a unit tangent vector of* α. *The frame* $\{\alpha(s), T(s), S(s)\}$ *is called the Sabban frame of* α *on* S^2, [27].

3. Spherical circles and constant angle surfaces

A circle of a sphere is a circle that lies on a sphere. A spherical circle can be formed as the intersection of a sphere and a plane, or two spheres. A circle on a sphere whose radius passes through the center of the sphere is called a great circle, otherwise this spherical circle is called the small circle. In this section, a method will be given to obtain constant angle ruled surfaces with the help of small circle on the sphere in Euclidean $3-$space $\mathbb{E}^3$.

Let $S^2 = \left\{(x, y, z) \in \mathbb{E}^3 \,\middle|\, x^2 + y^2 + z^2 = 1\right\}$ be a unit sphere in $\mathbb{E}^3$ and let $\{e_1, e_2\}$ be any orthonormal vectors in this space. We can express a circle on this sphere with the help of these orthonormal bases as follows

$$\alpha(v) = \cos\theta(\cos ve_1 + \sin ve_2) + \sin\theta(e_1 \times e_2). \tag{8}$$

If we take $\theta = 0$ in above equation, the expression

$$\alpha(v) = \cos ve_1 + \sin ve_2,$$

becomes the great circle with a radius of 1 on the unit sphere. Thus, in order to obtain other circles (small circles), we can construct circles with a certain angle θ and the normal $e_3 = e_1 \times e_2$ in the plane. By considering the study of Munteanu and Nistor in [2] from a different perspective, we obtain a ruled surface with a constant angle with respect to any direction. The fixed direction is directly related to $e_3 = e_1 \times e_2$. To find the tangent vector of the curve $\alpha(v)$ on the sphere for $\theta \neq 0$, we take the derivative of the Eq. (8) with respect to v

$$\alpha^{'}(v) = \cos\theta(-\sin ve_1 + \cos ve_2). \tag{9}$$

The norm of the above equation is that

$$\left\|\alpha^{'}(v)\right\| = \cos\theta.$$

So, the unit tangent vector of $\alpha(v)$ is obtained as follows

$$T(v) = \frac{\alpha^{'}(v)}{\|\alpha^{'}(v)\|} = -\sin ve_1 + \cos ve_2. \tag{10}$$

If we cross product the curve $\alpha(v)$ and the tangent vector $T(v)$, we get the expression

$$\begin{aligned} S(v) = \alpha(v) \times T(v) = -\sin\theta(\cos ve_1 + \sin ve_2) \\ + \cos\theta e_3. \end{aligned} \tag{11}$$

Thus, the Sabban frame is obtained on the unit sphere as $\{\alpha(v), T(v), S(v)\}$ as expressed in Definition 4 in the Preliminaries section. If the necessary calculations are taken, the derivative change of the frame is as follows

$$\begin{aligned} \frac{d}{dv}\begin{bmatrix} \alpha(v) \\ T(v) \\ S(v) \end{bmatrix} = \begin{bmatrix} 0 & \cos\theta & 0 \\ -\cos\theta & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{bmatrix} \\ \times \begin{bmatrix} \alpha(v) \\ T(v) \\ S(v) \end{bmatrix}. \end{aligned} \tag{12}$$

In addition, the Darboux vector of the spherical circle $\alpha(v)$ determines the fixed direction as

$$\omega = \sin\theta\alpha(v) + \cos\theta S(v). \tag{13}$$

In fact, if the necessary calculations are done here, it can be easily seen that

$$\omega = e_1 \times e_2 = e_3.$$

Theorem 4. *Let $\{e_1, e_2\}$ be any orthonormal vectors in $3-$dimensional Euclidean space. Let α be the small circle in the unit sphere given as*

$$\alpha(v) = \cos\theta(\cos ve_1 + \sin ve_2) + \sin\theta e_3, \quad \theta \neq 0.$$

The surface $\Phi(u,v)$ defined below is a ruled surface

$$\Phi(u,v) = \int\limits_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv + u\alpha(v), \tag{14}$$

where $f(v)$ and $g(v)$ are the differentiable functions.

Proof. Let $\alpha(v)$ be any small circle on the unit sphere S^2 and $\Phi(u,v)$ be the surface. Considering the definition of ruled surfaces, the curve

$$\int\limits_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv,$$

is defined as the ruled surface directrix (also called the base curve) and $\alpha(v)$ is defined as the direction vector of the surface. So, we can easily see that the surface $\Phi(u,v)$ is a ruled surface.

Corollary 1. *Let*

$$\Phi(u,v) = \int\limits_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv + u\alpha(v),$$

be the ruled surface in $3-$dimensional Euclidean space. $S(v) = \alpha(v) \times T(v)$ is the unit normal of ruled surface $\Phi(u,v)$ where $\alpha(v)$ is the small circle on the sphere and $T(v)$ is its unit tangent vector.

Proof. Let $\Phi(u,v)$ be the ruled surface in $3-$dimensional Euclidean space. To find the unit normal of the surface,

$$N = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|},$$

we firstly calculate the parameter curves of the surface. If the derivatives of Eq. (14) are taken with respect to u and v, respectively, we get

$$\Phi_u = \alpha(v), \tag{15}$$

and

$$\Phi_v = \alpha(v)f(v) + \alpha'(v)(g(v) + u). \tag{16}$$

If the following calculations are done to find the normal of the surface, we obtain

$$\begin{aligned}\Phi_u \times \Phi_v = &-(g(v)+u)\cos\theta\sin\theta\cos ve_1 \\ &-(g(v)+u)\cos\theta\sin\theta\sin ve_2 \\ &+(g(v)+u)\cos^2\theta(e_1 \times e_2),\end{aligned}$$

and

$$\|\Phi_u \times \Phi_v\| = (g(v)+u)\cos\theta.$$

So, we can easily find the normal of the surface as follows

$$N = -\sin\theta\cos ve_1 - \sin\theta\sin ve_2 + \cos\theta(e_1 \times e_2). \tag{17}$$

If necessary arrangements are made in the above expression, it can be seen that

$$\begin{aligned}&N = -\sin\theta(\cos ve_1 + \sin ve_2) + \cos\theta e_3, \\ &N = S.\end{aligned} \tag{18}$$

Thus, we can say that $S(v)$ is the unit normal to the ruled surface $\Phi(u,v)$.

Corollary 2. *Let the normal of the ruled surface $\Phi(u,v)$ defined in Eq. (14) be N and $\omega = (e_1 \times e_2) = e_3$ be the axis of the constant direction. Then, the surface $\Phi(u,v)$ is a constant angle ruled surface.*

Proof. Let $\Phi(u,v)$ be the ruled surface in $3-$dimensional Euclidean space. Considering Eq. (17) and axis of the constant direction $\omega = (e_1 \times e_2) = e_3$, we can easily write that

$$\langle N, \omega\rangle = \cos\theta = \text{constant}. \tag{19}$$

So, we can say that the surface $\Phi(u,v)$ is a constant angle ruled surface.

Corollary 3. *Let*

$$\Phi(u,v) = \int\limits_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv + u\alpha(v)$$

be the ruled surface in $3-$dimensional Euclidean space. The surface $\Phi(u,v)$ is a developable ruled surface.

Proof. Let $\Phi(u,v)$ be the ruled surface. If we rename the base curve of the surface $\Phi(u,v)$ as

$$\varphi = \int\limits_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv, \tag{20}$$

and use the developable ruled surface condition, we obtain that

$$\begin{aligned}\det(\varphi^{'}(v), \alpha(v), \alpha'(v)) = \det(&f(v)\alpha(v) \\ &+ g(v)\alpha'(v), \alpha(v), \alpha'(v)).\end{aligned}$$

If necessary calculations are made, it can be easily seen that this determinant value is zero. So, we can say that $\Phi(u,v)$ is a developable ruled surface.

Corollary 4. *Let*

$$\Phi(u,v)=\int_0^v [f(v)\alpha(v)+g(v)\alpha'(v)]\,dv+u\alpha(v)$$

be the ruled surface in 3−dimensional Euclidean space. The line of striction of the surface $\Phi(u,v)$ is as follows

$$\bar{\varphi}=\varphi-g(v)\alpha(v) \tag{21}$$

where

$$\varphi=\int_0^v [f(v)\alpha(v)+g(v)\alpha'(v)]\,dv.$$

Let $\Phi(u,v)$ be the ruled surface. The line of striction of the surface is calculated as follows

$$\bar{\varphi}=\varphi-\frac{\langle\alpha(v)\times\alpha'(v),\alpha(v)\times\varphi'(v)\rangle}{\|\alpha(v)\times\alpha'(v)\|^2}\alpha(v). \tag{22}$$

If the necessary calculations are made in the above expression, we get

$$\begin{aligned}\alpha(v)\times\alpha'(v)&=-\sin\theta\cos\theta\cos ve_1\\&\quad-\sin\theta\cos\theta\sin ve_2+\cos^2\theta(e_1\times e_2), \quad (23)\\ \alpha(v)\times\varphi'(v)&=-g(v)\cos\theta\sin\theta\cos ve_1-g(v)\cos\theta\\&\quad\times\sin\theta\sin ve_2+g(v)\cos^2\theta(e_1\times e_2). \quad (24)\end{aligned}$$

If the above equations are substituted in Eq. (22), line of striction is obtained as

$$\bar{\varphi}=\varphi-\frac{g(v)\cos^2\theta}{\cos^2\theta}\alpha(v),$$
$$\bar{\varphi}=\varphi-g(v)\alpha(v).$$

Corollary 5. *Let*

$$\Phi(u,v)=\int_0^v [f(v)\alpha(v)+g(v)\alpha'(v)]\,dv+u\alpha(v)$$

be the ruled surface in 3−dimensional Euclidean space. If $f(v)=0$ and $g(v)=$constant, $\Phi(u,v)$ is a cone surface.

Proof. Let $\Phi(u,v)$ be the ruled surface. If the expressions $f(v)=0$ and $g(v)=$constant are substituted in the surface equation above, we can easily see that

$$\Phi(u,v)=c\alpha(v)+u\alpha(v),\ c\text{ constant}$$
$$\Phi(u,v)=(c+u)\alpha(v),\ c\text{ constant}.$$

So, we can say that $\Phi(u,v)$ is a cone surface.

Theorem 5. *Let $\Phi: I\times J\to\mathbb{E}^3$,*

$$\Phi(u,v)=\int_0^v [f(v)\alpha(v)+g(v)\alpha'(v)]\,dv+u\alpha(v)$$

be a constant angle ruled surface and $f,g: I\to\mathbb{R}$ be smooth functions with

$$\frac{d}{dv}\left(\int_0^v [f(v)\alpha(v)+g(v)\alpha'(v)]\,dv\right)=f(v)\alpha(v)+g(v)\alpha'(v).$$

Also, let $(u_0,v_0)\in I\times J$ be a singular point of $\Phi(u,v)$ and

$$x_0=\int_0^v [f(v_0)\alpha(v_0)+g(v_0)\alpha'(v_0)]\,dv+u_0\alpha(v_0)=\Phi(u_0,v_0).$$

The germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to $C\times\mathbb{R}$ and SW. Also, the germ of $\Phi(u,v)$ at x_0 isn't locally diffeomorphic to CCR.

Proof. Let $\Phi: I\times J\to\mathbb{E}^3$ be a constant angle ruled surface and $f,g: I\to\mathbb{R}$ be smooth functions. Considering Theorem 3 in Preliminaries section, we are calculated that

$$\det(\alpha(v),\alpha'(v),\alpha''(v))=\sin\theta\cos^2\theta.$$

1. For $\theta\neq 0$ $(\theta\neq\frac{\pi}{2},\pi,...)$, $\det(\alpha(v),\alpha'(v),\alpha''(v))\neq 0$. Then;
   - **a.** Since $u_0=g(v_0)$ and $f(v_0)\neq g'(v_0)$, the germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to $C\times\mathbb{R}$.
   - **b.** Since $u_0=g(v_0)$, $f(v_0)=g'(v_0)$ and $f'(v_0)\neq g''(v_0)$ the germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to SW.
2. For $\theta=0$ $(\theta=\frac{\pi}{2},\pi,...)$, $\det(\alpha(v),\alpha'(v),\alpha''(v))=0$.

Although $u_0=g(v_0)$, $f(v_0)\neq g'(v_0)$, $\det(\alpha(v),\alpha'(v),\alpha^{(3)}(v))=0$. Hence, the germ of $\Phi(u,v)$ at x_0 isn't locally diffeomorphic to CCR.

Remark 1. *Considering the theory in the study, we can say that when we are given any axis, we can create a constant angle surface with the help of this axis. For example, let's examine the problem of creating a constant angle ruled surface with axis $k=e_3$. To find the circle $\alpha(v)$, the circle whose normal is $k=e_3$ must be written. This is found by writing the intersection curve of the unit sphere and the plane with e_3 normal. Let the $\{e_1,e_2\}$ be an orthonormal vector obtained by Gramm-Schmidt orthonormalization method in the plane whose normal is e_3. In this case, the intersection curve of the unit sphere and the plane is as follows*

$$\cos ve_1+\sin ve_2.$$

This curve is the great circle with radius length 1. *Small circles with radius* $r = \cos\theta$ *are as follows*

$$\alpha(v) = \cos\theta \cos v e_1 + \cos\theta \sin v e_2 + \sin\theta e_3.$$

The surface

$$\Phi(u,v) = \int_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv + u\alpha(v),$$

obtained by this spherical circle $\alpha(v)$ *is a constant angle ruled surface with the axis* $k = e_3$. *The normal to this surface is*

$$N = \sin\theta \cos v e_1 - \sin\theta \sin v e_2 + \cos\theta e_3,$$

and $\langle N, e_3 \rangle = \cos\theta$. *The angle that the surface makes with the axis is determined according to the state of the* θ *angle. Also, when the functions* f *and* g *are changed, they change on the constant angle surfaces.*

The equations and figures of the constant angle surfaces according to given any direction are discussed in the examples below.

Example 1. Let's get the equation of the constant angle surface with the axis $k = e_3 = (1/\sqrt{3})(1,1,1)$ and draw its figures as Fig. 3.

$$\tilde{e}_1 = (1,-1,0), \quad \tilde{e}_2 = (0,1,-1).$$

If the vectors perpendicular to the plane are made orthonormal with the Gramm-Schmidt method, the following vectors are obtained as

$$e_1 = \frac{1}{\sqrt{2}}(1,-1,0),\ e_2 = \frac{1}{\sqrt{6}}(1,1,-2).$$

In this case, the spherical circle $\alpha(v)$ *with radius* $r = \cos\theta$ *is obtained as follows for* $\theta = \pi/4$,

$$\begin{aligned}\alpha(v) = \Bigg(&\frac{1}{2}\cos v + \frac{1}{2\sqrt{3}}\sin v + \frac{1}{\sqrt{6}},\\ &\frac{-1}{2}\cos v + \frac{1}{2\sqrt{3}}\sin v + \frac{1}{\sqrt{6}},\\ &\frac{-1}{\sqrt{3}}\sin v + \frac{1}{\sqrt{6}}\Bigg).\end{aligned}$$

For the functions $f(v) = v + 1$ *and* $g(v) = v^2$, *if the necessary calculations are done, the equation of the constant angle ruled surface can be easily written as*

$$\Phi(u,v) = (\Phi_1, \Phi_2, \Phi_3)$$

where

$$\begin{aligned}\Phi_1(u,v) = &\frac{1}{6}[(\sqrt{3}+3v)\cos v + (-3+\sqrt{3}v)\sin v]\\ &+ \frac{1}{2}u\cos v + \frac{1}{2\sqrt{3}}u\sin v + \frac{1}{\sqrt{6}}u,\\ \Phi_2(u,v) = &\frac{1}{6}[(\sqrt{3}-3v)\cos v + (3+\sqrt{3}v)\sin v]\\ &- \frac{1}{2}u\cos v + \frac{1}{2\sqrt{3}}u\sin v + \frac{1}{\sqrt{6}}u,\\ \Phi_3(u,v) = &\frac{-1}{\sqrt{3}}(\cos v + v\sin v) - \frac{1}{\sqrt{3}}u\sin v + \frac{1}{\sqrt{6}}u.\end{aligned}$$

If we calculate the singular points for this surface according to Theorem 5, we can write that

$$\begin{aligned}&\det(\alpha(v), \alpha'(v), \quad \alpha''(v)) \neq 0\\ &\text{for} \quad \theta \neq 0 \quad \left(\theta \neq \frac{\pi}{2}, \pi, ...\right).\end{aligned}$$

a. For $f(v_0) = v_0 + 1$, $g'(v_0) = 2v_0$,

$$v_0 + 1 \neq 2v_0$$

and

$$v_0 \neq 1.$$

Since $u_0 = g(v_0)$, we can say that all points as (u_0, v_0) of $\Phi(u,v)$ satisfying the following condition are locally diffeomorphic to $C \times \mathbb{R}$

$$u_0 = v_0^2, \quad \text{for} \quad v_0 \neq 1.$$

b. For $f'(v) = 1$, $g'(v) = 2v$, $g''(v) = 2$,

$$\begin{aligned}f(v_0) &= v_0 + 1,\\ g'(v_0) &= 2v_0.\end{aligned}$$

From the equality of the above equations, we obtain that

$$v_0 = 1.$$

Considering the following equations,

$$\begin{aligned}f'(v_0) &= 1,\\ g''(v_0) &= 2,\end{aligned}$$

we can say that

$$f'(v_0) \neq g''(v_0).$$

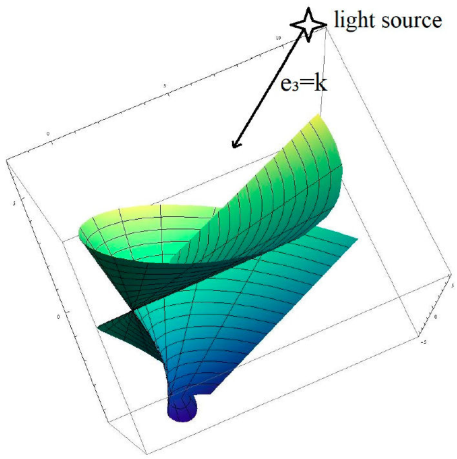
We obtain the other singular point as follows

$$u_0 = g(v_0) = g(1) = 1.$$

So, the point $\Phi(u_0, v_0) = \Phi(1,1)$ is locally diffeomorphic to SW. Also, according to Theorem 5, we know that the germ of $\Phi(u,v)$ isn't locally diffeomorphic to CCR.

Example 2. Let's get the equation of the constant angle surface with the axis $k = e_3 = (1/3)(2,1,2)$ and draw its figures as Fig. 4.

$$\tilde{e}_1 = (0,-2,1), \quad \tilde{e}_2 = (-1,0,1).$$



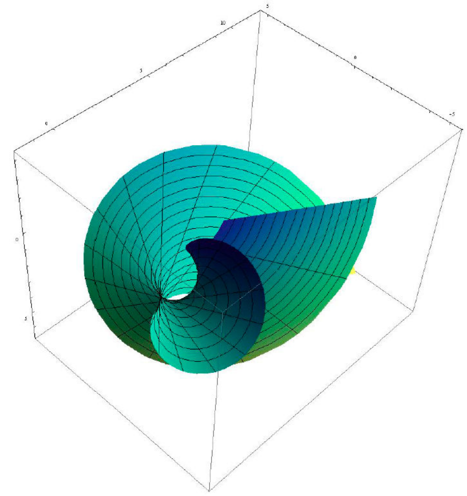


FIGURE 3. Constant angle surface for $\pi/4$.

If the vectors perpendicular to the plane are made orthonormal with the Gramm-Schmidt method, the following vectors are obtained as

$$e_1 = \frac{1}{\sqrt{5}}(0,-2,1), \quad e_2 = \frac{\sqrt{5}}{3}\left(-1,\frac{2}{5},\frac{4}{5}\right).$$

In this case, the spherical circle $\alpha(v)$ with radius $r = \cos\theta$ is obtained as follows for $\theta = \pi/3$,

$$\alpha(v) = \left(\frac{-\sqrt{5}}{6}\sin v + \frac{\sqrt{3}}{3}, \frac{-1}{\sqrt{5}}\cos v + \frac{\sqrt{5}}{15}\sin v + \frac{\sqrt{3}}{6}, \frac{1}{2\sqrt{5}}\cos v + \frac{2\sqrt{5}}{15}\sin v + \frac{\sqrt{3}}{3}\right). \tag{25}$$

For the functions $f(v) = \sin v$ and $g(v) = \cos v$, if the necessary calculations are done, the equation of the constant angle ruled surface can be easily written as

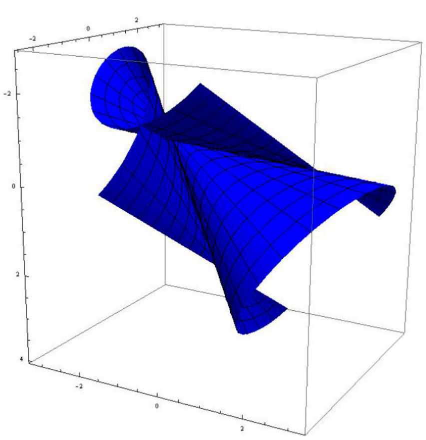
$$\Phi(u,v) = (\Phi_1, \Phi_2, \Phi_3)$$

where

$$\Phi_1(u,v) = \frac{-\sqrt{5}}{6}v - \frac{\cos v}{\sqrt{3}} + \frac{-\sqrt{5}}{6}u\sin v + \frac{\sqrt{3}}{3}u,$$

$$\Phi_2(u,v) = \frac{1}{30}(2\sqrt{5}v - 5\sqrt{3}\cos v) - \frac{1}{\sqrt{5}}u\cos v + \frac{\sqrt{5}}{15}u\sin v + \frac{\sqrt{3}}{6}u,$$

$$\Phi_3(u,v) = \frac{2v}{3\sqrt{5}} - \frac{\cos v}{\sqrt{3}} + \frac{1}{2\sqrt{5}}u\cos v + \frac{2\sqrt{5}}{15}u\sin v + \frac{\sqrt{3}}{3}u.$$



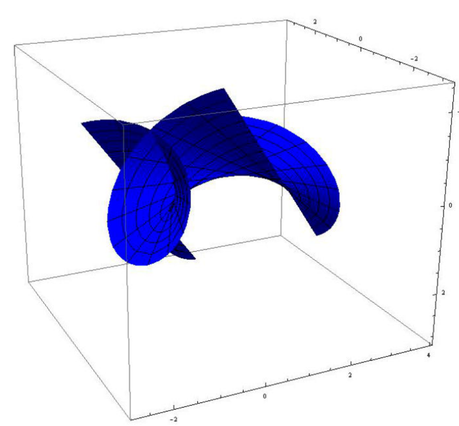


FIGURE 4. Constant angle surface for $\pi/4$.

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