Elko spinors revised

R. Romero

*Departamento de Ciencias Naturales, División de Ciencias Naturales e Ingeniería,
Universidad Autónoma Metropolitana unidad Cuajimalpa,
Avenida Vasco de Quiroga 4871, Col. Santa Fe Cuajimalpa.
Alcaldía Cuajimalpa de Morelos, 05348, Ciudad de México.
e-mail: rromero@cua.uam.mx*

Received 4 August 2022; accepted 14 September 2022

It is shown that c-number elko spinors obey the massless Dirac equation and are unitarily equivalent to Weyl bispinors. Therefore, they do not constitute a new spinor type with mass dimension one.

Keywords: Elko; Weyl bispinors; massless Dirac equation.

DOI: https://doi.org/10.31349/RevMexFis.69.020201

Elko spinors are a complete set of c-number bispinors that are eigenstates of the charge conjugation operator, a property from which they take their name (an acronym from the german *Eigenspinoren des Ladungs Konjugations Operators*). They were proposed in 2005 [1,2] as the expansion coefficients of a quantum field operator, the elko field, presented as a new type of fermion field with the exotic properties of not obeying the massive Dirac equation and having canonical mass dimension one, despite being fermionic. The latter feature led to their proposal as a dark matter candidate, since the elko field is assumed to couple only to the Higgs field to ensure renormalizability. To this day, elko spinors continue to appear in the scientific literature in various applications, see *e.g.* references [3-9] and references therein. However, elko spinors are just another type of massless bispinors satisfying the masslesss Dirac equation, and as such they can not constitute a new spinor type with mass dimension different from the known 3/2 mass dimension of fermions in the Standard Model Lagrangian [10]. In fact, they can be unitarily transformed to massless Weyl spinors, as is shown in this letter.

Let us first state the properties of masless four-component Weyl spinors. A complete treatment is given in Ref. [11], but here we reproduce the main properties for completeness. Plane wave solutions to the massless Dirac equation are given by $\Psi = u(\mathbf{p}) \exp\{i(\pm Et - \mathbf{x}\cdot\mathbf{p})\}$, with the bispinors

$$u^{(1)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_+(\mathbf{p}) \end{pmatrix}, \quad u^{(2)}(\mathbf{p}) = \begin{pmatrix} \chi_-(\mathbf{p}) \\ 0 \end{pmatrix},$$
$$u^{(3)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_-(\mathbf{p}) \end{pmatrix}, \quad u^{(4)}(\mathbf{p}) = \begin{pmatrix} \chi_+(\mathbf{p}) \\ 0 \end{pmatrix}, \tag{1}$$

and the two-component spinors $\chi_\pm(\mathbf{p})$ given by

$$\chi_+(\mathbf{p}) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix},$$
$$\chi_-(\mathbf{p}) = \begin{pmatrix} -e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2}$$

For definiteness, we use the gamma matrices Weyl representation, with the following definitions

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{3}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{\Sigma} \equiv \gamma^5\gamma^0\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{4}$$

where $\boldsymbol{\sigma} = \left(\sigma^1, \sigma^2, \sigma^3\right)$ are the standard Pauli matrices. Then the massless Dirac equation $i\gamma^\mu\partial_\mu\Psi = 0$ simplifies to

$$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\, u(\mathbf{p}) = \pm\gamma^5\, u(\mathbf{p}). \tag{5}$$

In Hamiltonian form Eq. (5) reads

$$\begin{aligned} \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\, u^{(s)}(\mathbf{p}) &= +\, u^{(s)}(\mathbf{p}), \\ \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\, u^{(s+2)}(\mathbf{p}) &= -\, u^{(s+2)}(\mathbf{p}), \end{aligned} \quad s = 1, 2 \tag{6}$$

and $u^{(1)}(\mathbf{p})$ and $u^{(2)}(\mathbf{p})$ are positive-energy bispinors, with both positive helicity an chirality for the former and negative for the latter, while $u^{(3)}(\mathbf{p})$ and $u^{(4)}(\mathbf{p})$ are negative-energy ones, with negative helicity and positive chirality for the former and the reversed values for the latter. These bispinors are orthonormal $\left[u^{(i)}(\mathbf{p})\right]^\dagger u^{(j)}(\mathbf{p}) = \delta_{ij}$. Taking the momentum $\hat{\boldsymbol{p}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ in the $\hat{\mathbf{z}}$ direction, which will be referred to as the canonical frame, the bispinors simplify to

$$u^{(1)}(p_z) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^{(2)}(p_z) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$u^{(3)}(p_z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u^{(4)}(p_z) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

The canonical frame bispinors and the general momentum ones are related by the rotation

$$\Lambda_1(\theta,\varphi) = \exp\left\{-\frac{\theta}{2}\left(\gamma^1\cos\varphi + \gamma^2\sin\varphi\right)\gamma^3\right\}, \quad (8)$$

since

$$\Lambda_1(\theta,\varphi)\, u^{(i)}(p_z) = u^{(i)}(\mathbf{p}), \qquad i = 1,\ldots,4. \quad (9)$$

Let us now define the elko bispinors following Ref. 9. These are

$$\lambda^{(1)}(\mathbf{p}) = \begin{pmatrix}\sigma^2\eta_+^*(\mathbf{p})\\ \eta_+(\mathbf{p})\end{pmatrix}, \quad \lambda^{(2)}(\mathbf{p}) = \begin{pmatrix}\sigma^2\eta_+^*(\mathbf{p})\\ -\eta_+(\mathbf{p})\end{pmatrix},$$
$$\lambda^{(3)}(\mathbf{p}) = \begin{pmatrix}-\sigma^2\eta_-^*(\mathbf{p})\\ \eta_-(\mathbf{p})\end{pmatrix}, \quad \lambda^{(4)}(\mathbf{p}) = \begin{pmatrix}\sigma^2\eta_-^*(\mathbf{p})\\ \eta_-(\mathbf{p})\end{pmatrix}, \quad (10)$$

where we have changed the notation and ignore the ad-hoc normalization used in the aforementioned reference, for simplicity. The two-component spinors $\eta_\pm(\mathbf{p})$ are given by

$$\eta_+(\mathbf{p}) = \begin{pmatrix}e^{-i\varphi/2}\cos\left(\frac{\theta}{2}\right)\\ e^{i\varphi/2}\sin\left(\frac{\theta}{2}\right)\end{pmatrix},$$
$$\eta_-(\mathbf{p}) = \begin{pmatrix}-e^{-i\varphi/2}\sin\left(\frac{\theta}{2}\right)\\ e^{i\varphi/2}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}, \quad (11)$$

which differ from the spinors in Eq. (2) by a phase

$$\chi_\pm = e^{\pm i\varphi/2}\eta_\pm. \quad (12)$$

The elko bispinors are eigenstates of the charge conjugation operator $\mathcal{C} \equiv \gamma^2\mathcal{K}$, which is their defining property, with $\mathcal{K}$ representing complex conjugation to the right

$$\mathcal{C}\lambda^{(1,4)}(\mathbf{p}) = +\,\lambda^{(1,4)}(\mathbf{p}),$$
$$\mathcal{C}\lambda^{(2,3)}(\mathbf{p}) = -\,\lambda^{(2,3)}(\mathbf{p}). \quad (13)$$

Now, a straightforward calculation shows that the elko bispinors are solutions to the massless Dirac equation

$$\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,\lambda^{(s)}(\mathbf{p}) = +\,\lambda^{(s)}(\mathbf{p}),$$
$$\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,\lambda^{(s+2)}(\mathbf{p}) = -\,\lambda^{(s+2)}(\mathbf{p}). \quad s = 1,2, \quad (14)$$

Hence, the correct field operator, expanded in terms of these spinors, would necessarily be that of a massless Dirac field, satisfying the massless Dirac equation, as Weinberg has shown in a seminal paper [12]. Furthermore, the associated propagator, either for the spinors in a Relativistic Quantum Mechanics framework or for the field operator in Quantum field Theory, would have to be the masless Dirac propagator. Therefore, a massive spin $1/2$ field operator in terms of elko spinors, as defined in reference Ref. 9, with mass dimension one and that does not obey the massive Dirac equation is just unphysical.

Having proved that Elko spinors obey the massless Dirac equation, we now have that both Elko and Weyl bispinors satisfy an eigenvalue equation with the same Hamiltonian and $\pm$ eigenvalues, as shown in Eqs. (6) and (14). Let us schematically write them as $\mathcal{H}u(\mathbf{p}) = \pm u(\mathbf{p})$ and $\mathcal{H}\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})$, with $\mathcal{H} = \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ the massless Dirac Hamiltonian. Then, there must be a unitary transformation $\Omega(\theta,\varphi)$ with the properties $\Omega\lambda(\mathbf{p}) = u(\mathbf{p})$ and $\Omega\mathcal{H} - \mathcal{H}\Omega = 0$, such that $\Omega\mathcal{H}\Omega^{-1}\Omega\lambda(\mathbf{p}) = \pm\Omega\lambda(\mathbf{p})$ implies $\mathcal{H}u(\mathbf{p}) = \pm u(\mathbf{p})$

To this end let us consider the rotation

$$\Lambda_2(\theta,\varphi) = \sin\left(\frac{\varphi}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)\gamma^1 + \sin\left(\frac{\theta}{2}\right)\gamma^3\right]\gamma^2$$
$$+\cos\left(\frac{\varphi}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)\mathbb{1} - \sin\left(\frac{\theta}{2}\right)\gamma^1\gamma^3\right], \quad (15)$$

which transforms the elko bispinors from the canonical frame to the general momentum bispinors in Eq. (10), that is

$$\Lambda_2(\theta,\varphi)\,\lambda^{(i)}(p_z) = \lambda^{(i)}(\mathbf{p}), \qquad i = 1,\ldots,4, \quad (16)$$

where the $\lambda^{(i)}(p_z)$ correspond to the $\lambda^{(i)}(\mathbf{p})$ with $\theta = \varphi = 0$. The canonical frame elko bispinors can be rotated to the corresponding Weyl bispinors, by means of the rotation

$$U = \frac{i}{2\sqrt{2}}\Big(-i\left(\mathbb{1}+\gamma^5\right) + \gamma^1\left(\mathbb{1}+i\gamma^2\right)$$
$$+\gamma^2 - \left(\mathbb{1} - i\left(\gamma^1+\gamma^2\right)\right)\gamma^0\gamma^3\Big), \quad (17)$$

yielding

$$U\lambda^{(1)}(p_z) = u^{(1)}(p_z),$$
$$U\lambda^{(2)}(p_z) = u^{(2)}(p_z),$$
$$U\lambda^{(3)}(p_z) = u^{(3)}(p_z),$$
$$U\lambda^{(4)}(p_z) = u^{(4)}(p_z). \quad (18)$$

Defining the rotation

$$\Omega(\theta,\varphi) \equiv \Lambda_1(\theta,\varphi)\,U\Lambda_2^\dagger(\theta,\varphi), \quad (19)$$

we obtain from Eqs. (9), (16) and (18), a relation between general momentum elko and massless Weyl bispinors

$$\Omega(\theta,\varphi)\,\lambda^{(1)}(\boldsymbol{p}) = u^{(1)}(\boldsymbol{p}),$$
$$\Omega(\theta,\varphi)\,\lambda^{(2)}(\boldsymbol{p}) = u^{(2)}(\boldsymbol{p}),$$
$$\Omega(\theta,\varphi)\,\lambda^{(3)}(\boldsymbol{p}) = u^{(4)}(\boldsymbol{p}),$$
$$\Omega(\theta,\varphi)\,\lambda^{(4)}(\boldsymbol{p}) = u^{(3)}(\boldsymbol{p}). \quad (20)$$

It can also be shown that, as required,

$$\left[\Omega(\theta,\varphi), \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\right] = 0. \quad (21)$$

Therefore, elko spinors can be obtained from Weyl bispinors by a rotation and vice versa, which is another way to show that the former do not constitute a new type of spinors, but are in fact equivalent to massless Weyl bispinors, obeying the massless Dirac equation.

1. D. V. Ahluwalia-Khalilova and D. Grumiller, *Phys. Rev. D* **72** (2005) 067701.
2. D. V. Ahluwalia-Khalilova and D. Grumiller, *Journal of Cosmology and Astroparticle Physics 2005* (2007) 012.
3. C.-Y. Lee, *Physics Letters B* **760** (2016) 164.
4. D. V. Ahluwalia, *Advances in Applied Clifford Algebras* **27** (2017) 2247.
5. C. H. C. Villalobos, R. J. B. Rogerio, and F. T. S. Sabbas, *The European Physical Journal C* **79** (2019) 308.
6. J. A. Nieto, *Modern Physics Letters A* **34** (2019) 1950211.
7. G. P. de Brito, J. M. Hoff da Silva, and V. Nikoofard, *The European Physical Journal Special Topics* **229** (2020) 2023.
8. C.-Y. Lee, *The European Physical Journal Special Topics* **229** (2020) 2003.
9. D. V. Ahluwalia, J. M. H. da Silva, C.-Y. Lee, Y.-X. Liu, S. H. Pereira, and M. M. Sorkhi, *Physics Reports* **967** (2022) 1.
10. S. Pokorski, Gauge Field Theories, Vol. Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2000).
11. R. Romero, *Rev. Mex. Fís. E* **17** (2020) 115.
12. S. Weinberg, *Phys. Rev.* **134** (1964) B882.