An Einstein-Maxwell interior solution obeying the Karmarkar condition

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For the Einstein-Maxwell equation system, with perfect fluid in a static and spherically symmetrical spacetime, we report an analytical internal solution which is obtained by imposing the Karmarkar condition, the behaviour of the solution is such that the density and pressures are monotonically decreasing functions while the electric field function is a monotonically increasing function that is adequate to represent compact objects. In particular we have these characteristics for the observational values of mass (1.29 ± 0.05) M$_\odot$ and radius (8.831 ± 0.09) km of the star SMC X-4. We will analyze the two extremes the one of minimum compactness $u_{\min} = 0.20523$ ($M = 1.24$ M$_\odot$, $R = 8.921$ km) and the one of maximum compactness $u_{\max} = 0.22635$ ($M = 1.34$ M$_\odot$, $R = 8.741$ km), resulting that the electric charge $Q_{u_{\min}} \in [1.5279, 1.8498]10^{20}$C and $Q_{u_{\max}} \in [1.6899, 1.9986]10^{20}$C respectively, implying that the case with higher compactness has a higher electric charge. Also in a graphic manner, it is shown that the causality condition is satisfied and that the solution is stable against infinitesimal radial adiabatic perturbation and also in regards to the Harrison-Novikov-Zeldovich criteria.

Keywords: Einstein - Maxwell; stars solutions; karmarkar condition.

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1. Introduction

In obtaining analytical interior solutions for a static and spherically symmetrical spacetime that represent compact objects like neutron stars and quark stars, we can utilize different mechanisms depending on the type of source of matter that is considered for the interior region. Solutions with perfect fluid tend to be more complicated to obtain than charged solutions or anisotropic solutions, *i.e.*, fluids that present differences between the radial and tangential pressures, since the number of restrictions that can be imposed is lower for the case of a perfect fluid. Even the solutions with perfect fluid [1-10] can be used as seed solutions to obtain generalizations to the charged [11-17] or anisotropic [18-24] cases. There is also a method to obtain the solution of a perfect fluid from a seed solution of perfect fluid, this mechanism utilizes the existence of a second order differential equation that relates the metric coefficients g_{tt} and g_{rr}, although this one can only generate an exact new solution, return to an exact solution that was already known or it can even be that the resulting integral equation does not admit a primitive function [25,26].

In the case of the construction of exact solutions with sources of matter from a fluid with anisotropic or charged pressures there exists a method that starts from a geometric property, related with embedding of the spacetime manifold in a 5-dimensional pseudo-Euclidean space E_5 called class 1 solutions, which originates a connection between the metric coefficients g_{tt} and g_{rr} through a differential equation known as the Karmarkar condition, which has allowed to obtain useful solutions to represent the interior of the stars [27-38]. A n dimensional manifold is said to be of class p if it is embedded in a Pseudo Euclidean spacetime of $n + p$ dimensions and a n-dimensional spacetime can always be embedded in $n(n + 1)/2$-dimensional Pseudo Euclidean space and the minimum dimension required to embed a manifold has to be lower or equal to $n(n - 1)/2$ [39], as such, the metric of the static and spherically symmetric spacetime is of class II al though the transformation of coordinates that generates the embed is not unique [40,41]. The space-time of Kerr is of class 5, the exterior Schwarzschild solution is of class 2, the Friedman-Lemaitre-Robertson-Walker space-time [42], the Scharzschild interior solution and the Kohler-Chao solution are of class 1 [43], these last two are the only solutions with perfect fluid that satisfy the Karmarkar condition.

The Karmarkar condition has also been employed to model the dissipative gravitational collapse with spherical symmetry [44], in the construction of solutions that represent gravastars [45], in obtaining wormhole solutions [46,47], in the study of charged wormholes and flat rotation curves in spiral galaxies [48]. Class 1 stellar models have also been analyzed in modified gravity theory [49-53].

On the other hand, considering charged stellar models allows for the representation of objects with a greater compactness compared to the case of chargeless perfect fluid [54,55], it favors the equilibrium of the configuration and its repulsive effect counteracts the gravitational attraction [56]. One of the forms in which we can have stellar objects with a net charge that is non zero is due to the accretion process. There are a variety of works in which they analyze the effect of the electric field's presence and construct stellar models that show aspects which contrast with the chargeless case [55,57-68], in one of these works there is the construction of a model with a compactness rate greater than the Buchdahl limit $u = GM/c^2R < 4/9$ present in the case of a perfect fluid [69].

Taking into account the necessity of including in the stellar models the effect of the charge and that the solutions which satisfy the Karmarkar condition result from it being a good tool in the construction of interior solutions, in this report we present an analytical charged perfect fluid solution of the Einstein-Maxwell field equations considering a spherically symmetric and static space-time. To satisfy the Karmarkar condition we chose the form of the metric potential $g_{rr} = [1 + Car^2 \arctan^2(d + ar^2)]^{-1}$, this was previously used to model the star EXO 1785-248 by means of a fluid with anisotropic pressures [70], in the current case the analysis of the solution is done by taking the observational data of mass $M =$ and radius $R =$ km of the star SMC X-4. The work we are presenting is organized as follows: in the Sec. 2 we show the field equations for a charged perfect fluid and we describe the origin of the Karmarkar condition. In the Sec. 3 we mention the required conditions for a solution to be physically acceptable and we obtain said solution, determining the integration constants that originate from imposing physical conditions. The Sec. 4 is dedicated to the graphic analysis of the solution, taking as observational data those corresponding to the star SMC X-4, showing also the stability of the solution and the analysis of the hydrostatic equilibrium. The Sec. 5 is dedicated to the discussion of the results and conclusions.

2. Solution with the Karmarkar condition

Our model considers that the equations which describe the interior behaviour of compact objects are given by the Einstein-Maxwell equations with a perfect fluid and that their geometry satisfies the Karmarkar condition, that is to say the 4 -dim spacetime can be embedded in a flat 5 -dim spacetime. To clarify each part, in this section we give the field equations, we discuss the Karmarkar condition and we obtain the solution of the equations.

2.1. The Einstein-Maxwell field equations

Einstein's equations are given by $G_{\alpha\beta} = kT_{\alpha\beta}$ with $G_{\alpha\beta}$ the components of the Einstein tensor and $T_{\alpha\beta}$ are the components of the stress energy tensor

$$T_{\alpha\beta} = (c^2\rho + P)u_\alpha u_\beta + Pg_{\alpha\beta} + \frac{1}{4\pi}(F_{\alpha\mu}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}), \tag{1}$$

with ρ and P being the energy density and pressure respectively, u^α represents the velocity of the fluid. The tensor $F_{\alpha\beta}$, represents the skew-symmetric electromagnetic tensor defined by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$

which satisfies the covariant Maxwell equations

$$\nabla_{[\alpha F\beta\gamma]} = 0, \quad \text{and} \quad \nabla_\beta F^{\beta\alpha} = 4\pi J^\alpha, \tag{2}$$

where J_α is the four electric current. Since out interest is centered in the presence of an electric field in a static and spherically symmetric spacetime, we chose the line element expressed in canonical coordinates $x^\alpha \equiv (t, r, \theta, \phi)$ as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2, \tag{3}$$

then the vectorial potential $A^\alpha = \Phi(r)\delta^\alpha_t$, while the four electric current is expressed by $J^\alpha = e^{-\nu/2}\sigma\delta^\alpha_t$, where σ is representing the charge density. As such the only non zero components are $F_{tr}(r) = -F_{rt}(r)$ and from the Eq. (2) and by means of the relativistic Gauss's law we define the charge $q(r)$ [59,71]

$$q(r) = 4\pi\int_0^r \sigma(x)x^2 e^{\lambda(x)/2}dx = E(r)r^2, \tag{4}$$

where $E(r)$ is the electric field. With these elements the non zero components of Einstein's equations, with stress energy tensor (1), generate the equation system:

$$kc^2\rho + E^2 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{5}$$

$$kP - E^2 = e^{-\lambda}\left(\frac{\nu'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{6}$$

$$kP + E^2 = \frac{1}{4}e^{-\lambda}\left[2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right]. \tag{7}$$

The conservation of the energy momentum tensor $\nabla_\alpha T^\alpha{}_\beta = 0$ implies

$$-P' - \frac{\nu'}{2}(c^2\rho + P) + 2\frac{E}{r^2}[r^2E]' = 0, \tag{8}$$

and although it can be obtained from the Eqs. (5)-(7), its relevance is that it allows us to express the equilibrium between the forces that are present:

$$F_h = -P', \;\; F_g = -\frac{\nu'}{2}(c^2\rho + P), \;\; F_e = 2\frac{E}{r^2}[r^2E]', \tag{9}$$

the first term represents the force of the pressure gradient, the second term is the gravitational force and the last term is the electric force, this expression will be employed in the Sec. 4 to show the repulsive effect of the electric force.

2.2. The karmarkar condition

The Karmarkar equation is the result of the integrability conditions which guarantee that if $({}^n g_{\alpha\beta}, \Omega_{\alpha\beta})$ satisfies the equations [72-74]

$$R_{\alpha\beta\gamma\delta} = \epsilon(\Omega_{\alpha\gamma}\Omega_{\beta\delta} - \Omega_{\alpha\delta}\Omega_{\beta\gamma}) + K_0(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \tag{10}$$

$$\nabla_\alpha\Omega_{\beta\gamma} - \nabla_\beta\Omega_{\alpha\gamma} = 0, \tag{11}$$

then, the manifold $M,{}^n g)$ of dimension n can be immersed in a manifold $(N,{}^{n+1} g)$ of dimension $n+1$ of curvature K_0. Where $\epsilon = \pm 1$ whenever the normal to the manifold is spacelike (+1) or time-like (-1). For the deduction of the Karmarkar condition we will solve the Eqs. (10) and (11) in the particular case of the spacetime described by the metric (3), $n = 4$ for the embedding in a flat 5 dimension space, that is to say $K_0 = 0$. The non zero components of the Riemann tensor for (3) are:

$$R_{trtr} = -\frac{e^\nu}{4}[2\nu'' + \nu'^2 - \lambda'\nu'],$$

$$R_{t\theta t\theta} = R\sin^2\theta = -\frac{r}{2}\nu' e^{\nu-\lambda},$$

$$R_{\theta\phi\theta\phi} = e^{-\lambda}r^2(1 - e^{-\lambda})\sin^2\theta,$$

$$R_{r\phi r\phi} = R_{r\theta r\theta}\sin^2\theta = \frac{r}{2}\lambda'.$$

Replacing these components of $R_{\alpha\beta\gamma\delta}$ in the Eq. (10), and then from algebra, we obtain the equation:

$$R_{trtr}R_{\theta\phi\theta\phi} - R_{t\theta t\theta}R_{r\phi r\phi} = 0, \tag{12}$$

with the restriction $R_{\theta\phi\theta\phi} \neq 0$ [74,75], from this equation we obtain what is called the Karmarkar condition for the metric (3):

$$2(1 - e^\lambda)\nu'' + (1 - e^\lambda)\nu'^2 + \lambda'\nu' e^\lambda = 0, \tag{13}$$

This differential equation can be integrated for λ or ν, giving as a result [28,29]:

$$e^\lambda = 1 + C\nu'^2 e^\nu, \tag{14}$$

$$e^\nu = \left[C_1 + C_2\int\sqrt{(e^\lambda - 1)}dr\right]^2, \tag{15}$$

(C, C_1, C_2) are integration constants. The Eq. (15) will be our starting point to solve the equation system.

3. The physical conditions and the solution

How it can be observed from the equation (14) or (15) obtaining a solution requires the adequate assignation of one of the functions λ or ν. Which will immediately generate a solution to Einstein's equations (5)-(7), however, this does not mean that the functions (μ, ν, ρ, P, E) are physically acceptable, it is required that they satisfy special properties. a) the functions ρ, P must be positive, regular and monotonically decreasing, the only value in which the pressure is zero is on the surface of the star $r = R$; b) the causality conditions must not be violated; c) the magnitude of the electric field $E(r)^2$ must be zero on the center and must be given by a regular and positive function; d) the geometry must be regular and the metric functions $g_{tt} = -e^{\nu(r)} < 0$ and $g_{rr} = e^{\lambda(r)} > 0$ with $g_{rr}(0) = 1$ e) The Israel-Darmois conditions must be satisfied [76], this is, the metric functions and their second fundamental form of the interior (i) and exterior (e) metric, on the surface, must be continuous

$$g_{tt}^{(i)}\Big|_{r=R} = g_{tt}^{(e)}\Big|_{r=R} \quad \text{and} \quad g_{rr}^{(i)}\Big|_{r=R} = g_{rr}^{(e)}\Big|_{r=R}, \tag{16}$$

$$\frac{\partial}{\partial r}g_{tt}^{(i)}\Big|_{r=R} = \frac{\partial}{\partial r}g_{tt}^{(e)}\Big|_{r=R}, \tag{17}$$

where the exterior geometry corresponds to the Reissner - Nordström solution:

$$ds^2 = -\left[1 - \frac{2GM}{c^2r} + \frac{Q^2}{r^2}\right]dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2. \tag{18}$$

f) The energy conditions must be satisfied; g) the solution must be stable against disturbances, in our case we will verify the adiabatic index condition and the Zeldovich condition. Although it seems like a simple task the integration of the equations starting from the Karmarkar condition, not every solution satisfies the conditions previously mentioned. In the case of chargeless perfect fluid most of the solutions are not physically acceptable [77], some are singular, don't satisfy the integrability condition, the density or pressure functions are not monotonically decreasing or both are not positive [78], which exemplifies the difficulty in obtaining solutions that are physically acceptable. In our case we chose:

$$e^{\lambda} = 1 + Car^2 \arctan^2(d + ar^2), \tag{19}$$

and replacing in the Eq. (15) we obtain the function:

$$e^{\nu} = (C_1 + C_2[2\{d + ar^2\} \arctan\{d + ar^2\} - \ln\{1 + (d + ar^2)^2\}])^2. \tag{20}$$

Once we have determined the form of the metric functions we replace these in the Eqs. (5)-(7). The electric field function E^2 we obtain from subtracting to the Eq. (7) the Eq. (6). And adding the Eqs. (6) and (7) we determine the pressure, finally replacing the form of the electric field in the Eq. (5) we find the density, each one of these functions ends up expressed as:

$$c^2k\rho(r) = \frac{(5CS(r) - 4C_2e^{-\nu/2})(H(r)S(r) + 2ar^2)a}{2H(r)e^{2\lambda}} + \frac{1}{2}aS(r)(CS(r) + 4C_2e^{-\nu/2})e^{2\lambda}, \tag{21}$$

$$kP(r) = \frac{(4C_2e^{-\nu/2} - CS(r))(H(r)S(r) + 2ar^2)a}{2H(r)e^{2\lambda}} - \frac{1}{2}aS(r)(CS(r) - 12C_2e^{-\nu/2})e^{-\lambda}, \tag{22}$$

$$E(r)^2 = \frac{(2 - CH(r)S(r)^3)(4C_2e^{-\nu/2} - CS(r))a^2r^2}{2H(r)e^{2\lambda}}, \tag{23}$$

where $S(r) = \arctan(d + ar^2)$ and $H(r) = 1 + (d + ar2)^2$ Other quantities that are relevant in the analysis of the interior solutions are the speed of sound and the adiabatic index defined by:

$$v^2 = \frac{\partial P(\rho)}{\partial \rho},$$

$$\gamma \frac{P + c^2\rho}{c^2P} \frac{\partial P(\rho)}{\partial \rho},$$

however, given it's typographic extension this is not reported but it will be analyzed in the following section. Imposing, on the surface of the star $r = R$, the continuity conditions of the metric (Eq. (16)) and the continuity of the second fundamental form (Eq. (17) or equivalently that the pressure is zero on the surface) and that $E(R)^2 = Q^2/R^4$ we obtain expressions for the constants C, C_1 and C_2 given by

$$C = \frac{2u(2H(R)S(R) + aR^2)}{(2 - 3u)H(R)S(R)^3aR^2},$$

$$C_1 = \sqrt{1 - 2u + q^2} - (2[d + aR^2]S(R) - \ln H(R))C_2,$$

$$C_2 = \frac{([2 - u]H(R)S(R) + 2[1 - u]aR^2)u}{4S(R)\sqrt{(2 - 3u)H(R)S(R)}NaR^2},$$

$$q^2 = 2(H(R)S(R)u - [1 - 2u]aR^2)N^{-2}u,$$

where $N = \sqrt{(2 + u)H(R)S(R) + 2uaR^2}$, $q = Q/R$ and $u = GM/c^2R$. Once we have obtained the integration constants and determined the functions we observe that the solution depends of the dimensionless parameters u (compactness) and of the parameters ($d, s = aR^2$).

4. Graphic analysis for the star SMC X-4

It results easier to visualize, through a graphic analysis, the behaviour of the constructed solution. And we will do this specially for the case of the star SMC X-4, although it can also be developed for another star with a compactness rate $u < 0.3224$. In the graphic analysis it's convenient to define the dimensionless functions of the speed of sound v^2/c^2, density $kc^2R^2\rho$, pressure kR^2P and electric field R^2E^2.

For the case of the star SMC X-4 which has values of mass (1.29 ± 0.05) $M_\odot$ and radius (8.831 ± 0.09) km we will analyze the two extremes, the one of minimum compactness $u_{\min} = 0.20523$ ($M = 1.24$ $M_\odot$, $R = 8.921$ km) and the one of maximum compactness $u_{\max} = 0.22635$ ($M = 1.34$ $M_\odot$, $R = 8.741$ km). Once we set the compactness parameter u we chose the adequate value for the parameter d and we determine the value of $s = aR^2$ imposing the conditions that must be satisfied for the solution to be physically acceptable. For the star SMC X-4 it results convenient to set the parameter $d = 1/2$, and we observe that the interval of the parameter $s = aR^2$ is determined by a) the causality condition, that is to say, that the speed of sound is a positive function that is equal or lower than the speed of light in the vacuum and b) that the function of the speed of sound is a monotonically decreasing function. If these conditions are met, then the rest of the conditions are satisfied. For $u_{\min} = 0.20523$ we have that $s[0.03427, 0.10800]$ and this implies that the interval for $q \in [0.1476, 0.1786]$, meanwhile that for $u_{\max} = 0.22635$ we have $s \in [0.03863, 0.12019]$ or $q \in [0.1665, 0.1969]$.

In the graphic representation we will employ the tag of q which has a more direct association with the dimensionless measure of the charge-radius rate. And the graphs in purple color correspond to the case of compactness $u = 0.22635$

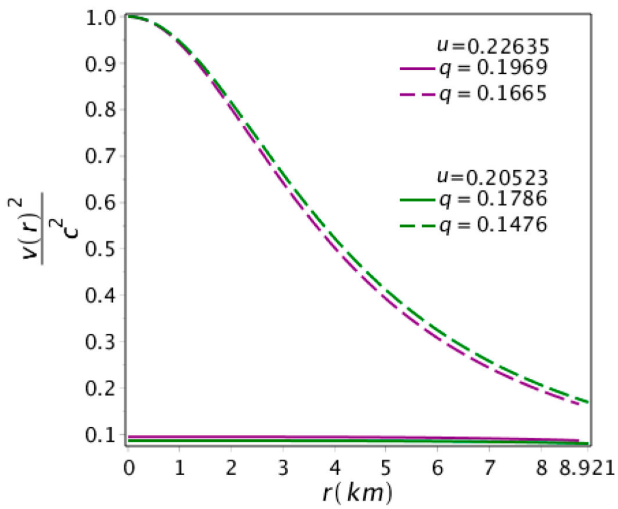

FIGURE 1. Behaviour of the speed of sound for the two values of maximum and minimum compactness considering the extreme values of q.

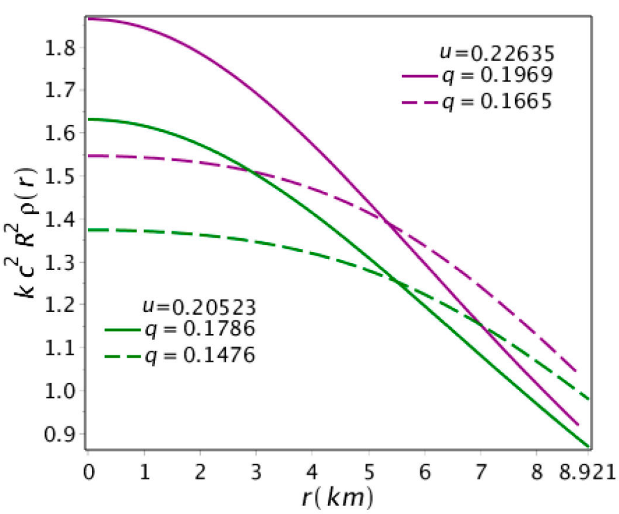

FIGURE 2. Behaviour of the density for the compactness data $(u_{\rm max}, u_{\rm min})$ corresponding to the star SMC X-4.

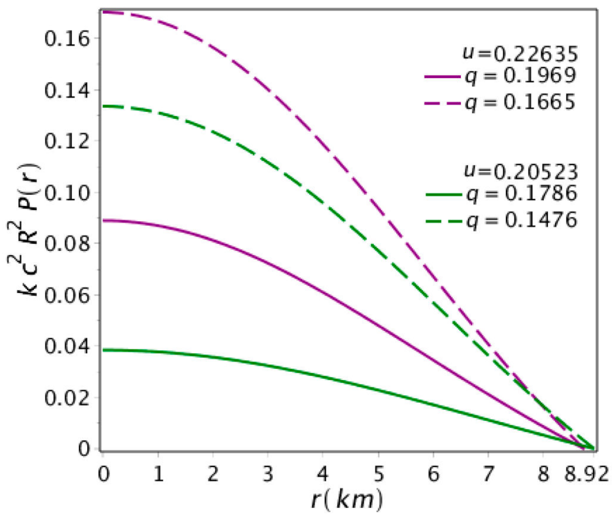

FIGURE 3. Behaviour of the pressure in which we can appreciate that the pressure on the surface is zero.

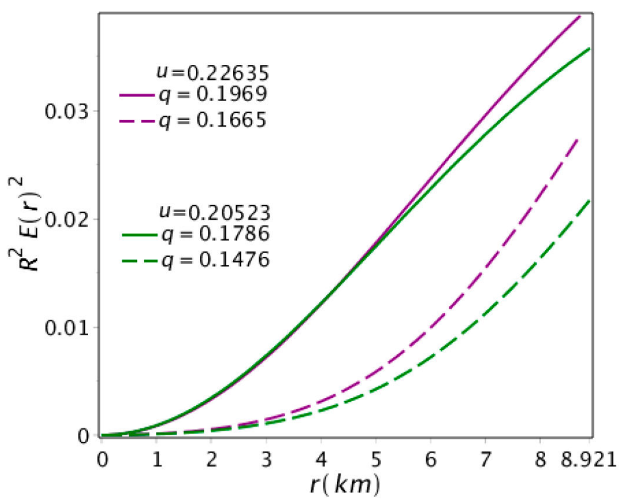

FIGURE 4. Behaviour of the electric field magnitude.

and the ones in green to the case of compactness $u = 0.20523$, meanwhile the solid line will be associated to the value of greater charge-radius rate q and the dotted line (dash) represents the value of the lower charge-radius rate. In the graph we can appreciate the value of the radius for the star with greater radius $R = 8.921$ km (green colored lines) and it can be noticed that the graphs of the star with lower radius $R = 8.741$ km end slightly before (purple colored lines).

In the Fig. 4 it is shown the positive and monotonically decreasing behaviour of the speed of sound function. The maximum value of the speed of sound, equal to the value of the speed of light in the vacuum, we have that for the value of lower rate $q = q_{\rm min}$. If we take values of $q < q_{\rm min}$ we have that $v^2 > c^2$ which would imply the violation of the causality, on the other hand for $q > q_{\rm max}$ the function of the speed of sound stops being a monotonically decreasing function.

In the Fig. 2 we graph the density for the compactness values, from this we can observe that it has a monotonically decreasing behaviour. The central density is greater for the case in which we have a greater compactness value and rate q and it exists the possibility that the density is greater for the case of lower compactness if the rate q is taken as the maximum (green solid line and purple dotted line), this as a result of the presence of a greater charge in the star.

The pressure is graphed in the Fig. 3, in it we observe that the pressure is greater for a greater compactness rate, and that for a fixed value of the compactness (same color lines) in the presence of a greater charge-radius rate q the pressure is lower as a result of the presence of the charge.

From the magnitude of the electric field $E(r)^2$, represented in the Fig. 4, as it was to be expected, we observe that for a greater value of the rate q the magnitude $E(r)^2$ is greater.

From the set of graphs presented in the Figs. 4 we have that the model is consistent with what is expected for this type of star, however it is necessary to review some other properties like the energy conditions and the stability of the solution, which will be done in the following.

4.1. Energy conditions

The energy conditions that are imposed on the interior solutions are

1. Null energy condition (NEC): $c^2\rho + P \geq 0$, $kc^2\rho + kP + 2E^2 \geq 0$.
2. Weak energy condition (WEC): $c^2\rho + P \geq 0$, $kc^2\rho + E^2 \geq 0$ and $kc^2\rho + kP + 2E^2 \geq 0$.
3. Strong energy condition (SEC): $c^2\rho + Pr \geq 0$, $kc^2\rho + 3kP + 2E^2 \geq 0$, $kc^2\rho + kP + 2E^2 \geq 0$.
4. Dominant energy condition (DEC): $c^2\rho + P \geq 0$, $kc^2\rho + E^2 \geq 0$, $c^2\rho - P \geq 0$, $kc^2\rho - kP + 2E^2 \geq 0$, $kc^2\rho + kP + 2E^2 \geq 0$.
5. Trace energy condition (TEC): $c^2\rho - 3P \geq 0$.

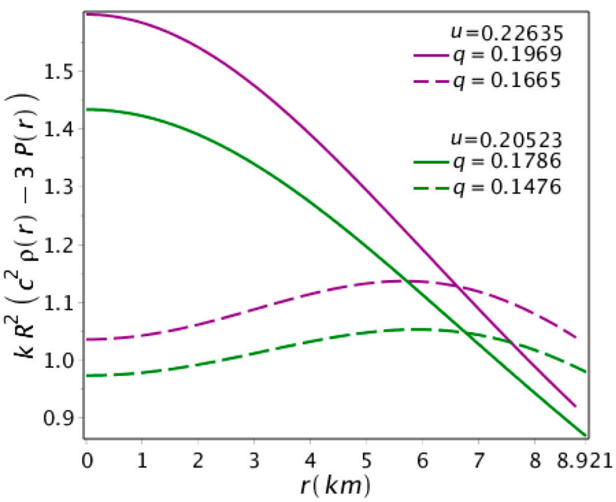

FIGURE 5. Behaviour of the trace energy condition for the model.

In the graphic analysis that was done in the previous subsection it was shown that the density, pressure and electric field $(E(r)^2)$ are non negative functions, as such, the first three energy conditions (NEC, WEC, SEC) are met. To verify the other two conditions, in the Fig. 5 we present the graph of $c^2\rho - 3P$ and from it we obtain that the TEC is satisfied and as a consequence the DEC is also satisfied (for this we will observe that if $c^2\rho - 3P > 0$ then $c^2\rho - P > 0$ and since $E^2 \geq 0$ then $kc^2\rho - kP + 2E^2 > 0$), so all the energy conditions are met.

4.2. Hydrostatic equilibrium and stability

In the case of a chargeless perfect fluid, the equation of the hydrostatic equilibrium is given by the TOV equation [81,82], the difference with the charged case is the existence of a term associated with the electric force (repulsive) as a result of the presence of the charge. So in the charged case the hydrostatic equilibrium is given by the attractive effect of the gravitational force F_g, the repulsive hydrostatic force due to the pressure gradient F_h and the electric force F_e resulting of the presence of the charge, each one of these terms have already been mentioned in the Eqs. (8) and (9). The graphic behaviour of these forces is shown in the Fig. 6, in it we verify the attractive behaviour of the gravitational force and re-

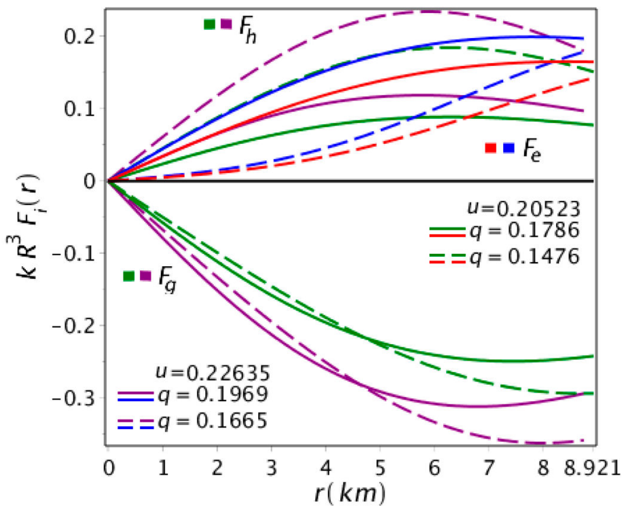

FIGURE 6. The gravitational force corresponds to the lines below the horizontal axis (attractive force). The hydrostatic force is represented by the lines that are found above the horizontal axis in green and purple (repulsive force) colors. The electric force is represented by the lines above the horizontal axis in red and blue (repulsive force) colors.

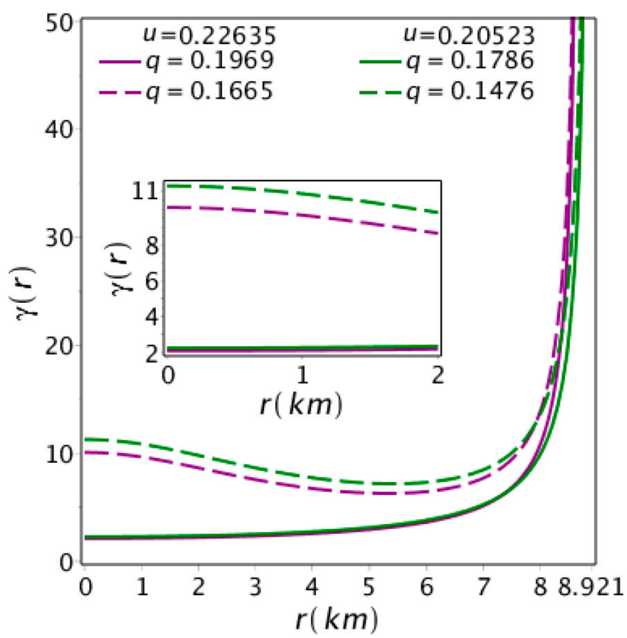

FIGURE 7. Behaviour of the adiabatic index for the solution. In the graph it's shown that the smallest value is greater than 2, which guarantees the stability.

pulsive of the electrical and hydrostatic force. We also observe that for a greater compactness rate (Purple color lines) we have a greater gravitational force and that as the rate q increases the electric force has a greater contribution in the repulsive effect.

In the determination of the stability of the solution we begin by showing that the solution is stable according to the Harrison-Zeldovich-Novikov criteria [79,80] which indicates to us that a configuration of a fluid is stable if the mass is an increasing function in relation to the central density $\rho(0) = \rho c$, that is to say, if $\partial M(\rho_c)/\partial\rho_c > 0$. From the Eq. (21) evaluating in $r = 0$ and replacing the constants C, C_2 we obtain

$$M(\rho c) = \frac{2S(R)^3 H(R) R_c^2 \rho c}{3G}\{H(R)S(R)^3\rho c + 2[aR^2 + 2S(R)H(R)]\arctan^2 d\}^{-1}, \tag{24}$$

and deriving this function in regards to the ρc results

$$\frac{\partial M(\rho c)}{\partial \rho c} = \frac{3(2H(R)S(R) + aR^2)GM(\rho c)^2 \arctan^2 d}{H(R)S(R)^3 Rc^2 \rho c^2}, \tag{25}$$

this function is positive since for the model $d = 1/2$ and $aR^2 > 0$ as such $H(R)$, $S(R) > 0$. Which shows that the solution is stable according to the Harrison-Zeldovich-Novikov criteria. Another criteria that gives us information in relation to the stability is the one of the adiabatic index $\gamma = (c^2\rho + P)v_r^2/(c^2 P)$, that for the case of a perfect fluid, guarantees that the solution is stable against infinitesimal radial adiabatic perturbation if $\gamma > 4/3$. In the Fig. 7 we present the graph of the adiabatic index for the cases considered of compactness for the values of the rate q, from it we observe that the solution satisfies the

5. Results and conclusions

In previous sections, we have obtained an analytical solution which is physically acceptable and applied the model to describe the behaviour of a compact object's interior taking as observational data of mass and radius those corresponding to the star SMC X-4, it was shown that the solution satisfies the criteria of stability in relation to the adiabatic index and the Harrison-Novikov-Zeldovich index. And by means of graphic representation in each one of the terms of the generalized TOV equation for the charged case was shown that, as it was expected, the presence of the charge has a repulsive effect. In a complementary manner in the Tables I and II we report the physical values of the densities in the center ρc and on the surface ρ_b, the central pressure P_c, the values of the speed of sound in the center v_c^2 and on the surface v_b^2 as well as the net charge. From these tables we have that the orders of magnitude of each one of these hydrostatic functions are consistent with what is to be expected for these type of stars.

TABLE I. Interior behavior of the physical values for the star SMC X-4, with minimal compactness.

Theoretical results for the star SMC X-4		
$M = 1.24$ M	$R = 8.921$ km	$u = 0.20523$
q	0.1476	0.1786
ρ_c 10^{18} kg/m^3	0.9256	1.0990
ρ_b 1017 kg/m^3	6.5967	5.8481
P_c 10^{33} Pa	8.0842	3.9956
v_c^2 (c2)	1.0000	0.0879
v_b^2 (c2)	0.1707	0.0821
Q 10^{20} C	1.5279	1.8498

TABLE II. Interior behavior of the physical values for the star SMC X-4, with maximal compactness.

$M = 1.34$ M	$R = 8.741$ km	$u = 0.22635$
q	0.1665	0.1969
ρ_c 10^{18} kg/m^3	0.0851	1.3088
ρ_b 1017 kg/m^3	0.0.6448	1
P_c 10^{33} Pa	10.735	5.6165
v_c^2 (c2)	1.0000	0.0960
v_b^2 (c2)	0.1664	0.0883
Q 10^{20} C	1.6899	1.9986

Another relevant observation of this model and that can be deduced from the graphs and complemented with the values of the tables is that for a fixed value of the compactness value the central density is greater as the electric charge increases, meanwhile the pressure has the opposite behaviour, that is to say, the pressure is lower for a greater net charge. Which manifests the consistency of the model with the expected behaviour from the charge contributing to counteract the attractive gravitational force, diminishing the hydrostatic pressure and allowing for a greater density in the interior of the star.

In conclusion, we have that the model satisfies the criteria required for a solution to be physically acceptable, stable and consistent with the expected behaviour of the electric charge present in the interior. On the other hand, in regard to the Karmarkar condition which has a geometric origin, it leaves open the possibility of being able to do a posterior analysis for the same choice of the metric function grr but in which a new model will include sources of matter that are quintessence type or ordinary matter, with a state equation for any of the two cases, which would allow to clarify what is the effect of these sources on the behaviour of the hydrostatic functions. Questions that can be developed in future investigations.

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