Electromagnetic curves and Rytov's law in the optical fiber with Maxwellian evolution via alternative moving frame

S. Kaya Nurkan[a], H. Ceyhan[b], Z. Özdemir[c], and İ. Gök[b]
[a]*Department of Mathematics, Faculty of Arts and Sciences, Usak University, 64200 Usak, Turkey.*
e-mail: hazallceyhan@gmail.com
[b]*Department of Mathematics, Science Faculty, Ankara University, 06100 Ankara, Turkey.*
e-mail: zehra.ozdemir@amasya.edu.tr
[c]*Department of Mathematics, Arts and Science Faculty, Amasya University, 05189 Amasya, Turkey.*
e-mail: igok@science.ankara.edu.tr

Received 22 September 2022; accepted 18 May 2023

In this study, we research the behavior of a linearly-polarized light wave in an optical fiber and the rotation of the polarization plane through the alternative moving frame $\{N, C, W\}$ in Minkowski 3d-space. Then Berry's phase equations are discussed for electromagnetic curves in the $\{\mathrm{C}\}$ and $\{W\}$ directions along an optic fiber via alternative moving frame in Minkowski 3d-space. Moreover, electromagnetic curve's $\{\mathrm{C}\}$ and $\{W\}$ Rytov parallel transportation laws are defined. Finally, we examine the electromagnetic curve's Maxwellian evolution by Maxwell's equation.

Keywords: Applications to physics; magnetic flows; vector fields; ordinary differential equations; electromagnetic theory; Maxwell's equation.

DOI: https://doi.org/10.31349/RevMexFis.69.061301

1. Introduction

Differential geometry is one of the largest fields used by many disciplines in analysis. One of these fields of science is undoubtedly physics. Recently, the most interesting subject in the field of physics is electromagnetic theory. Electromagnetic theory is also studied by sub-branches of mathematics, for example topology, geometry, etc.

The first mathematical perspective on quantum theory was in the field of topology [1]. Then, a new perspective on quantum theory was developed and its geometrical phase was examined [2]. The Rytov's curve and Rytov's law can be defined by the rotation of the polarization plane and particle's motion along an optical fiber. The motion of a particle that enters the magnetic field and the rotation of polarization through the geometric phase are examined in Refs. [3–5]. These papers led to the research of the trajectory of polarized light along the optical fiber. The phase in which the quantum theory was examined from the geometric point of view was studied as Berry's phase [6]. And then, the relation between Berry's phase and Fermi-Walker transport was explained [7]. With the introduction of the Gauss-Landau-Hall magnetic field concept on a Riemann surface, this problem has begun to be studied in the field of geometry [8, 9]. The following article that collects and presents all this information in this field is one of the leading ones [10]. This subject, which draws attention with great momentum, is researched by making use of all areas of geometry. Important studies have been presented to the scientific world with interesting results that emerged by examining the magnetic field in different spaces. Some examples of these studies are: involving the study of the magnetic field and magnetic flow in complex space [11, 12], which allows new search to show up by moving this field to a new space with a 3D Semi-Riemannian [13], that showed magnetic flows in Riemmanian surface [14], and the other study on Sasakian manifold that researched contact magnetic flows [15]. Examining the trajectories of geometrically charged particles also enabled the study of curves, which is the most important subject of geometry. In this case, the geometric properties of the curves formed by the magnetic helices [16] and the trajectories of the charged particles are examined [17, 18]. The magnetic field has been investigated in different spaces as well as on several surfaces and diverse frames [19, 20]. Bjorgum [21] stated: "In this paper, it is suggested that a study of special vector fields, properly chosen, might prove as fruitful for application to phenomena described by vector fields as has been the study of special functions for problems expressed by scalar quantities." Using this significant paper, vector fields can be studied from many perspectives. Magnetic fields are vector fields, and this work has formed a very important basis for the mathematical investigation of Maxwell's equations. Then Marris showed vector field relations expressing dynamical, electromagnetic or other considerations [22]. Later magnetic theory and Maxwell equations were studied by several researchers [23, 24].

Recently, optical fibers have become a very important field that receive care in mathematics and geometry. Polarized light is generally thought as a transport of an electromagnetic wave. When it is assumed to propagate within the optical fiber, it is well-defined, owing to the Maxwell's equations. The set of Maxwell's equations implicitly shows how electromagnetic field vectors propagate and explicitly show the sources of the fields. In the optical fiber configuration of uniform, isotropic, nonconducting, free-from charge, non-dispersive, etc., the evolution of the space curve is a very influential way to understand many physical processes such as vortex filament, dynamics of Heisenberg spin chain, integrable systems, soliton equation theory, sigma models, relativity, water wave theory, fluid dynamics, field theories, linear and nonlinear optics. Evolution systems and equations possess important geometric applications and meanings. For example, the sine-Gordon equation, which is used as a model in nonlinear optics, field theories, and dislocation of crystals. The Betchov-Da Rios equation or the filament equation, is an idealized example of the evolution of the centerline of a thin vortex tube in a 3D inviscid incompressible fluid [26,27]. This equation also constrains the evolution of curves in magnetohydrostatic and steady hydrodynamic problems of nested toroidal flux surfaces [28, 29]. The relationship between the solutions of the cubic nonlinear Schrödinger equation and the localized solutions of the induction equation was discovered by Hasimoto [30]. He described a special transformation, including complex curvature and torsion functions of the curve. At the same time, many application areas have been found in the field of physics in this case. If we refer to the studies on the above-mentioned issues, Aritra K. Mukhopadhyay *et al*. found the rogue soliton equivalent in the spin system to link about non-linear Schrödinger equation with the continuum Heisenberg spin chain [31]. Banica and Milot reviewed the vortex filaments studies done so far and showed a collection of new situations of filaments collapse [32]. M. Grbovic and E. Nesovic defined Backlund transformation of a null Cartan curvein Minkowski 3d-space as a transformation which maps a null Cartan helix to another null Cartan helix and via the Da Rios vortex filament equation derived the vortex filament equation for a null Cartan curve [33]. Amor *et al*. studied the relationship between the Burgers' equation and the pseudo-null vortex filament equation in Minkowski space [34]. Hesami *et al.*, researched the evolution of two profiles, in a positive Kerr medium and the effect of initial beam-width [35]. Authors indicated numerically that it is possible to generate bright spatial quasi-solitons, in media that are defined via the (1+1)-Dimensional local and nonlocal Nonlinear Schrödinger Equation, via initial condition field distributions that are aside from the analytical solution [36]. In Ref. [37], via Nonlinear Schrodinger Equation, the relationship between two higher order bright spatial solitons were researched by authors, where the solitons launched in a parallel way. N. Gürbüz investigated the nonlinear Schrödinger of repulsive type for timelike curves and nonlinear heat systems in a general intrinsic geometric setting including a normal congruence in 3-dimensional Minkowski space [38].

2. Maxwell evolution of alternative moving frame $\{$N,C,W$\}$

Let E_1^3 be Minkowski 3d-space given by the standard metric

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ [4].

Let γ be a curve in E^3 that has one of three casual characters depending on the tangent vector of the curve. So that this tangent vector being v, if $\langle v, v\rangle > 0$ or $v = 0$, the curve is spacelike, if $\langle v, v\rangle < 0$, the curve is called timelike, and then $\langle v, v\rangle = 0$ and $v \neq 0$, it is null.

Let the $\{\ N, C, W\ \}$ frame with the curvatures $f(s)$ and $g(s)$ along $\gamma : I \subset \mathbb{R} \to E_1^3$ is a non-null regular curve in Minkowski 3d-space. The alternative moving frame's vectors are $\{N, C, W\}$, the principal normal vector field, the derivative of principal normal vector field, and Darboux vector field, respectively. Derivating of the alternative moving frame is:

$$\begin{pmatrix} N_s \\ C_s \\ W_s \end{pmatrix} = \begin{pmatrix} 0 & f(s) & 0 \\ -\varepsilon_N\varepsilon_C f(s) & 0 & g(s) \\ 0 & -\varepsilon_C\varepsilon_W g(s) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}, \quad (1)$$

where

$$\langle N, N\rangle = \varepsilon_N \quad \langle C, C\rangle = \varepsilon_C \quad \langle W, W\rangle = \varepsilon_W\ . \quad (2)$$

For all that, the vector products of alternative moving frame's vectors are given,

$$N \times C = W, \quad C \times W = \varepsilon_N\varepsilon_W N,$$
$$N \times W = -\varepsilon_C\varepsilon_W C, \quad (3)$$

in Ref. [25]. In this study we assume that the curve is a non-null curve. Therefore, frame's vector fields consist of two spacelike and one timelike vectors on which the curve lies. Thus, we can write $\varepsilon_N\varepsilon_C\varepsilon_W = -1$.

3-dimensional vectors fields and the geometry of curvature and torsion of vector lines applications used to get these vectors as nonholonomic coordinates are shown in Minkowski space via the alternative moving frame. $\delta/\delta s, \delta/\delta c$ and $\delta/\delta w$ are the directional derivatives in the N, C and W, directions, respectively, for the alternative moving frame of a non-null curve γ in E_1^3 [22]. Here we assume that $\gamma(s, c, w)$ is a non-null curve lying in the 3-D Minkowski space.

$$\frac{\delta}{\delta c} = \varepsilon_C C grad,$$
$$\frac{\delta}{\delta w} = \varepsilon_W W grad,$$
$$\frac{\delta}{\delta s} = \varepsilon_N N grad. \quad (4)$$

Assume that a directional derivative of an arbitrary vector A with respect to directional $\eta \in$ $\{$N,C,W$\}$ and assuming the directional derivative $\delta A/\delta\eta$ as follow:

$$\frac{\delta A}{\delta \eta} = \left(N \cdot \frac{\delta A}{\delta \eta}\right) N + \left(C \cdot \frac{\delta A}{\delta \eta}\right) C + \left(W \cdot \frac{\delta A}{\delta \eta}\right) W,$$

from Ref. [39].

Thus, we can calculate the derivatives of the frame vectors in the C direction and in the W direction, respectively, and choose $\{N, C, W\}$ instead of the A vector in the above equation, and c, w, s instead of η , and write the anholonomic coordinates.

Other geometric equations in terms of anholonomic coordinates are given as:

$$\theta_{CS} = C\frac{\delta}{\delta c}N,$$

$$\theta_{WS} = W\frac{\delta}{\delta w}N, \tag{5}$$

$$\text{div}N = \varepsilon_C \theta_{CS} + \varepsilon_W \theta_{WS}, \tag{6}$$

$$divC = -\varepsilon_N f(s) + \varepsilon_W W \frac{\delta}{\delta w} C, \tag{7}$$

$$divW = \varepsilon_C C \frac{\delta}{\delta c} W, \tag{8}$$

$$\Omega_N = curlN \cdot N = -\left(-W \cdot \frac{\delta}{\delta c}N + C.\frac{\delta}{\delta w}N\right),$$

$$\Omega_C = curlC \cdot C = \varepsilon_N \left(N \cdot \frac{\delta}{\delta w}C + g(s)\right),$$

$$\Omega_W = curlW \cdot W = \varepsilon_N \left(-N \cdot \frac{\delta}{\delta c}W + g(s)\right). \tag{9}$$

With the help of these geometric properties and equations (1-9) along c-directional and w-directional the derivatives of alternative moving frame's fields are calculated as follows,

$$\frac{\delta}{\delta c}\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_C \theta_{CS} & \varepsilon_W(\varepsilon_N \Omega_W - g(s)) \\ -\varepsilon_C \theta_{CS} & 0 & \varepsilon_N divW \\ -\Omega_W + \varepsilon_N g(s) & divW & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}, \tag{10}$$

$$\frac{\delta}{\delta w}\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon_C(\varepsilon_N \Omega_C - g(s)) & \varepsilon_W \theta_{WS} \\ \varepsilon_N(\varepsilon_N \Omega_C - g(s)) & 0 & divC + \varepsilon_N f(s) \\ -\varepsilon_N \theta_{WS} & \varepsilon_N divC + f(s) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}. \tag{11}$$

The Lorentz force of a magnetic vector field V is defined by the skew symmetric operator Φ and is given by

$$\Phi(X) = V \times X,$$

When a charged-point particle enters the magnetic field under the influence of the Lorentz force, it follows a new trajectory called a magnetic trajectory. The magnetic trajectories of the magnetic vector field V satisfy the following equation

$$\Phi(\mathbf{t}) = V \times \mathbf{t} = \nabla_{\mathbf{t}} \mathbf{t},$$

in Ref. [40].

2.1. Maxwell evolution for two cases of electric field

Berry's phase in the directions c and w arises with the propagation of an electromagnetic wave along with the optic fiber for the alternative moving frame of the non-null curve γ.

Let optic fiber be defined as a curve that is a non-null curve $\gamma(s, c, w)$ via alternative moving frame in Minkowski space. The electromagnetic wave propagation is in the direction of $N = (s, c, w)$ the polarization of the electromagnetic wave is mentioned by the direction of the electric field vector $E = (s, c, w)$ and magnetic field is shown as $V = (s, c, w)$. Here basically the electric field will be shown perpendicular to N. Then, the cases where E is perpendicular to the direction of C and perpendicular to the direction of W will be examined.

Case 1. The variation of the electric field vector E between any two points in the c direction for the alternative moving frame $\{N, C, W\}$ of the non-null curve $\gamma(s, c, w)$ can be expressed as,

$$\frac{\delta}{\delta c}\vec{E}(s, c, w) = \lambda_1 N + \lambda_2 C + \lambda_3 W. \tag{12}$$

The electric field is right angle to N and if we assume that because of absorption, there is no loss mechanism in the optical fiber, we can write the following equations:

$$\langle N, E\rangle = 0 \;\; \langle E, E\rangle = c. \tag{13}$$

If we take the derivative of the first (13) and use (11), we get:

$$\left\langle \frac{\delta N}{\delta s}, E\right\rangle = -\varepsilon_N \lambda_1.$$

When the necessary calculations are made, we get the following:

$$\lambda_1 = -\varepsilon_N \varepsilon_C \theta_{CN} E^C - \varepsilon_N \varepsilon_W(\varepsilon_N \Omega_W - g(s))E^W.$$

Considering $\varepsilon_N \varepsilon_C \varepsilon_W = -1$, that $-\varepsilon_N \varepsilon_W = \varepsilon_C$ and $-\varepsilon_N \varepsilon_C = \varepsilon_W$, we can arrange:

$$\lambda_1 = \varepsilon_W \theta_{CN} E^C + \varepsilon_C(\varepsilon_N \Omega_W - g(s))E^W. \tag{14}$$

If we take the derivative of the second (13), we can get:

$$\left\langle \frac{\delta E}{\delta c}, E \right\rangle = 0.$$

bringing together (14), (13), and (11) we get

$$\overrightarrow{E}_c = (\varepsilon_W \theta_{CN} E^C + \varepsilon_C(\varepsilon_N \Omega_W - g(s))E^W)N + \lambda(E \times N), \tag{15}$$

where λ is a constant term.

With the last equation, we can find the rotation of the electric field in the c direction around $\overrightarrow{n}$. Moreover, we can suppose that $\lambda = 0$, hereby we can finalize which E is Rytov parallel transport in the c direction with the conditions given above.

$$\overrightarrow{E}_c = -\varepsilon_N(E, N_c)N. \tag{16}$$

Furthermore, the Fermi-Walker transportation law is calculated in Minkowski space as:

$$B_c^{FW} = B_c + \varepsilon_N(B.N_c)N - (B.N)N_c. \tag{17}$$

Generally, we can write:

$$E = \varepsilon_N E^C C + \varepsilon_W E^W W, \tag{18}$$

where E^C and E^W are optionally smooth components of the $\overrightarrow{c}$ and $\overrightarrow{w}$. Derivating of (18) and combining of (10) we can write:

$$\frac{\delta}{\delta c}\overrightarrow{E} = (-\varepsilon_N \varepsilon_C \theta_{CN} E^C - \varepsilon_N \varepsilon_W(\varepsilon_N \Omega_W - g(s))E^W)N + (\varepsilon_N E_c^C + \varepsilon_C \varepsilon_W divW.E^W)C + (W.E^C)W. \tag{19}$$

If the electric field is assumed to be Rytov parallel transported in c direction, then comparing (16) and (19) satisfies that;

$$\begin{pmatrix} E_c^C \\ E_c^W \end{pmatrix} = \begin{pmatrix} 0 & divW \\ -divW & 0 \end{pmatrix} \begin{pmatrix} E^C \\ E^W \end{pmatrix}. \tag{20}$$

Therefore, we can accomplish that (20) describes the rotation of the polarization plane in the c direction along the optic fiber thus Berry's phase $r = (s, c, w)$ in the c direction is defined by:

$$\frac{\delta}{\delta c} r = divW.$$

We can state the magnetic field vector V in relation to the ingredient of the electric field as;

$$V = \varepsilon_N E^W.C - \varepsilon_C E^C.W, \tag{21}$$

that satisfies the following conditions:

$$V \perp E \quad \text{and} \quad V \perp N, \tag{22}$$

where

$$V^C = E^W \quad V^W = \varepsilon_C E^C.$$

Using (22) and (10), derivating (21), we can get,

$$\frac{\delta V}{\delta c} = (\varepsilon_W \theta_{CN} E^W - \varepsilon_C E^C(-\Omega_W + \varepsilon_N g(s))N + (\varepsilon_N E_c^W - \varepsilon_C divW.E^C)C + (-\varepsilon_C E_c^C + divW.E^W)W, \tag{23}$$

which satisfies

$$\left\langle \frac{\delta V}{\delta c}, E \right\rangle + \left\langle \frac{\delta E}{\delta c}, V \right\rangle = 0,$$

and

$$\left\langle \frac{\delta V}{\delta c}, N \right\rangle + \left\langle \frac{\delta N}{\delta c}, V \right\rangle = 0.$$

When we consider all this, we can say that magnetic field and electric field have alike Berry's phase in the same conditions as follows,

$$V_c = -\varepsilon_N(V.N_c).N. \tag{24}$$

We see that E is the Rytov parallel transported along the c direction if and only if it is Fermi-Walker parallel transported in the c direction along with optic fiber via alternative moving frame of the non-null curve in Minkowski space.

The Lorentz force is the force acting on a charged particle moving in a non-null electromagnetic field in Minkowski space. At that time, the electromagnetic field in the c direction along non-null curve via alternative moving frame with respect to anholonomic coordinates help of Lorentz equation $\phi(E) = X \times E$ where X is a Killing magnetic field in Minkowski space and (10) is given as follows,

$$\langle \phi_c(E), N \rangle = -\langle \phi(N), E_C \rangle = \left\langle \frac{\delta E}{\delta c}, N \right\rangle = -\varepsilon_C \theta_{CN} E^C - \varepsilon_W(\varepsilon_N \Omega_W - g(s))E^W.$$

When necessary arrangements are made, we can write,

$$\phi_c(N) = \varepsilon_C \theta_{CN} E^C + \varepsilon_W(\varepsilon_N \Omega_W - g(s))E^W + a_1 E^N, \tag{25}$$

$$\phi_c(C) = -\lambda E^W + a_2 E^N, \tag{26}$$

$$\phi_c(W) = \lambda E^C + a_3 E^N. \tag{27}$$

Taking into account Eqs. (25-27) and (10), the Lorentz force in the C direction along with optic fiber that is determined non-null curve for the alternative moving frame implies the following matrix form:

$$\begin{pmatrix} \phi_c(N) \\ \phi_c(C) \\ \phi_c(W) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_C\theta_{CN} & \varepsilon_W(\varepsilon_N\Omega_W - g(s)) \\ -\varepsilon_N\theta_{CN} & 0 & -\lambda \\ -\varepsilon_N(\varepsilon_N\Omega_W - g(s)) & \lambda & 0 \end{pmatrix} \begin{pmatrix} E^N \\ E^C \\ E^W \end{pmatrix}. \tag{28}$$

Case 2. The variation of the electric field vector E between any two points in the w direction for the alternative moving frame $\{N, C, W\}$ of the non-null curve $\gamma(s, c, w)$ can be expressed as,

$$\frac{\delta}{\delta w}\overrightarrow{E}(s, c, w) = \lambda_1 N + \lambda_2 C + \lambda_3 W. \tag{29}$$

The electric field is perpendicular to N and if we assume that because of absorption there is no loss mechanism in the optical fiber, we can get:

$$\langle N, E\rangle = 0 \;\; \langle E, E\rangle = c. \tag{30}$$

Derivating of the first (29) and utilizing the (11), we compute

$$\left\langle \frac{\delta N}{\delta w}, E \right\rangle = -\varepsilon_N\lambda_1.$$

If abbreviations and necessary calculations are made, we can write the following:

$$\lambda_1 = -\varepsilon_N\varepsilon_W\theta_{WN}E^W - \varepsilon_N\varepsilon_C(\varepsilon_N\Omega_C - g(s))E^C.$$

Considering $\varepsilon_N\varepsilon_C\varepsilon_W = -1$, $-\varepsilon_N\varepsilon_W = \varepsilon_C$ and $-\varepsilon_N\varepsilon_C = \varepsilon_W$, we can arrange:

$$\lambda_1 = \varepsilon_C\theta_{WN}E^W + \varepsilon_W(\varepsilon_N\Omega_C - g(s))E^C. \tag{31}$$

If we take the derivative of the second (29), we can organize:

$$\left\langle \frac{\delta E}{\delta c}, E \right\rangle = 0.$$

After that we collected (29), (30), and (11) we get

$$\overrightarrow{E}_w = (\varepsilon_C\theta_{WN}E^W + \varepsilon_W(\varepsilon_N\Omega_C - g(s))E^C)N + \mu(E \times N), \tag{32}$$

where μ is a constant term.

Considering the last equation we get the rotation of E in the w direction around $\overrightarrow{n}$. Furthermore, we assume that $\mu = 0$, in this manner we can conclude that E is non-null parallel transport in the w direction with the above terms.

$$\overrightarrow{E}_w = -(E, N_w)N. \tag{33}$$

Additionally, this motion can be defined through the Fermi-Walker transportation law in Minkowski space is as follows:

$$B_w^{FW} = B_w + \varepsilon_N(B.N_c)N - (B.N)N_w. \tag{34}$$

Generally, we get;

$$E = \varepsilon_N E^C C + \varepsilon_W E^W W, \tag{35}$$

where E^C and E^W are optionally smooth components of $\overrightarrow{c}$ and $\overrightarrow{w}$. Derivating of (34) and combining of (11) we can calculate:

$$\frac{\delta}{\delta w}\overrightarrow{E} = (-\varepsilon_N\varepsilon_W\theta_{WN}E^W + (\varepsilon_N\Omega_C - g(s))E^C)N + (\varepsilon_N E_w^C + \varepsilon_W(\varepsilon_N divW + f(s))E^W)C + (\varepsilon_w E_w^W + \varepsilon_N(\varepsilon_N f(s) + divC)E^C)W. \tag{36}$$

If the electric field is presumed to be Rytov parallel transported in w direction, then comparing (32) and (35) implies that:

$$\begin{pmatrix} E_w^C \\ E_w^W \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_C(\varepsilon_N divC + f(s)) \\ \varepsilon_C(divC + \varepsilon_N f(s)) & 0 \end{pmatrix} \begin{pmatrix} E^C \\ E^W \end{pmatrix}.$$

Therefore, we can accomplish that (36) describes the rotation of the polarization plane in the w direction along the optic fiber thus a Berry's phase $r = (s, c, w)$ in the w direction is described by:

$$\frac{\delta}{\delta w}r = divC + f(s).$$

We can state the magnetic field vector in relation to the ingredient of the electric field as:

$$V = E^W.C - E^C.W, \tag{37}$$

that satisfies the following conditions:

$$V \perp E \quad \text{and} \quad V \perp N,$$

where

$$V^C = E^W \quad V^W = E^C.$$

Using (11), $V \perp E$, $V \perp N$ and derivating (37), we can get,

$$\frac{\delta V}{\delta w} = (\Omega_C - \varepsilon_N g(s)E^W + \varepsilon_N\theta_{WN}E^C)N + (E_w^W - \varepsilon_N(divC + f(s))E^C)C + (-E_w^C + (divC + \varepsilon_N f(s))E^W)W, \tag{38}$$

which satisfies

$$\left\langle \frac{\delta V}{\delta w}, E \right\rangle + \left\langle \frac{\delta E}{\delta w}, V \right\rangle = 0,$$

and

$$\left\langle \frac{\delta V}{\delta w}, N \right\rangle + \left\langle \frac{\delta N}{\delta w}, V \right\rangle = 0.$$

When we consider all this, we can say that magnetic field and electric field have alike Berry's phase in the same conditions as follows;

$$V_w = -(V.N_w).N. \tag{39}$$

E is the Rytov parallel transported in the w direction if and only if it is Fermi-Walker parallel transported in the w direction along with optic fiber via the alternative moving frame of the non-null curve in Minkowski space.

Then, the electromagnetic field in the w direction along non-null curve via alternative moving frame with respect to anholonomic coordinates help of Lorentz equation in Minkowski space and (11) is given as follows,

$$\langle \phi_w(E), N \rangle = -\langle \phi(N), E_W \rangle,$$
$$\left\langle \frac{\delta E}{\delta w}, N \right\rangle = -\varepsilon_W \theta_{WN} E^W + \varepsilon_N (\varepsilon_N \Omega_C - g(s)) E^C. \tag{40}$$

When necessary arrangements are made, we can write;

$$\phi_w(N) = \varepsilon_W \theta_{WN} E^W - \varepsilon_N (\varepsilon_N \Omega_C - g(s)) E^C + a_1 E^N, \tag{41}$$
$$\phi_w(C) = -\lambda E^W + a_2 E^N, \tag{42}$$
$$\phi_w(W) = \lambda E^C + a_3 E^N. \tag{43}$$

Taking into account Eqs. (41-43) and (11), the Lorentz force in the w direction along with optic fiber that is determined non-null curve for the alternative moving frame implies the following matrix form:

$$\begin{pmatrix} \phi_w(N) \\ \phi_w(C) \\ \phi_w(W) \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon_N(\varepsilon_N \Omega_C - g(s)) & \varepsilon_W \theta_{WN} \\ \varepsilon_C(\varepsilon_N \Omega_C - g(s)) & 0 & -\lambda \\ -\varepsilon_N \theta_{WN} & \lambda & 0 \end{pmatrix} \begin{pmatrix} E^N \\ E^C \\ E^W \end{pmatrix}. \tag{44}$$

2.2. Maxwell's equation for electromagnetic waves in Minkowski 3d-space

Maxwell's equations emerged by combining Faraday's and Gauss's law and finding a new equation. It has been a groundbreaking breakthrough in understanding electromagnetic theory. Maxwell's equations, consisting of these four equations have been an important method for studying electromagnetic field vectors. Maxwell's equations which are called Gauss's law, Magnetic monopoles, Ampere-Maxwell law, and Faraday's law are given as follows,

$$\nabla \cdot E = 0, \tag{45}$$
$$\nabla \cdot V = 0, \tag{46}$$
$$\nabla \times V = \epsilon v \frac{\delta E}{\delta u}, \tag{47}$$
$$\nabla \times E = -\frac{\delta V}{\delta u}, \tag{48}$$

where ϵ and v have the same values at all points and (s, c, w) and u space, time variables, where E is electric field and V is magnetic field. If we consider that the electric field is perpendicular to the tangent direction and (19), (35), and (45), we can calculate,

$$\nabla \cdot E = \left(N \cdot \frac{\delta}{\delta s} + C \frac{\delta}{\delta c} - W \frac{\delta}{\delta w} \right) \cdot E,$$
$$N \cdot \frac{\delta E}{\delta s} + C \cdot \frac{\delta E}{\delta c} - W \cdot \frac{\delta E}{\delta w} = 0,$$

that satisfies:

$$E_c^C - E_w^W = -E^C divC + E^W divW. \tag{49}$$

In the same way, noting that E is right angle to the tangent directional and (19),(35), and (46), we can compute

$$\nabla \cdot V = \left(N \cdot \frac{\delta}{\delta s} + C \frac{\delta}{\delta c} - W \frac{\delta}{\delta w} \right) \cdot V,$$

$$N \cdot \frac{\delta V}{\delta s} + C \cdot \frac{\delta V}{\delta c} - W \cdot \frac{\delta V}{\delta w} = 0,$$

which implies that,

$$E_c^W - E_w^C = E^C divW - E^W divC. \tag{50}$$

If we think more (49) and (50), then it is calculated that Laplacian-like equations through c-lines and w-lines of the electromagnetic waves are as follows,

$$\begin{aligned}\frac{\delta^2}{\delta c^2}E^W - \frac{\delta^2}{\delta w^2}E^W &= E^C((divW)_c - (divC)_w)\\ &+ E^W((divW)_w - (divcC)_c)\\ &+ divW(E^C_c + E^W_w)\\ &- divC(E^W_c + E^C_w),\end{aligned}$$

$$\begin{aligned}\frac{\delta^2}{\delta c^2}E^C - \frac{\delta^2}{\delta w^2}E^C &= E^C((divW)_w - (divC)_c)\\ &- E^W((divW)_w + (divC)_c)\\ &+ divW(E^C_w + E^W_c)\\ &- divC(E^C_c + E^W_w).\end{aligned}$$

If we consider that the electric field is perpendicular to the tangent direction and (19), (35), and (47), we get that

$$\nabla \times V = \epsilon v\frac{\delta E}{\delta u} = \left(N\cdot\frac{\delta}{\delta s} + C\frac{\delta}{\delta c} - W\frac{\delta}{\delta w}\right)\times V,$$

$$\epsilon v\frac{\delta E}{\delta u} = \left(N\times\frac{\delta}{\delta s}V + C\times\frac{\delta}{\delta c}V - W\times\frac{\delta}{\delta w}V\right),$$

which satisfies that,

$$\begin{aligned}\epsilon v\frac{\delta E}{\delta u} &= \left(-E^C_c + E^W divW - E^W_w + E^C(f(s) + divC\right)N\\ &+ \left(E^C_s - \Omega_C E^W - \theta_{WS}E^W\right)C\\ &+ \left(-E^W_s + \Omega_W E^C - \theta_{CS}E^W\right)W.\end{aligned}$$

In the same way, noting that E is right angle to the tangent directional and (19),(35), and (48), we can write

$$-\frac{\delta}{\delta u}V = \nabla\times E = \left(N\frac{\delta}{\delta s} + C\frac{\delta}{\delta c} - W\frac{\delta}{\delta w}\right)\times E,$$

$$-\frac{\delta}{\delta u}V = N\times\frac{\delta}{\delta s}E + C\times\frac{\delta}{\delta c}E - W\times\frac{\delta}{\delta w}E,$$

which implies that,

$$\begin{aligned}\frac{\delta}{\delta u}V &= \left(E^W_c - E^C divW + E^C_w - E^W(f(s) + divC)\right)N\\ &+ (-E^W_s + \Omega_C E^C + \theta_{WS}E^W)C\\ &+ (E^C_s - \Omega_W E^W + \theta_{CS}E^C)W.\end{aligned}$$

3. Electromagnetic theory

Let us consider the variation of the electric field vector E in the s direction by means of the alternative moving frame $\{N, C, W\}$.Then we can express the following

$$\frac{\delta}{\delta s}\vec{E} = \lambda_1 N + \lambda_2 C + \lambda_3 W. \tag{51}$$

Case 1. In this case, it was presumed that the vector E lies on a plane perpendicular to N. Thus we have

$$\langle N, E\rangle = 0.$$

Now taking the derivative of this equation we acquire

$$\left\langle\frac{\delta N}{\delta s}, E\right\rangle + \left\langle\frac{\delta E}{\delta s}, N\right\rangle = 0.$$

This equation satisfies that:

$$\left\langle\frac{\delta N}{\delta s}, E\right\rangle = -\left\langle\frac{\delta E}{\delta s}, N\right\rangle.$$

After that we use Eq. (51) and last equation, we can calculated

$$\left\langle\frac{\delta N}{\delta s}, E\right\rangle = -\varepsilon_N\lambda_1. \tag{52}$$

Using Eq. (51) and Eq. (52), we can compute

$$\lambda_1 = -\varepsilon_N f(s)\langle E, C\rangle.$$

Presuming there is no loss mechanism due to absorption, we have

$$\langle E, E\rangle = c,$$

where c is a constant. Then taking the derivative of this equation we get,

$$\left\langle\frac{\delta E}{\delta s}, E\right\rangle = 0.$$

Thus, the coefficients in Eq. (51) are determined as follows

$$\lambda_2 = \lambda\langle E, W\rangle, \quad \lambda_3 = -\lambda\langle E, C\rangle.$$

Lastly, from Eq. (51), we can write

$$\begin{aligned}\frac{\delta}{\delta s}\vec{E} &= -\varepsilon_N f(s)\langle E, C\rangle N\\ &+ \lambda\langle E, W\rangle C - \lambda\langle E, C\rangle W.\end{aligned} \tag{53}$$

By thinking N is parallel transported we get $\lambda = 0$.

Then, if we use the equation in which the electric field vector is written in terms of alternative moving frames, we have

$$E = \varepsilon_C\langle E, C\rangle C + \varepsilon_W\langle E, W\rangle W. \tag{54}$$

In the last equation, we can take the derivative and consider alternative moving frame equations, thus we obtain

$$\begin{aligned}\frac{\delta}{\delta s}\vec{E} &= -\varepsilon_N f(s)\langle E, C\rangle N + (\varepsilon_C\langle E, C\rangle'\\ &- \varepsilon_C g(s)\langle E, W\rangle)C + (\varepsilon_W\langle E, W\rangle'\\ &+ \varepsilon_C g(s)\langle E, C\rangle)W.\end{aligned} \tag{55}$$

Now, let us take the derivative of Eq. (53) and compare with Eq. (55) to get the matrix form below:

$$\begin{pmatrix} \langle E,C\rangle' \\ \langle E,W\rangle' \end{pmatrix} = \begin{pmatrix} 0 & g(s) \\ -\varepsilon_W\varepsilon_C g(s) & 0 \end{pmatrix} \begin{pmatrix} \langle E,C\rangle \\ \langle E,W\rangle \end{pmatrix}.$$

On the other hand, using the definition of electromagnetic field via alternative moving frame with respect anholonomic coordinates and with the help of Lorentz equation in Minkowski space, we get

$$\langle \phi(E),N\rangle = -\langle \phi(N),E\rangle = \left\langle \frac{\delta E}{\delta s},N\right\rangle = -\varepsilon_N f(s)\langle E,C\rangle,$$

$$\langle \phi(E),C\rangle = -\langle \phi(C),E\rangle = \left\langle \frac{\delta E}{\delta s},C\right\rangle = \varepsilon_C\lambda\langle E,W\rangle,$$

$$\langle \phi(E),W\rangle = -\langle \phi(W),E\rangle = \left\langle \frac{\delta E}{\delta s},W\right\rangle = -\varepsilon_W\lambda\langle E,C\rangle.$$

If necessary calculations are made, we can write

$$\phi(N) = \varepsilon_N f(s)C + a_1 N,$$
$$\phi(C) = \varepsilon_C\lambda W + a_2 N,$$
$$\phi(W) = \varepsilon_W\lambda C + a_3 N.$$

Then, Lorentz force along with optical fiber that is determined non-null curve for the alternative moving frame means the following matrix form:

$$\begin{pmatrix} \phi(N) \\ \phi(C) \\ \phi(W) \end{pmatrix} = \begin{pmatrix} 0 & f(s) & 0 \\ -\varepsilon_C\varepsilon_N f(s) & 0 & \varepsilon_C\lambda \\ 0 & \varepsilon_W\lambda & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}.$$

This matrix has a structure that relates the Lorentz force to the $\{N,C,W\}$ frame.We know that Lorentz equation $\phi(E) = X \times E$ where X is a Killing magnetic field in Minkowski space. So we can write

$$\phi(N) = V \times N.$$

Also, if we express V in terms of the frame $\{N,C,W\}$ as follows

$$V = b_1 N + b_2 C + b_3 W,$$

and calculate the coefficients, we obtain

$$V = \varepsilon_C\lambda N + \varepsilon_C\varepsilon_W f(s)W.$$

Case 2. In this case, it was presumed that the vector E lies on a plane perpendicular to C. Thus we have

$$\langle C,E\rangle = 0.$$

Now taking the derivative of this equation and using the derivative of the alternative moving frame, we acquire

$$\left\langle \frac{\delta C}{\delta s},E\right\rangle = -\varepsilon_C\lambda_2. \tag{56}$$

Using Eq. (51) and Eq. (56), we compute the following

$$\lambda_2 = \varepsilon_N f(s)\langle E,N\rangle + \varepsilon_C g(s)\langle E,W\rangle.$$

Presuming there is no loss mechanism due to absorption, we have

$$\langle E,E\rangle = c,$$

where c is a constant. Then taking the derivative of this equation we get,

$$\left\langle \frac{\delta E}{\delta s},E\right\rangle = 0.$$

Thus the coefficients in Eq. (51) are determined as follows

$$\lambda_1 = \lambda\langle E,W\rangle,\ \lambda_3 = -\lambda\langle E,N\rangle.$$

Lastly, from Eq. (51), we can write

$$\frac{\delta}{\delta s}\vec{E} = (\varepsilon_N f(s)\langle E,N\rangle + \varepsilon_C g(s)\langle E,W\rangle)C - \lambda\langle E,N\rangle W + \lambda\langle E,W\rangle N$$
$$E = \varepsilon_N\langle E,N\rangle N + \varepsilon_W\langle E,W\rangle W. \tag{57}$$

In the last equation, we can take the derivative and consider alternative moving frame equations, thus we obtain

$$\frac{\delta}{\delta s}\vec{E} = (\varepsilon_N\langle E,N\rangle')N + (\varepsilon_N f(s)\langle E,N\rangle - \varepsilon_C g(s)\langle E,W\rangle)C + (\varepsilon_W\langle E,W\rangle')W. \tag{58}$$

Now, let us take the derivative of Eq. (57) and compare this derivative with Eq. (58) to get the matrix form below:

$$\begin{pmatrix} \langle E,N\rangle' \\ \langle E,W\rangle' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \langle E,N\rangle \\ \langle E,W\rangle \end{pmatrix}.$$

On the other hand, using the definition of electromagnetic field via alternative moving frame with respect anholonomic coordinates help of Lorentz equation in Minkowski space, we get

$$\langle \phi(E),N\rangle = -\langle \phi(N),E\rangle = \left\langle \frac{\delta E}{\delta s},N\right\rangle = \varepsilon_N\lambda\langle E,W\rangle.$$

If necessary calculations are made, we can write

$$\phi(N) = -\varepsilon_N\lambda W + a_1 C,$$
$$\phi(C) = -\varepsilon_N\varepsilon_C f(s)N - g(s)W + a_2 C,$$
$$\phi(W) = \varepsilon_W\lambda N + a_3 C.$$

Then, Lorentz force along with optical fiber that is determined non-null curve for the alternative moving frame means the following matrix form:

$$\begin{pmatrix} \phi(N) \\ \phi(C) \\ \phi(W) \end{pmatrix} = \begin{pmatrix} 0 & f(s) & -\varepsilon_N\lambda \\ \varepsilon_W f(s) & 0 & -g(s) \\ \varepsilon_W\lambda & -\varepsilon_N g(s) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}.$$

This matrix has a structure that relates the Lorentz force to the $\{N, C, W\}$ frame. We know that Lorentz equation $\phi(E) = X \times E$ where X is a Killing magnetic field in Minkowski space. So we can write

$$V \times N = \phi(N).$$

Also, if we express V in terms of the frame $\{N, C, W\}$ as follows

$$V = b_1 N + b_2 C + b_3 W,$$

and calculate the coefficients, we obtain

$$V = -g(s)N + \varepsilon_N \lambda C + \varepsilon_C \varepsilon_W f(s)W.$$

Case 3. In this case, it was presumed that the vector E lies on a plane perpendicular to W. Thus we have

$$\langle W, E\rangle = 0.$$

Now taking the derivative of this equation and using the derivative of the alternative moving frame, we acquire

$$\left\langle \frac{\delta W}{\delta s}, E \right\rangle = -\varepsilon_W \lambda_3. \tag{59}$$

If we take notice it using Eq. (51) and Eq. (59), we compute the following

$$\lambda_3 = \varepsilon_C g(s)\langle E, C\rangle.$$

Presuming there is no loss mechanism due to absorption, we have

$$\langle E, E\rangle = c,$$

where c is a constant. Then taking the derivative of this equation we get,

$$\left\langle \frac{\delta E}{\delta s}, E \right\rangle = 0.$$

Thus the coefficients in Eq. (51) are determined as follows

$$\lambda_1 = \lambda\langle E, C\rangle,\ \lambda_2 = -\lambda\langle E, N\rangle.$$

Lastly, from Eq. (51), we can write

$$\frac{\delta}{\delta s}\vec{E} = \lambda\langle E, C\rangle N - \lambda\langle E, N\rangle C + (\varepsilon_N\langle E, N\rangle N + \varepsilon_C\langle E, C\rangle C)W. \tag{60}$$

By thinking W is parallel transported we get $\lambda = 0$. Then, if we use the equation in which the electric field vector is written in terms of alternative moving frame, we have

$$E = \varepsilon_N\langle E, N\rangle N + \varepsilon_C\langle E, C\rangle C.$$

In the last equation, we can take the derivative and consider alternative moving frame equations, thus we obtain

$$\frac{\delta}{\delta s}\vec{E} = (\varepsilon_N\langle E, N\rangle' - \varepsilon_N f(s)\langle E, C\rangle)N + (\varepsilon_C\langle E, C\rangle' + \varepsilon_N f(s)\langle E, N\rangle)C + (\varepsilon_C g(s)\langle E, C\rangle)W. \tag{61}$$

Now, let us take the derivative of Eq. (60) and compare this derivative with Eq. (61) to get the matrix form below:

$$\begin{pmatrix} \langle E, N\rangle' \\ \langle E, C\rangle' \end{pmatrix} = \begin{pmatrix} 0 & f(s) \\ -\varepsilon_C\varepsilon_N f(s) & 0 \end{pmatrix} \begin{pmatrix} \langle E, N\rangle \\ \langle E, C\rangle \end{pmatrix}.$$

On the other hand, using the definition of electromagnetic field via alternative moving frame with respect anholonomic coordinates help of Lorentz equation in Minkowski space, we get

$$\langle \phi(E), N\rangle = -\langle \phi(N), E\rangle = \langle \frac{\delta E}{\delta s}, N\rangle = \varepsilon_N \lambda\langle E, C\rangle.$$

If necessary calculations are made, we can write

$$\phi(N) = -\varepsilon_N \lambda C + a_1 W,$$
$$\phi(C) = \lambda\varepsilon_C N + a_2 W,$$
$$\phi(W) = -\varepsilon_C\varepsilon_W g(s)C + a_3 W.$$

Then, Lorentz force along with optic fiber that is determined non-null curve for the alternative moving frame means the following matrix form:

$$\begin{pmatrix} \phi(N) \\ \phi(C) \\ \phi(W) \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon_N\lambda & 0 \\ \lambda\varepsilon_C & 0 & g(s) \\ 0 & -\varepsilon_W\varepsilon_C g(s) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}.$$

This matrix has a structure that relates the Lorentz force to the $\{N, C, W\}$ frame. We know that Lorentz equation $\phi(E) = X \times E$ where X is a Killing magnetic field in Minkowski space. So we can write

$$V \times N = \phi(N).$$

Also, if we express V in terms of the frame $\{N, C, W\}$ as follows

$$V = b_1 N + b_2 C + b_3 W,$$

and calculate the coefficients, we obtain

$$V = g(s)N - \varepsilon_N \lambda W.$$

4. Conclusion

In this study, we found the movement of polarized light along optical fiber by calculating the equations of the electric field and magnetic field in cases where the frame of the space is at a right angle with respect to the vector fields. Thus, we had the opportunity to examine the action of light in the field of geometry. In this way, the relationship of the action of light in space with special curves, which is an important subject of geometry, can be investigated. At the same time, we investigated the geometric phase issue and Maxwell's equations together. We have obtained two important cases. These situations gave us the chance to examine the motion of light in the c-direction and in the direction of the Darboux vector in Minkowski 3d-space. We also gave their connections with Fermi-Walker parallel transportation laws in Minkowski space. For further research, we aim to research Maxwellian evolution equations relationship between spherical coordinates to better understand the solutions of the equations.

Acknowledgment

The authors would like to express their sincere gratitude to the referees for the valuable suggestions to improve the paper.

1. V. V. Vladimirski, *Dokl. Akad. Nauk. SSSR* **31** (1941) 222; reprinted in B. Markovski, S.I. Vinitsky (eds) Topological Phases in Quantum Theory, World Scientific, Singapore (1989).
2. Y. A. Kravtsov and Y. I. Orlov, Geometrical Optics of Inhomogeneous Media, (Springer-Verlag, Berlin, 1990).
3. E. M. Frins, W. Dultz, Rotation of the polarization plane in optical fibers, *J. Lightwave Technol.* **15** (1997) 144. https://doi.org/10.1109/50.552122.
4. J .N. Ross, The rotation of the polarization in low briefrigence monomode optical fibres due to geometric effects, *Opt. Quantum Electron.* **16** (1984) 455. https://doi.org/10.1007/BF00619638.
5. M. V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. Roy. Soc. London A*. **392** (1984) 45. https://doi.org/10.1098/rspa.1984.0023.
6. M. Kugler, S. Shtrikman, Berry's phase, locally inertial frames, and classical analogues, *Phys. Rev. D*. **37** (1988) 934. https://doi.org/10.1103/physrevd.37.934.
7. R. Dandoloff, Berry's phase and Fermi–Walker parallel transport, *Phys. Lett. A*. **139** (1989) 19. https://doi.org/10.1016/0375-9601(89)90599-9.
8. A. Comtet, On the Landau Hall levels on the hyperbolic plane, *Ann. Phys*. **173** (1987) 185. https://doi.org/10.1016/0003-4916(87)90098-4.
9. M. Barros, A. Romero, J. L. Cabrerizo, M. Fern´ andez, The Gauss-Landau-Hall problem on Riemanniansurfaces. *J. Math. Phys*. **46** (2005) 112905, https://doi.org/10.1063/1.2136215.
10. M. Barros, J. L. Cabrerizo, M. Fern´andez, A. Romero, Magnetic vortex flament flows, *J. Math. Phys*. **48** (2007) 082904, https://doi.org/10.1063/1.2767535.
11. T. Adachi, Kahler magnetic on a complex projective space, *Proc. Jpn. Acad. Ser. A Math. Sci.* **70** (1994) 12. https://doi.org/10.3792/pjaa.70.12.
12. T. Adachi, Kahler magnetic flow for a manifold of constant holomorphic sectional curvature, *Tokyo J. Math.* **18** (1995) 473. https://doi.org/10.3836/tjm/1270043477.
13. T. Körpınar, R.C. Demirkol, Electromagnetic curves of the linearly polarized light wave along an optical fiber in a 3D semi-Riemannian manifold. *Journal of Modern Optics*, **66** (2019) 857. https://doi.org/10.1080/09500340.2019.1579930.
14. T. Sunada, Magnetic flows on a Riemann surface, In Proceedings of the KAIST Mathematics Workshop:Analysis and Geometry, *Taejeon, Korea*, **8** (1993) 93. https://cir.nii.ac.jp/crid/1574231873874473728.
15. J. L Cabrerizo, M. Fernandez and J. S. Gómez, The contact magnetic flow in 3D Sasakian manifolds, *J. Phys. A: Math. Theor.* **42** (2009) 195201. https://doi.org/10.1088/1751-8113/42/19/195201.
16. M. Barros, General helices and a theorem of Lancret, *Proc. Amer. Math. Soc*. **125** (1997) 1503. https://www.jstor.org/stable/2162098.
17. Z. Bozkurt, İ. Gök, Y. Yaylı F. N. Ekmekci, A new approach for magnetic curves in 3D Riemannian manifolds, *J. Math. Phys*. **55** (2014) 053501. https://doi.org/10.1063/1.4870583.
18. Z. Özdemir, A New Calculus for the Treatment of Rytov's Law in the Optical Fiber, *Optik* **216** (2020) 164892, https://doi.org/10.1016/j.ijleo.2020.164892.
19. H. Ceyhan, Z. Özdemir, İ. Gök, F. N. Ekmekci, Electromagnetic Curves and Rotation of the Polarization Plane through Alternative Moving Frame, *European Physical Journal Plus* **135** (2020) 1. https://doi.org/10.1140/epjp/s13360-020-00881-z.
20. J.L. Cabrerizo, Magnetic fields in 2D and 3D sphere. *J. Nonlinear Math. Phys*. **20** (2013) 440 https://doi.org/10.1080/14029251.2013.855052.
21. O. Bjorgum G. Thore, On Beltrami vector fields and flows, Part 1. Universitet I Bergen, Arbok (1951), Naturvitenskapelig rekke nr. 1.
22. A. W. Marris, Addendum to, Vector fields of solenoidal vector-line rotation, A class of permanent flows of solenoidal vector-line rotation. *Arch. Rational Mech. Anal.* **27** (1967) 195.
23. N. E. Gurbuz, The pseudo-null geometric phase along optical fiber, *Int. J. of Geometric Methods in Modern Phys.* **18** (2021) https://doi.org/10.1142/S0219887821502303.
24. T. Körpınar, R. C. Demirkol, Z. Körpinar, V. Asil, Maxwellian evolution equations along the uniform optical fiber in Minkowski space, *Rev. Mex. Fis.* **66** (2020) 431, https://doi.org/10.31349/revmexfis.66.431.
25. F. Ateş, E. Kocakuşaklı, İ. Gök, F. N. Ekmekci, Tubular Surfaces Formed by Semi-spherical Indicatrices in E_1^3, Mediterr. *J. Math.*, **17** (2020) 127, https://doi.org/10.1007/s00009-020-01561-z.

26. R. Betchov, On the curvature and torsion of an isolated vortex filament, *Journal of Fluid Mechanics*, **22** (1965) 471 https://doi.org/10.1017/S0022112065000915.

27. L. S. Da Rios, Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque, Rend, *Circ. Matem. Palermo*, **22** (1906) 117 https://doi.org/10.1007/BF03018608.

28. W. K. Schief, Hidden integrability in ideal magnetohydrodynamics: The Pohlmeyer-Lund-Regge Model, *Phys. of Plasmas*, **10** (2003) 2677 https://doi.org/10.1063/1.1577347.

29. W. K. Schief, C. Rogers, The Da Rios system under a geometric constraint: the Gilbarg problem, *Jour. of Geometry and Physics*, **54** (2005) 286 https://doi.org/10.1016/j.geomphys.2004.10.001.

30. H. Hasimoto, A soliton on a vortex filament, *Jour. of Fluid Mech.*, **51** (1972) 477 https://doi.org/10.1017/S0022112072002307.

31. A. Mukhopadhyay, V. Vyas, P. Panigrahi, Rogue waves and breathers in Heisenberg spin chain, *Eur. Phys. J. B* **88** (2015) 188 https://doi.org/10.1140/epjb/e2015-60229-8.

32. V. Banica, E. Miot, Evolution, interaction and collisions of vortex filaments, *Differential Integral Equations*, **26** (3/4) (2013) 355 https://doi.org/10.48550/arXiv.1202.2580.

33. M. Grbovic, E. Nesovic, On Backlund transformation and vortex filament equation for pseudo-null curves in Minkowski 3-space, *Int. J. Geom. Methods Mod. Phys*. **13** (2016) 1650077 https://doi.org/10.1142/S0219887816500778.

34. J. Amor, A. Gimenez, P. Lucas, Integrability aspects of the vortex filaments equation for pseudo-null curves, *Int. J. Geom. Methods Mod. Phys.* **14** (2017) 1750090, https://doi.org/10.1142/S0219887817500906.

35. M. Hesami, M. Avazpour, M. M. Méndez Otero, J. J. A. Rodríguez, Evolution of rectangular and triangular initial beam profiles in positive Kerr local medium, *Supl. Rev. Mex. Fis*. **1** (2020) 13 https://doi.org/10.31349/SuplRevMexFis.1.13.

36. M. Hesami, M. Avazpour, M. M. Méndez Otero, J. Arriaga, M. D. I. Castillo, S. C. Cerda, Generation of bright spatial quasi-solitons by arbitrary initial beam profiles in local and nonlocal (1+1)-dimensional nonlinear media, *Optik* **202** (2020) 163504 https://doi.org/10.1016/j.ijleo.2019.163504.

37. M. Hesami, M. Avazpour, M. D. I. Castillo, H. Nadgaran, E. Alvarado-Mendez, Observation of a different type of splitting solitons induced by interaction of second order spatial solitons, *Optik* **245** (2021) 167647 https://doi.org/10.1016/j.ijleo.2021.167647.

38. N. Gürbüz, Intrinsic geometry of the NLS equation and heat system in 3-dimensional Minkowski space, *Adv. Studies Theor. Phys.* **4** (2010) 557 https://m-hikari.com/astp/astp2010/astp9-12-2010/gurbuzASTP9-12-2010.pdf.

39. M. Ungs, L. P. and A. Gize, The Theory of Quantum Torus Knots: Its Foundation in Differential Geometry- Volume II, (2020) https://books.google.com.tr/books?id=XtuEzQEACAAJ.

40. H. Ceyhan, Z. Özdemir, İ. Gök, F. N. Ekmekci, A Geometric Interpretation of Polarized Light and Electromagnetic Curves Along an Optical Fiber with Surface Kinematics, *Mediterr. J. Math.* **19** (2022) 265. https://doi.org/10.1007/s00009-022-02160-w.