Solitary wave type solutions of nonlinear improved mKdV equation by modified techniques

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In this paper, the improved and modified version of the Sardar sub-equation method (IMSSEM) and the improved generalized Riccati equation mapping method (IGREMM) are manipulated effectively and generously to determine the exact solitary wave soliton solutions of the improved modified KdV (mKdV) equation. The purpose of this study is to provide novel exact solutions to the improved mKdV equation. Specifically, we utilized IMSSEM and IGREMM to study different solutions of the nonlinear improved mKdV equation, focusing on exponential, trigonometric, and trigonometric hyperbolic type solutions. Furthermore, the plotting of various solutions for direct viewing analysis is provided in two and three-dimensional graphs. The new strategies are straightforward, quick, and efficient and have many other advantages, whereas, they provide the most accurate and unique solution to many other types of nonlinear partial differential equations (NPDEs), which usually arise in engineering and applied sciences. It should be noted that these methodologies are novel mathematical instruments that have shown to be the most effective mathematical tools for solving higher-order nonlinear partial differential equations in mathematical physics. Symbolic computation was used to validate all of the solutions that were established. Thus, it is also hoped that these techniques will ultimately reduce the cumbersome workload involved during the process of solutions to complicated NLPDEs.

Keywords: IMSSEM and IGREMM; improved mKdV equation; solitary wave solution; NLPDEs.

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1. Introduction

The nonlinear partial differential equations (NLPDEs) have been used extensively for the simulation of many problems of physical nature, whereas, they have exactly described and controlled the behavior of such phenomena. However, such NLPDEs are used frequently for the simulation of problems in fluid mechanics, structural engineering, optical fiber, plasma physics, biology, solid state, and physical sciences. Besides that the techniques for finding the exact solutions to NLPDEs are rare. On the other hand, few handsome strategies have been devised for the solutions of NLPDEs in recent years, however, it is a big miracle in the world of mathematics. Note that the details of such techniques can be found in the literature, *e.g*. Tan-cot [1, 2], sine-cosine [3, 4, 12], extended trail method [5, 6], new auxiliary equation [7–9], Jacobi elliptic ansatz [10, 11], Hirota's direct [14, 15], extended direct algebraic [13, 16, 17], generalizes Bernoulli sub-ODE [18, 19], function variable [20, 21], sub-function [22, 23], and others [24–37]. Strictly speaking, we categorically emphasized the effectiveness of the two important and modified techniques, *i.e*. IMSSEM and IGREMM and they are utilized here wisely to produce the solitary wave solutions of the nonlinear improved mKdV equation. These solitary wave solutions of the improved mKdV equation are important in the application of KdV equations in science, engineering, and physics. The improved mKdV equation is:

$$\mathrm{u_t}+a\mathrm{u}^2\mathrm{u_x}+b\mathrm{u_{xxt}}+\beta\mathrm{u_{xxx}}=0, \tag{1}$$

where the unknown function $\mathrm{u(x,t)}$ is the dependent variable and it has varied with x and t. The improved mKdV equation has an extra dispersive term, *i.e.* the part of Eq. (1) which contains b This modification creates many significant changes in the solution structure. The improved mKdV equation is one of the perfect models useful in electromagnetic and elastic media, whereas, it explains the nonlinear wave propagation in polarity and symmetric systems. Therefore, we focused its solutions on improved and modified algorithms which are very effective and efficient, whereas, they may solve other nonlinear problems of a complex nature in a very easy way and preserve the physical characteristics of the problem.

Furthermore, the article is briefly managed: In Sec. 2, we explained the description of the proposed methodology IMSSEM and IGREMM. In Sec. 3, we applied the IMSSEM and IGREMM to construct the novel exact, most accurate solitary wave solutions of the improved mKdV equation. In Sec. 4, the 2D and 3D graphs are plotted for direct viewing study and in Sec. 5, we outlined the conclusion.

2. Methodology of the proposed improved techniques

Analyze the nonlinear PDEs

$$\mathrm{F}(\mathrm{u},\mathrm{u_t},\mathrm{u_x},\mathrm{u_{xx}},\ldots), \tag{2}$$

where the function $\mathrm{u}=\mathrm{u}(\mathrm{x},\mathrm{t})$ is an unknown function. At the moment, we introduced the following wave transformation

$$\mathrm{u}=\mathrm{u}\left(\xi\right), \xi\!=\mathrm{x}-c\mathrm{t}, \tag{3}$$

where $c\neq 0$, by using the transformation in (3) into (2) to reduce the nonlinear PDE and to convert it into a nonlinear ODE with an integral order.

$$\mathrm{N}\left(\mathrm{u}^{'},\mathrm{u}^{''},\mathrm{u}^{'''},\ldots\right). \tag{4}$$

We have solved the above non-linear ODE by using IMSSEM and IGREMM these techniques have the following standard forms:

$$\mathrm{u}\left(\xi\right)=a_0+\sum_{j=1}^{N}a_j\phi^j\left(\xi\right)\, a_N\neq 0, \tag{5}$$

where, a_j $(j=0,1,2,3,\ldots.N)$.

The value of N can be determined by balancing the highest order derivative term and the highest order nonlinear term in Eq. (5). Therefore, the highest degree of $\mathrm{d^r u/d\xi^r}$ is classified as:

$$\mathrm{O}\left(\frac{\mathrm{d^r u}}{\mathrm{d}\xi^{\mathrm{r}}}\right)=\mathrm{n}+\mathrm{r},\;\; \mathrm{r}=1,2,3,\ldots. \tag{6}$$

$$\mathrm{O}\left(\mathrm{u^q}\frac{\mathrm{d^r u}}{\mathrm{d}\xi^{\mathrm{r}}}\right)=(\mathrm{q}+1)\,\mathrm{n}+\mathrm{r},\;\; \mathrm{q}=0,\ 1,2,\ldots, \tag{7}$$

and $\mathrm{r}=1,2,3,\ldots$.

2.1. The enhanced Sardar sub-equation approach

The $\phi(\xi)$ in Eq. (5) is considered the solution to the following equation:

$$\left(\phi^{'}\right)^2\left(\xi\right)=\delta_2\phi^4\left(\xi\right)+\delta_1\phi^2\left(\xi\right)+\delta_0, \tag{8}$$

where $\delta_i, i=0,1,2$ are constants to be determined. The following set of solutions satisfies Eq. (8) where C is the integration constant:

1. For $\delta_0=\delta_1=0$ and $\delta_2>0$, we obtained the rational solutions:

$$\phi_1^{\pm}(\xi)=\pm\frac{1}{\sqrt{\delta_2}(\xi+C)}, \tag{9}$$

2. For $\delta_0=0$ and $\delta_1>0$, the exponential solutions will be of the form:

$$\phi_2^{\pm}(\xi)=\frac{4\delta_1 \mathrm{e}^{\pm\sqrt{\delta_1}(\xi+\mathrm{C})}}{\mathrm{e}^{\pm 2\sqrt{\delta_1}(\xi+C)}-4\delta_1\delta_2}, \tag{10}$$

$$\phi_3^{\pm}(\xi)=\frac{\pm 4\delta_1 \mathrm{e}^{\pm\sqrt{\delta_1}(\xi+\mathrm{C})}}{1-4\delta_1\delta_2\mathrm{e}^{\pm 2\sqrt{\delta_1}(\xi+\mathrm{C})}}. \tag{11}$$

3. The trigonometric hyperbolic solutions are as follows:

4. For $\delta_0=0$, $\delta_1>0$ and $\delta_2\neq 0$, we have

$$\phi_4^{\pm}(\xi)=\pm\sqrt{-\frac{\delta_1}{\delta_2}}\mathrm{sech}\left(\sqrt{\delta_1}\left(\xi+C\right)\right), \tag{12}$$

$$\phi_5^{\pm}(\xi)=\pm\sqrt{\frac{\delta_1}{\delta_2}}\mathrm{csch}\left(\sqrt{\delta_1}\left(\xi+C\right)\right). \tag{13}$$

5. For $\delta_0={\delta_1}^2/4\delta_2$, $\delta_1<0$ and $\delta_2>0$, we have

$$\phi_6^{\pm}(\xi)=\pm\sqrt{-\frac{\delta_1}{2\delta_2}}\tanh\left(\sqrt{-\frac{\delta_1}{2}}\left(\xi+C\right)\right), \tag{14}$$

$$\phi_7^{\pm}(\xi)=\pm\sqrt{-\frac{\delta_1}{2\delta_2}}\coth\left(\sqrt{-\frac{\delta_1}{2}}\left(\xi+C\right)\right), \tag{15}$$

$$\phi_8^{\pm}(\xi)=\pm\sqrt{-\frac{\delta_1}{2\delta_2}}(\tanh\left(\sqrt{-2\delta_1}\left(\xi+C\right)\right) \pm \mathrm{i}\,\mathrm{sech}\left(\sqrt{-2\delta_1}\left(\xi+C\right)\right), \tag{16}$$

$$\phi_9^{\pm}(\xi)=\pm\sqrt{-\frac{\delta_1}{2\delta_2}}(\coth\left(\sqrt{-2\delta_1}\left(\xi+C\right)\right) \pm \mathrm{csch}\left(\sqrt{-2\delta_1}\left(\xi+C\right)\right), \tag{17}$$

$$\phi_{10}(\xi)=\pm\sqrt{-\frac{\delta_1}{8\delta_2}}(\tanh\left(\sqrt{-\frac{\delta_1}{8}}\left(\xi+C\right)\right) + \coth\left(\sqrt{-\frac{\delta_1}{8}}\left(\xi+C\right)\right). \tag{18}$$

6. The solutions which have the form of trigonometric functions are presented below:

7. For $\delta_0 = 0$, $\delta_1 < 0$ and $\delta_2 \neq 0$, we have

$$\phi_{11}^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{\delta_2}}\sec\left(\sqrt{-\delta_1}\,(\xi + C)\right), \qquad (19)$$

$$\phi_{12}^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{\delta_2}}\csc\left(\sqrt{-\delta_1}\,(\xi + C)\right). \qquad (20)$$

8. For $\delta_0 = {\delta_1}^2/4\delta_2$, $\quad \delta_1 > 0$ and $\delta_2 > 0$, we have

$$\phi_{13}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\tan\left(\sqrt{\frac{\delta_1}{2}}\,(\xi + C)\right), \qquad (21)$$

$$\phi_{14}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\cot\left(\sqrt{\frac{\delta_1}{2}}\,(\xi + C)\right), \qquad (22)$$

$$\phi_{15}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}(\tan\left(\sqrt{2\delta_1}\,(\xi + C)\right) \pm\sec\left(\sqrt{2\delta_1}\,(\xi + C)\right), \qquad (23)$$

$$\phi_{16}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}(\cot\left(\sqrt{2\delta_1}\,(\xi + C)\right) \pm\csc\left(\sqrt{2\delta_1}\,(\xi + C)\right), \qquad (24)$$

$$\phi_{17}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{8\delta_2}}\left(\tan\left(\sqrt{\frac{\delta_1}{8}}\,(\xi + C)\right) - \cot\left(\sqrt{\frac{\delta_1}{8}}\,(\xi + C)\right)\right), \qquad (25)$$

Remember that we have substituted Eqs. (5)-(8) into Eq. (4) and equated all the coefficients of each power of $\phi(\xi)$ to zero and solved the resultant system of algebraic equations with the help of Maple. Eventually, we incorporated these constants (coefficients) into Eq. (5) and obtained the solution of distinct types as shown in Eqs. (9)-(25). As a result, we obtained different exact solutions for NPDEs.

2.2. The enhanced generalized Riccati equation mapping method

The $\phi(\xi)$ in Eq. (5) is the solution of

$$\phi'(\xi) = \beta_2\phi^2(\xi) + \beta_1\phi(\xi) + \beta_0, \qquad (26)$$

where $\beta_{\mathrm{i}}, \mathrm{i} = 0, 1, 2$ are constants and they need to be determined later. The following set of solutions is obtained with the integration constant c :

1. For $\beta_0 = \beta_1 = 0$ and $\beta_2 \neq 0$, the rational solutions will be of the form:

$$\phi_1^{\pm}(\xi) = \pm\frac{1}{\beta_2(\xi + \mathrm{C})}, \qquad (27)$$

2. For $\beta_0 = 0$, the solution of the exponential type is simply obtained as:

$$\phi_2(\xi) = -\frac{\beta_1\phi}{\beta_1(\mathrm{e}^{-\beta_1(\xi + C)} + \varphi)}, \qquad (28)$$

$$\phi_3(\xi) = -\frac{\beta_1\mathrm{e}^{\beta_1(\xi + C)}}{\beta_2(\mathrm{e}^{\beta_1(\xi + C)} + \varphi)}, \qquad (29)$$

3. For $\rho = \beta_1^2 - 4\beta_0\beta_1 > 0$, $\beta_1\beta_2 \neq 0$ or $\beta_0\beta_2 \neq 0$, and p and q be nonzero real constants, the solutions presented in the form of trigonometric hyperbolic functions are given below:

$$\phi_4(\xi) = -\frac{\sqrt{\rho}}{2\beta_2}\tanh\left(\frac{\sqrt{\rho}}{2}\,(\xi + C)\right) - \frac{\beta_1}{2\beta_2}, \qquad (30)$$

$$\phi_5(\xi) = -\frac{\sqrt{\rho}}{2\beta_2}\coth\left(\frac{\sqrt{\rho}}{2}\,(\xi + C)\right) - \frac{\beta_1}{2\beta_2}, \qquad (31)$$

$$\phi_6^{\pm}(\xi) = -\frac{\sqrt{\rho}}{2\beta_2}(\tanh(\sqrt{\rho}\,(\xi + C)) \pm \mathrm{i}\,\mathrm{sech}(\sqrt{\rho}\,(\xi + C)) - \frac{\beta_1}{2\beta_2}, \qquad (32)$$

$$\phi_7^{\pm}(\xi) = -\frac{\sqrt{\rho}}{2\beta_2}\left(\coth(\sqrt{\rho}\,(\xi + C)) \pm \mathrm{csch}(\sqrt{\rho}\,(\xi + C))\right) - \frac{\beta_1}{2\beta_2}, \qquad (33)$$

$$\phi_8(\xi) = -\frac{\sqrt{\rho}}{4\beta_2}(\tanh\left(\frac{\sqrt{\rho}}{4}\,(\xi + C)\right) + \coth\left(\frac{\sqrt{\rho}}{4}\,(\xi + C)\right) - \frac{\beta_1}{2\beta_2}, \qquad (34)$$

$$\phi_9^{\pm}(\xi) = \frac{\pm\sqrt{\rho(\mathrm{p}^2 + \mathrm{q}^2)} - \mathrm{p}\sqrt{\rho}\cosh(\sqrt{\rho}\,(\xi + C))}{2\beta_2\left(\mathrm{p}\sinh\left(\sqrt{\rho}\,(\xi + C)\right) + \mathrm{q}\right)} - \frac{\beta_1}{2\beta_2}, \qquad (35)$$

$$\phi_{10}(\xi) = \frac{2\beta_0\cosh\left(\frac{\sqrt{\rho}}{2}\,(\xi + C)\right)}{\sqrt{\rho}\sinh\left(\frac{\sqrt{\rho}}{2}\,(\xi + C)\right) - \beta_1\cosh\left(\frac{\sqrt{\rho}}{2}\,(\xi + C)\right)}, \qquad (36)$$

$$\phi_{11}(\xi)=\frac{2\beta_0\sinh\left(\frac{\sqrt{\rho}}{2}(\xi+C)\right)}{\sqrt{\rho}\cosh\left(\frac{\sqrt{\rho}}{2}(\xi+C)\right)-\beta_1\sinh\left(\frac{\sqrt{\rho}}{2}(\xi+C)\right)}, \tag{37}$$

$$\phi_{12}^{\pm}(\xi)=\frac{2\beta_0\cosh\left(\sqrt{\rho}(\xi+C)\right)}{\sqrt{\rho}\sinh\left(\sqrt{\rho}(\xi+C)\right)-\beta_1\cosh\left(\sqrt{\rho}(\xi+C)\right)\pm\mathrm{i}\sqrt{\rho}}, \tag{38}$$

$$\phi_{13}^{\pm}(\xi)=\frac{2\beta_0\sinh\left(\sqrt{\rho}(\xi+C)\right)}{\sqrt{\rho}\cosh\left(\sqrt{\rho}(\xi+C)\right)-\beta_1\sinh\left(\sqrt{\rho}(\xi+C)\right)\pm\sqrt{\rho}}, \tag{39}$$

$$\phi_{14}(\xi)=\frac{2\beta_0\sinh\left(\frac{\sqrt{\rho}}{4}(\xi+C)\right)\cosh\left(\frac{\sqrt{\rho}}{4}(\xi+C)\right)}{2\sqrt{\rho}\cosh^2\left(\frac{\sqrt{\rho}}{4}(\xi+C)\right)-2\beta_1\sinh\left(\frac{\sqrt{\rho}}{4}\sqrt{\rho}(\xi+C)\right)\cosh\left(\frac{\sqrt{\rho}}{4}(\xi+C)\right)-\sqrt{\rho}}. \tag{40}$$

4. For $\rho=\beta_1^2-4\beta_0\beta_2<0$, $\beta_1\beta_2\neq 0$ or $\beta_0\beta_2\neq 0$, the solutions of the trigonometric form are demonstrated as follows:

$$\phi_{15}(\xi)=\frac{\sqrt{-\rho}}{2\beta_2}\tan\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)-\frac{\beta_1}{2\beta_2}, \tag{41}$$

$$\phi_{16}(\xi)=-\frac{\sqrt{-\rho}}{2\beta_2}\cot\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)-\frac{\beta_1}{2\beta_2}, \tag{42}$$

$$\phi_{17}^{\pm}(\xi)=\frac{\sqrt{-\rho}}{2\beta_2}\left(\tan\left(\sqrt{-\rho}(\xi+C)\right)\pm\sec\left(\sqrt{-\rho}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{43}$$

$$\phi_{18}^{\pm}(\xi)=-\frac{\sqrt{-\rho}}{2\beta_2}\left(\cot\left(\sqrt{-\rho}(\xi+C)\right)\pm\csc\left(\sqrt{-\rho}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{44}$$

$$\phi_{19}(\xi)=\frac{\sqrt{-\rho}}{4\beta_2}\left(\tan\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{-\rho}}{4}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{45}$$

$$\phi_{20}^{\pm}(\xi)=\frac{\pm\sqrt{-\rho(\mathrm{p}^2-\mathrm{q}^2)}-\mathrm{p}\sqrt{-\rho}\cos\left(\sqrt{-\rho}(\xi+C)\right)}{2\beta_2\left(\mathrm{p}\sin(\sqrt{-\rho}(\xi+C)+\mathrm{q}\right)}-\frac{\beta_1}{2\beta_2}, \tag{46}$$

$$\phi_{21}(\xi)=-\frac{2\beta_0\cos\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)}{\sqrt{-\rho}\sin\left(\frac{\sqrt{-\rho}}{2}(\xi+\mathrm{C})\right)+\beta_1\cos\left(\frac{\sqrt{-\rho}}{2}(\xi+\mathrm{C})\right)}, \tag{47}$$

$$\phi_{22}(\xi)=\frac{2\beta_0\sin\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)}{\sqrt{-\rho}\cos\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)-\beta_1\sin\left(\frac{\sqrt{-\rho}}{2}(\xi+C)\right)}, \tag{48}$$

$$\phi_{23}^{\pm}(\xi)=-\frac{2\beta_0\cos\left(\sqrt{-\rho}(\xi+\mathrm{C})\right)}{\beta_1\cos\left(\sqrt{-\rho}(\xi+\mathrm{C})\right)+\sqrt{-\rho}\sin\left(\sqrt{-\rho}(\xi+C)\right)\pm\sqrt{-\rho}}, \tag{49}$$

$$\phi_{24}^{\pm}(\xi)=\frac{2\beta_0\sin\left(\sqrt{-\rho}(\xi+\mathrm{C})\right)}{\beta_1\sin\left(\sqrt{-\rho}(\xi+\mathrm{C})\right)-\sqrt{-\rho}\cos\left(\sqrt{-\rho}(\xi+C)\right)\pm\sqrt{-\rho}}, \tag{50}$$

$$\phi_{25}(\xi)=\frac{4\beta_0\sin\left(\frac{\sqrt{-\rho}}{4}(\xi+C)\right)\cos\left(\frac{\sqrt{-\rho}}{4}(\xi+C)\right)}{2\sqrt{-\rho}\cos^2\left(\frac{\sqrt{-\rho}}{4}(\xi+\mathrm{C})\right)-2\beta_1\sin\left(\frac{\sqrt{-\rho}}{4}(\xi+C)\right)\cos\left(\frac{\sqrt{-\rho}}{4}(\xi+C)\right)-\sqrt{-\rho}}. \tag{51}$$

Note that we have substituted Eqs. (5) and (26) into Eq. (4) and equated all the coefficients of each power of $\phi^{\mathrm{i}}(\xi)$ to zero and solved the resultant system of algebraic equations with the help of Maple. Eventually, we incorporated these constants (coefficients) into Eq. (5) and obtained the solution of distinct types as shown in Eqs. (27)-(51). As a result, we obtained different exact solutions for NPDEs.

3. Improved Modified KdV Equation and its solutions

In this section, IMSSEM and IGREMM are used to find the new exact solitary solution of the improved mKdV equation. The KdV equation describes the development of lengthy waves on the surface of the fluid. The nonlinear and dispersive term in the KdV equation is quantifying the distribution of long waves, which are of small but finite amplitude in dispersive media. The KdV equation comes from a generic model to study weakly non-linear long waves, to incorporate the leading order, non-linearity, and diffusion. The non-linear KdV equation has a vital role to study the dispersion of water waves having a low amplitude in shallow water bodies and the arrangement of long internal ocean waves in separate layers.

Consider the improved modified KdV Eq. (1) and it has been usually expressed as:

$$\mathrm{u_t}+a\mathrm{u}^2\mathrm{u_x}+b\mathrm{u_{xxt}}+\beta\mathrm{u_{xxx}}=0. \tag{52}$$

The modified KdV equation explains nonlinear wave propagation in a polarity symmetric system. The Improved mKdV equation is useful in electromagnetic, wave propagation in size quantized films and elastic media. One of the best models for examining the characteristics and behavior of shallow water waves is the Improved Modified Kortewege de Vries (mKdV) equation. The equation also depicts phenomena that are frequently observed in plasma physics.

Consider the wave variable

$$\mathrm{u}=\mathrm{u}\left(\xi\right),\xi=\mathrm{x}-c\mathrm{t}. \tag{53}$$

We use the wave variable $\xi=\mathrm{x}-c\mathrm{t}$, where $c\neq0$, the variable ξ transforms the equation under consideration into the ordinary differential equation (ODE):

$$-c\frac{\mathrm{du}}{\mathrm{d}\xi}+a\mathrm{u}^2\frac{\mathrm{du}}{\mathrm{d}\xi}+\left(\beta-bc\right)\frac{\mathrm{d}^3\mathrm{u}}{\mathrm{d}\xi^3}=0 \tag{54}$$

Integrating once and taking the constant as zero, the above equation becomes

$$-\mathrm{cu}+\frac{a}{3}\mathrm{u}^3+\left(\beta-bc\right)\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{d}\xi^2}=0 \tag{55}$$

By balancing procedure, we obtained that $\mathrm{n}=1$, thus the value of " n " is substituted in Eq. (5) and finally we get

$$\mathrm{u}\left(\xi\right)=a_0+a_1\phi\left(\xi\right) \tag{56}$$

3.1. New exact solutions using enhanced Sardar sub-equation method

Now, the Eqs. $(8\ \&\ 55)$ are substituted into Eq. (55) and we get

$$\begin{aligned}&-ca_0-ca_1\phi\left(\xi\right)+\frac{1}{3}aa_0^3+aa_0^2a_1\phi\left(\xi\right)+aa_0a_1^2\phi\left(\xi\right)^2\\&+\frac{1}{3}aa_1^3\phi\left(\xi\right)^3+2a_1\phi\left(\xi\right)^3\beta\delta_2+a_1\phi\left(\xi\right)\beta\delta_1\\&-2a_1\phi\left(\xi\right)^3bc\delta_2-a_1\phi\left(\xi\right)bc\delta_1=0.\end{aligned} \tag{57}$$

By collecting various power of $\phi\left(\xi\right)^{\mathrm{i}}$, we get the system below:

$$\phi\left(\xi\right)^0:\ \ -\mathrm{c}a_0+\frac{1}{3}aa_0^3=0, \tag{58}$$

$$\phi\left(\xi\right)^1:\ \ -\mathrm{c}a_1+aa_0^2a_1+a_1\beta\delta_1-a_1bc\delta_1=0, \tag{59}$$

$$\phi\left(\xi\right)^2:\ \ \mathrm{a}a_0a_1^2=0, \tag{60}$$

$$\phi\left(\xi\right)^3:\ \frac{1}{3}aa_1^3+2a_1\beta\delta_2-2a_1bc\delta_2=0. \tag{61}$$

The above system has been solved with the help of Maple and finally, we get the coefficients involved in the series (55) as:

$$a_0=0, \tag{62}$$

$$a_1=a_1, \tag{63}$$

$$\delta_1=\frac{c}{\beta-bc}, \tag{64}$$

$$\delta_2=\frac{-aa_1^2}{6(\beta-\mathrm{bc})}. \tag{65}$$

Using Eq. $(61-64)$ in combination with Eq. $(9-25)$ $\&$ (55), we get the following solutions.

1. For $\delta_0=0$ and $\delta_1>0$,

$$\mathrm{u}_1^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\frac{4a_1\left(\frac{c}{\beta-bc}\right)\mathrm{e}^{\pm\sqrt{\frac{c}{(\beta-bc)}}(\xi+C)}}{\mathrm{e}^{\pm2\sqrt{\frac{c}{(\beta-bc)}}(\xi+C)}+\frac{4\mathrm{ca}a_1^2}{(\beta-bc)^2}}, \tag{66}$$

$$\mathrm{u}_2^{\pm}\left(\mathrm{x,t}\right)=\frac{\pm\frac{4a_1c}{(\beta-bc)}\mathrm{e}^{\pm\sqrt{\frac{c}{(\beta-bc)}}(\xi+C)}}{1+\frac{4\mathrm{ca}a_1^2}{(\beta-bc)^2}\mathrm{e}^{\pm2\sqrt{\frac{c}{(\beta-bc)}}(\xi+C)}}. \tag{67}$$

2. For $\delta_0=0$, $\delta_1>0$ and $\delta_2\neq0$ we have

$$\mathrm{u}_3^{\pm}\left(\mathrm{x,t}\right)=\pm\sqrt{\frac{6\mathrm{c}}{a}}\mathrm{sech}\left(\sqrt{\frac{c}{(\beta-bc)}}\left(\xi+C\right)\right), \tag{68}$$

$$\mathrm{u}_4^{\pm}\left(\mathrm{x,t}\right)=\pm\sqrt{\frac{-6\mathrm{c}}{a}}\mathrm{csch}\left(\sqrt{\frac{c}{(\beta-bc)}}\left(\xi+C\right)\right). \tag{69}$$

3. For $\delta_0={\delta_1}^2/4\delta_2$, $\delta_1<0$ and $\delta_2>0$ we have

$$\mathrm{u}_5^{\pm}\left(\mathrm{x,t}\right)=\pm\sqrt{\frac{3\mathrm{c}}{a}}\tanh\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\left(\xi+C\right)\right), \tag{70}$$

$$\mathrm{u}_6^{\pm}\left(\mathrm{x,t}\right)=\pm\sqrt{\frac{3\mathrm{c}}{a}}\coth\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\left(\xi+C\right)\right), \tag{71}$$

$$\mathrm{u}_7^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{3\mathrm{c}}{a}}(\tanh\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\pm\mathrm{sech}\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right), \tag{72}$$

$$\mathrm{u}_8^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{3\mathrm{c}}{a}}(\coth\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\pm\mathrm{csch}\left(\sqrt{-\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right), \tag{73}$$

$$\mathrm{u}_9\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{3\mathrm{c}}{4\mathrm{a}}}(\tanh\left(\sqrt{-\frac{8\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)+\coth\left(\sqrt{-\frac{8\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right), \tag{74}$$

$$\mathrm{u}_{10}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{6\mathrm{c}}{a}}\sec\left(\sqrt{-\frac{c}{(\beta-bc)}}\,(\xi+C)\right), \tag{75}$$

$$\mathrm{u}_{11}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{6\mathrm{c}}{a}}\csc\left(\sqrt{-\frac{c}{(\beta-bc)}}\,(\xi+C)\right), \tag{76}$$

$$\mathrm{u}_{12}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{-3\mathrm{c}}{a}}\tan\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right), \tag{77}$$

$$\mathrm{u}_{13}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{-3\mathrm{c}}{a}}\cot\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right), \tag{78}$$

$$\mathrm{u}_{14}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{-\frac{3\mathrm{c}}{a}}\left(\tan\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\pm\sec\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\right), \tag{79}$$

$$\mathrm{u}_{15}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{-\frac{3\mathrm{c}}{a}}\left(\cot\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\pm\csc\left(\sqrt{\frac{2\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\right), \tag{80}$$

$$\mathrm{u}_{16}^{\pm}\,(\mathrm{x},\mathrm{t})=\pm\sqrt{\frac{-3\mathrm{c}}{4\mathrm{a}}}\left(\tan\left(\sqrt{\frac{8\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)-\cot\left(\sqrt{\frac{8\mathrm{c}}{(\beta-bc)}}\,(\xi+C)\right)\right). \tag{81}$$

3.2. New exact solutions using enhanced generalized Riccati equation mapping method

Now, the Eqs. $(26\ \&\ 55)$ are substituted into Eq. (55) and we obtained the following equations with the help of Maple:

$$\begin{aligned}&-ca_1-ca_1\phi\left(\xi\right)+\frac{1}{3}aa_0^3+aa_0^2a_1\phi\left(\xi\right)+aa_0a_1^2\phi^2\left(\xi\right)+\frac{1}{3}aa_1^3\phi^3\left(\xi\right)+a_1\beta\beta_1\beta_0+a_1\beta\beta_1^2\phi\left(\xi\right)+3a_1\beta\beta_1\beta_2\phi^2\left(\xi\right)\\&+2a_1\beta\beta_2\phi\left(\xi\right)\beta_0+2a_1\beta\beta_2^2\phi^3\left(\xi\right)-a_1bc\beta_1\beta_0-a_1bc\beta_1^2\phi\left(\xi\right)-3a_1bc\beta_1\beta_2\phi^2\left(\xi\right)\\&-2a_1bc\beta_2\phi\left(\xi\right)\beta_0-2a_1bc\beta_2^2\phi^3\left(\xi\right)=0.\end{aligned} \tag{82}$$

By collecting the various coefficients of $\phi^{\mathrm{i}}(\xi)$, we obtain

$$\phi^0\left(\xi\right):\ -\mathrm{c}a_0+\frac{1}{3}aa_0^3+a_1\beta\beta_1\beta_0-a_1bc\beta_1\beta_0=0, \tag{83}$$

$$\phi^1\left(\xi\right):\ -\mathrm{c}a_1+aa_0^2a_1+a_1\beta\beta_1^2+2a_1\beta\beta_2\beta_0-a_1bc\beta_1^2-2a_1bc\beta_2\beta_0=0, \tag{84}$$

$$\phi^2\left(\xi\right):\ \mathrm{a}a_0a_1^2+3a_1\beta\beta_1\beta_2-3a_1bc\beta_1\beta_2=0, \tag{85}$$

$$\phi^3\left(\xi\right):\ \frac{1}{3}aa_1^3+2a_1\beta\beta_2^2-2a_1bc\beta_2^2=0. \tag{86}$$

Solving the above system of Eq. (83)-(86) with the help of Maple, we get the following coefficients involved in series (55)

$$a_0=\pm\,\frac{3\beta_1\left(-\beta+bc\right)}{a\sqrt{-\frac{6\beta-6bc}{a}}}, \tag{87}$$

$$a_1=\pm\sqrt{-\frac{6\beta-6bc}{a}}\beta_2, \tag{88}$$

$$\beta_1=\beta_1, \tag{89}$$

$$\beta_2=\beta_2, \tag{90}$$

$$\beta_0=\frac{1}{4}\frac{bc\beta_1^2-2\mathrm{c}-\beta\beta_1^2}{\beta_2(-\beta+bc)}. \tag{91}$$

Using Eq. $(86-90)$ in combination with Eq. (27), (51) and (55) we get the following solutions

1. For $\rho=\beta_1^2-4\beta_0\beta_1>0$, $\beta_1\beta_2\neq0$ or $\beta_0\beta_2\neq0$, the trigonometric hyperbolic form solutions of Eq. (1) are

$$\mathrm{u}_{17}\left(\mathrm{x},\mathrm{t}\right)=\pm\ \frac{3\beta_1\left(\beta+bc\right)}{\sqrt{-6\mathrm{a}\left(\beta+bc\right)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{\rho}}{2}\tanh\left(\frac{\sqrt{\rho}}{2}\left(\xi+C\right)\right)\right)-\frac{\beta_1}{2}, \tag{92}$$

$$\mathrm{u}_{18}^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\pm\ \frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{\rho}}{2}\coth\left(\frac{\sqrt{\rho}}{2}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right), \tag{93}$$

$$\mathrm{u}_{19}^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{\rho}}{2}\left(\tanh\left(\sqrt{\rho}\left(\xi+C\right)\right)\pm i\mathrm{sech}\left(\sqrt{\rho}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right)\right), \tag{94}$$

$$\mathrm{u}_{20}^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{\rho}}{2}\left(\coth\left(\sqrt{\rho}\left(\xi+C\right)\right)\pm\mathrm{csch}\left(\sqrt{\rho}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right)\right), \tag{95}$$

$$\mathrm{u}_{21}^{\pm}(\mathrm{x},\mathrm{t})=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}\left(\beta+bc\right)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{\rho}}{4}\left(\tanh\left(\frac{\sqrt{\rho}}{4}\left(\xi+C\right)\right)+\coth\left(\frac{\sqrt{\rho}}{4}\left(\xi+C\right)\right)\right)\right)-\frac{\beta_1}{2}. \tag{96}$$

2. For $\rho=\beta_1^2-4\beta_0\beta_2<0$, $\beta_1\beta_2\neq0$ or $\beta_0\beta_2\neq0$, the trigonometric form solutions of Eq. (1) are as follows:

$$\mathrm{u}_{22}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{-\rho}}{2}\tan\left(\frac{\sqrt{-\rho}}{2}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right), \tag{97}$$

$$\mathrm{u}_{23}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{-\rho}}{2}\cot\left(\frac{\sqrt{-\rho}}{2}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right), \tag{98}$$

$$\mathrm{u}_{24}^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{-\rho}}{2}\left(\tan\left(\sqrt{-\rho}\left(\xi+C\right)\right)\pm\sec\left(\sqrt{-\rho}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right)\right), \tag{99}$$

$$\mathrm{u}_{25}^{\pm}\left(\mathrm{x},\mathrm{t}\right)=\pm\frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{-\rho}}{2}\left(\cot\left(\sqrt{-\rho}\left(\xi+C\right)\right)\pm\csc\left(\sqrt{-\rho}\left(\xi+C\right)\right)-\frac{\beta_1}{2}\right)\right), \tag{100}$$

$$\mathrm{u}_{26}\left(\mathrm{x},\mathrm{t}\right)=\pm\ \frac{3\beta_1\left(-\beta+bc\right)}{\sqrt{-6\mathrm{a}(\beta+bc)}}\pm\sqrt{-\frac{6\beta-6bc}{a}}\left(\frac{\sqrt{-\rho}}{4}\left(\tan\left(\frac{\sqrt{-\rho}}{2}\left(\xi+C\right)\right)-\cot\left(\frac{\sqrt{-\rho}}{4}\left(\xi+C\right)\right)\right)-\frac{\beta_1}{2}\right). \tag{101}$$

4. Figures and discussion of the solutions

In this section, we have plotted the graphs of the solitary wave solutions. At the moment, we assigned a set of appropriate values to obtain different soliton structures. Moreover, for the Sardar sub-equation solutions, and also for justification, we used $\beta=1,\ a=0.1,\ b=2,\ c=4$,$b_1=4$, β_1=β_3=1, $\beta_2=-0.5$, and $C=1$ uniformly to plot Figs. 1-5. Similarly, for the solutions derived via Ricatti, we $\beta=1,\ a=0.1,\ b=2,\ c=4$,$b_1=4$, β_1=$\beta_3=1$, $\beta_2=-0.5$, $\rho=1$, and $C=1$. We finally derived the following soliton structures.

The recovered soliton structures in Figs. 1-24 for both approaches included singular, dark, bright, kink, anti-kink, and mixed solitons. For example, $\mathrm{u}_1\left(\mathrm{x},\mathrm{t}\right)$ and $\mathrm{u}_2\left(\mathrm{x},\mathrm{t}\right)$ correspond to kink soliton solutions, $\mathrm{u}_3\left(\mathrm{x},\mathrm{t}\right)$ corresponds to bright soliton solution, $\mathrm{u}_5\left(\mathrm{x},\mathrm{t}\right)$ and $\mathrm{u}_{17}\left(\mathrm{x},\mathrm{t}\right)$ correspond to dark soliton solutions, $\mathrm{u}_6\left(\mathrm{x},\mathrm{t}\right)$ and $\mathrm{u}_{18}\left(\mathrm{x},\mathrm{t}\right)$ correspond to singular soliton

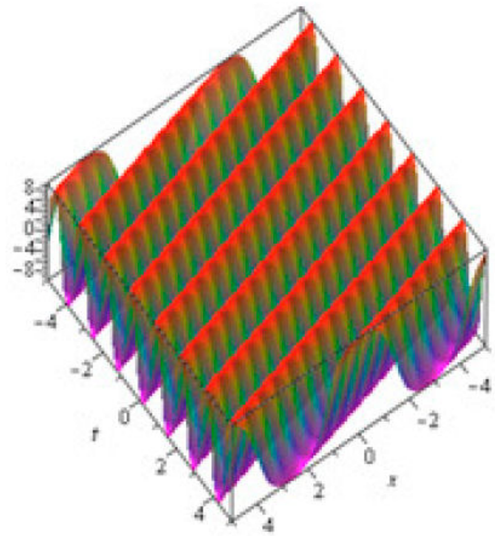

FIGURE 1. The 3D plot of $\mathrm{Re}\left(u_2^{\pm}(x,t)\right)$.

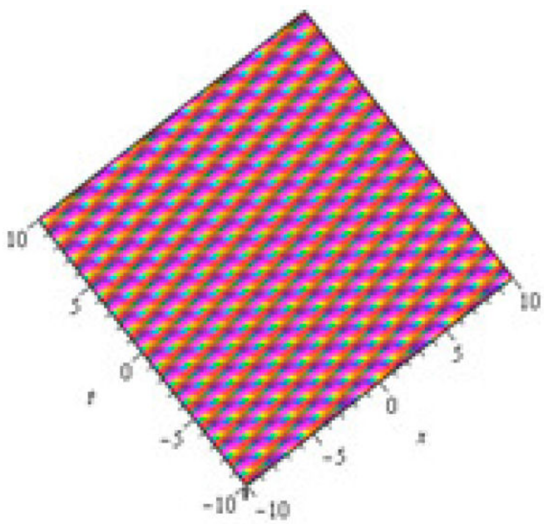

FIGURE 2. The contour plot of $\mathrm{Re}\left(u_2^{\pm}(x,t)\right)$.

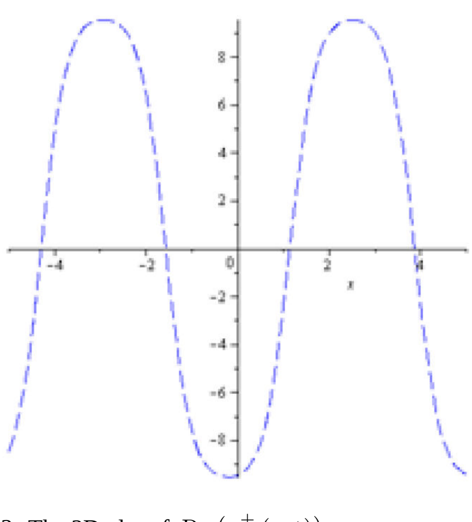

FIGURE 3. The 2D plot of $\mathrm{Re}\left(u_2^{\pm}(x,t)\right)$.

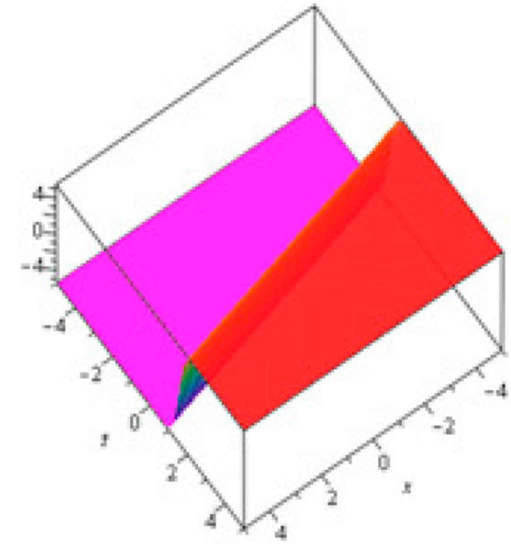

FIGURE 4. The 3D plot of $\mathrm{Re}\left(u_{13}^{\pm}(x,t)\right)$.

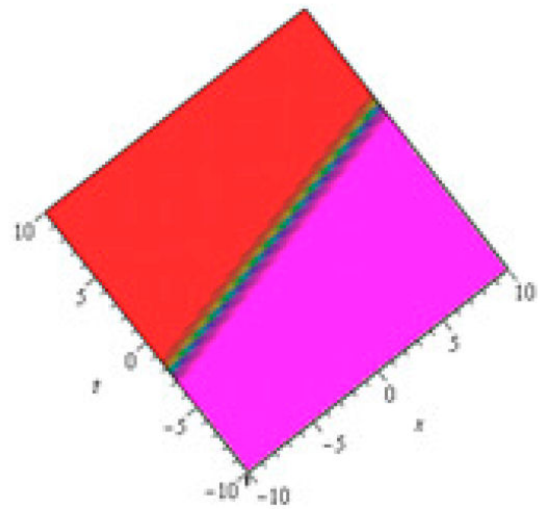

FIGURE 5. The contour plot of $\mathrm{Re}\left(u_{13}^{\pm}(x,t)\right)$.

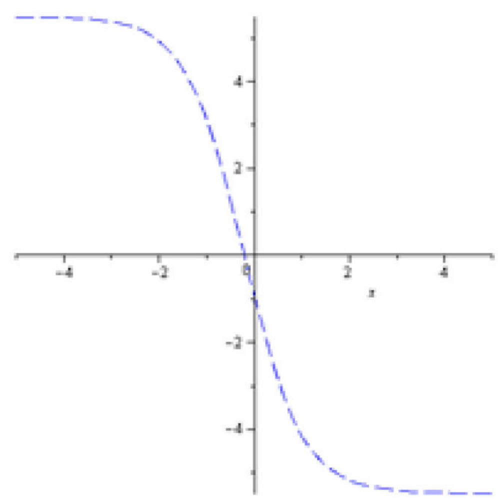

FIGURE 6. The 2D plot of $\mathrm{Re}\left(u_{13}^{\pm}(x,t)\right)$.

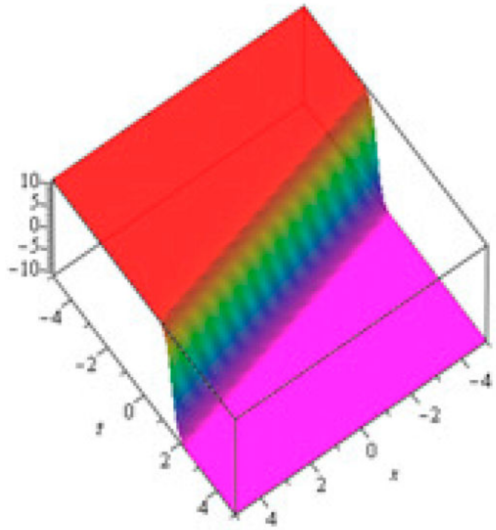

FIGURE 7. The 3D plot of $\mathrm{Re}\left(\mathrm{u}_6^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right)$.

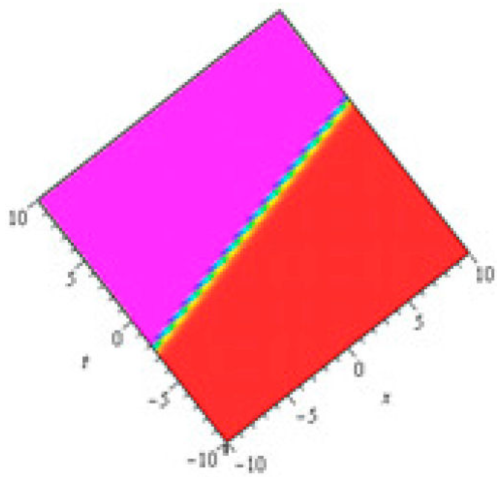

FIGURE 8. The contour plot of $\mathrm{Re}\left(\mathrm{u}_6^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right)$.

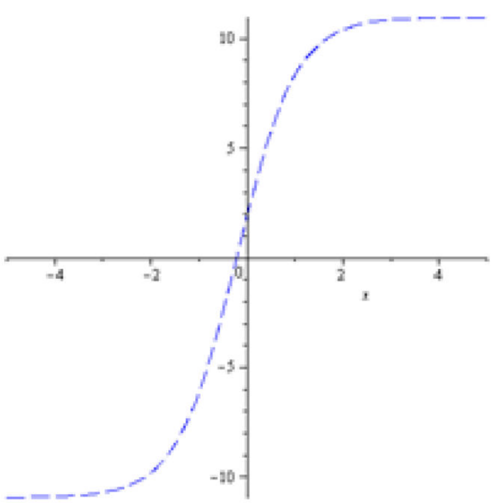

FIGURE 9. The 2D plot of $\mathrm{Re}\left(\mathrm{u}_6^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right)$.

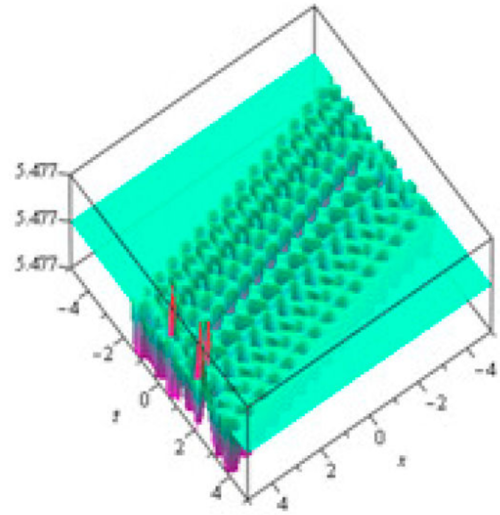

FIGURE 10. The 3D plot of $\left|\mathrm{u}_{15}^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right|$.

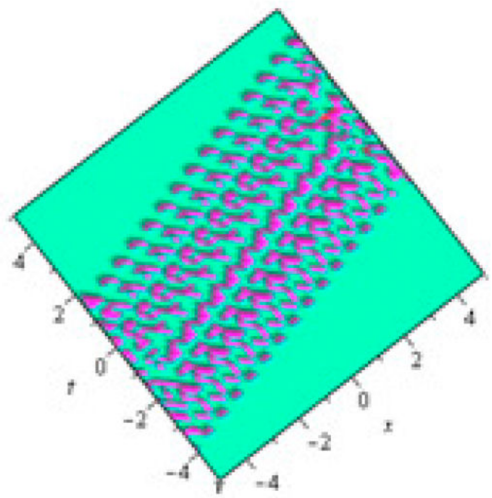

FIGURE 11. The contour plot of $\left|\mathrm{u}_{15}^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right|$.

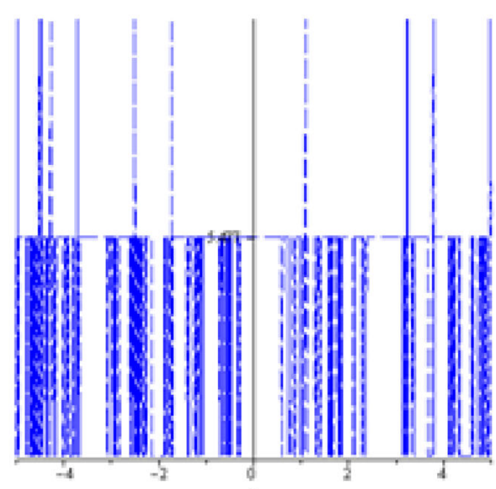

FIGURE 12. The 2D plot of $\left|\mathrm{u}_{15}^{\pm}\left(\mathrm{x},\mathrm{t}\right)\right|$.

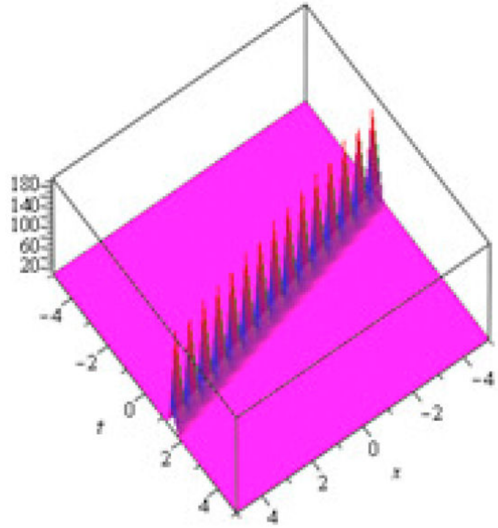

FIGURE 13. The 3D plot of $\left|u_{16}^{\pm}(x,t)\right|$.

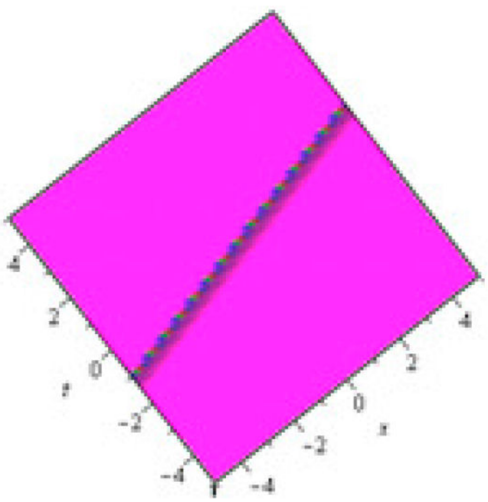

FIGURE 14. The contour plot of $\left|u_{16}^{\pm}(x,t)\right|$.

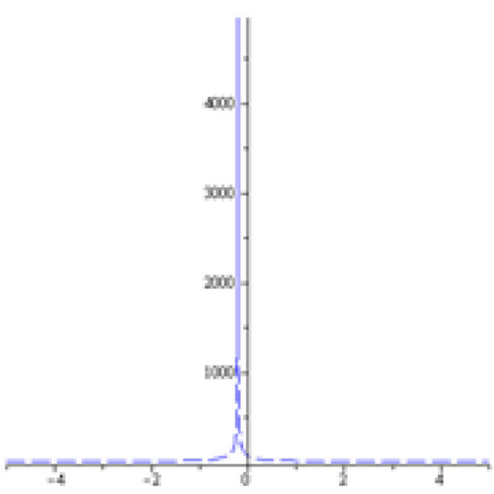

FIGURE 15. The 2D plot of $\left|u_{16}^{\pm}(x,t)\right|$.

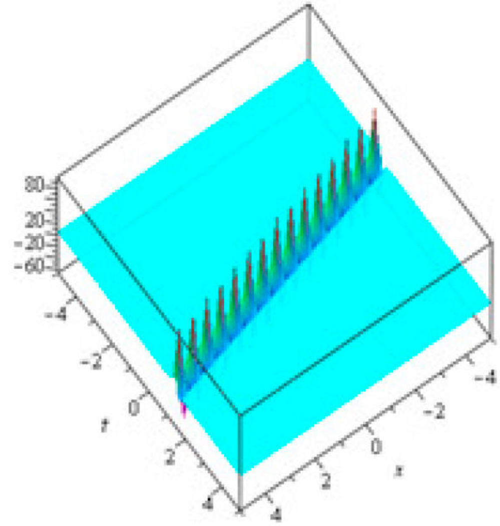

FIGURE 16. The 3D plot of $\mathrm{Re}\left(u_{12}^{\pm}(x,t)\right)$.

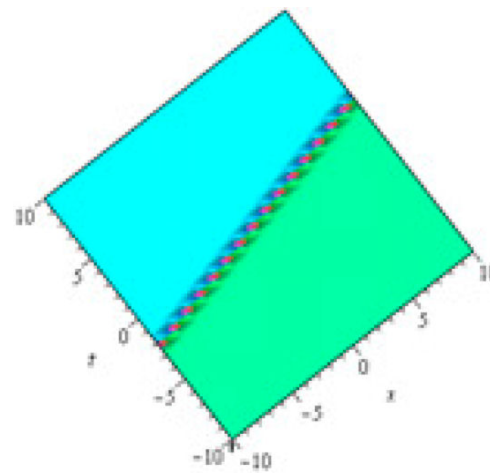

FIGURE 17. The contour plot of $\mathrm{Re}\left(u_{12}^{\pm}(x,t)\right)$.

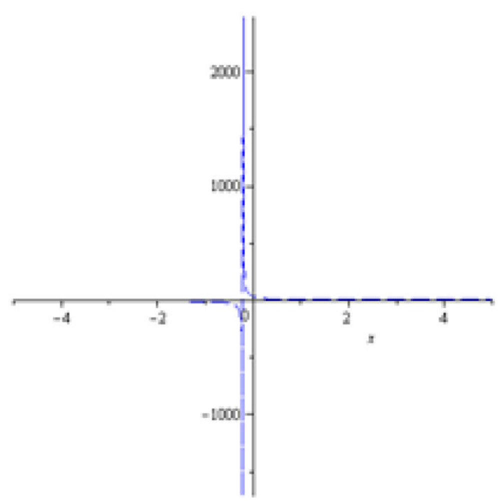

FIGURE 18. The 2D plot of $\mathrm{Re}\left(u_{12}^{\pm}(x,t)\right)$.

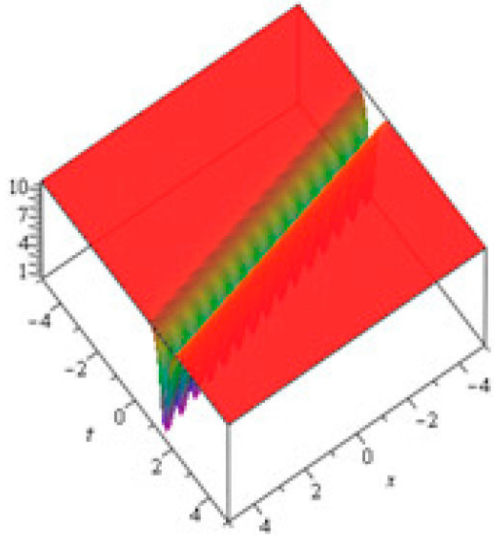

FIGURE 19. The 3D plot of $\left|u_{17}^{\pm}(x,t)\right|$.

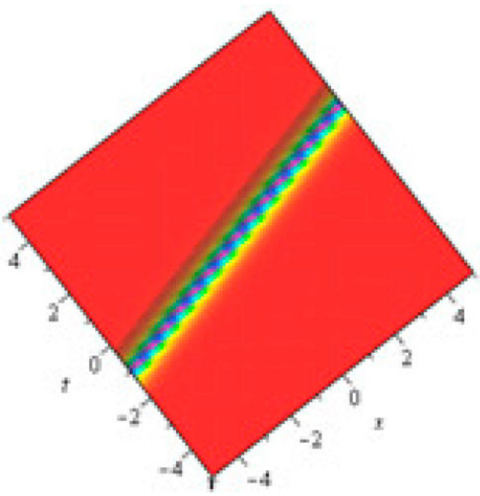

FIGURE 20. The contour plot of $\left|u_{17}^{\pm}(x,t)\right|$.

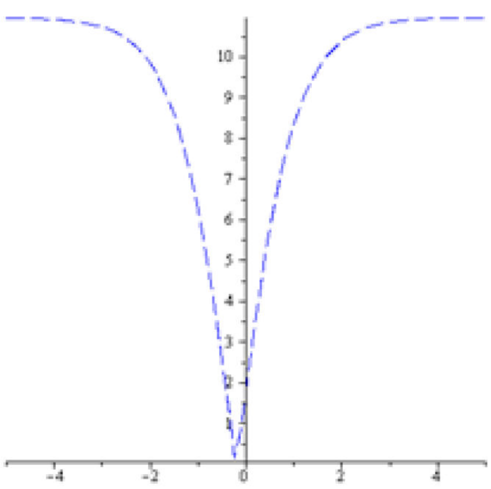

FIGURE 21. The 2D plot of $\left|u_{17}^{\pm}(x,t)\right|$.

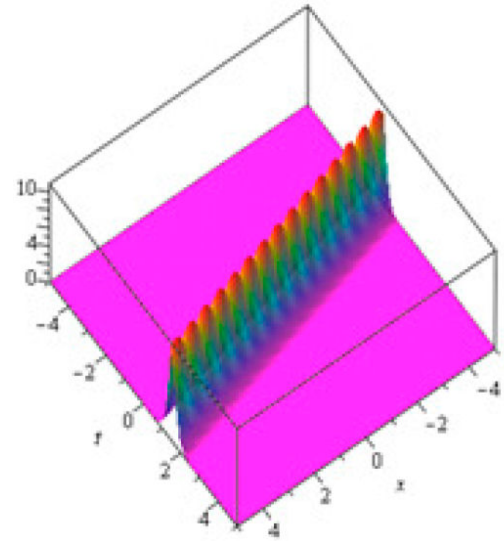

FIGURE 22. The 3D plot of $\mathrm{Im}\left(u_{24}^{\pm}(x,t)\right)$.

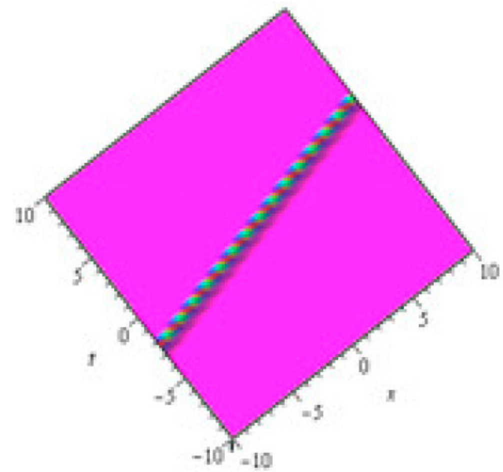

FIGURE 23. The contour plot of $\mathrm{Im}\left(u_{24}^{\pm}(x,t)\right)$.

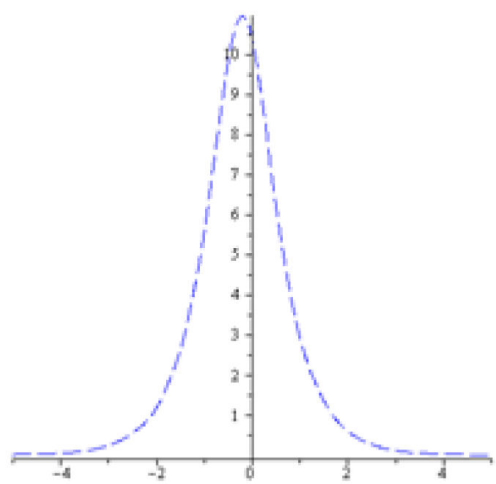

FIGURE 24. The 2D plot of $\mathrm{Im}\left(u_{24}^{\pm}(x,t)\right)$.

solutions, $u_7(x,t)$ and $u_{19}(x,t)$ correspond Bright-dark soliton solutions, $u_9(x,t)$ and $u_{21}(x,t)$ correspond dark-singular soliton solutions, and $u_{10}(x,t)$ corresponds to periodic soliton solutions. The structures in Figs. 1-8 are extremely useful in mathematical physics. Similarly, the same structures and beyond can be obtained using the appropriate values on the solutions derived via the two improved methods.

5. Conclusion

In this paper, we categorically emphasized the effectiveness and generality of the two well-known, well-established, and classified techniques. Therefore, we employed the improved and modified Sardar sub-equation approach and improved generalized Riccati equation mapping method to investigate and analyze the new formats of exact solutions to the nonlinear improved mKdV equation. The techniques have been incorporated gently and applied to this well-known equation. However, the set of solutions, obtained by these techniques, has multiple and popular types of the form, *i.e*. rational, exponential, trigonometric, and trigonometric hyperbolic solutions of the improved mKdV equation. The methods are quick and highly effective in nature, whereas, in the first phase, we used the wave variable to transform the NPDEs into nonlinear ordinary differential equations (ODEs) with integer order after adapting the most general and simple techniques the IMSSEM and IGREMM are used to construct the novel solutions of the improved mKdV equation. Our findings imply that the approach is a strong, well-defined algorithm that is exceedingly efficient. It is confirmed from the profiles (soliton structures for both approaches) that the novel solutions preserved the qualities of singular, dark, bright, kink, anti-kink, and mixed solitons. Therefore, these methods applicable to solve various nonlinear PDEs arise in different areas of research and soliton. Additionally, these results may be helpful in the KdV equations family and application in engineering and mathematical science. We also presented a direct-viewing analysis by providing both two-dimensional and three-dimensional solution figures. Future studies will also concentrate on several fascinating findings connected to the suggested model, such as the physical feasibility, modulational stability, and the analysis of the lie symmetry of the solutions.

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