Application of the SU$(1,1)$ spinors in the study of the Lorentz transformations

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We show that the orthochronous proper Lorentz transformations that preserve the condition $z = 0$ can be parametrized by (two-component) SU$(1,1)$ spinors in such a way that the Wigner angle associated with a pair of non-collinear boosts is given by one of the scalar products defined between these spinors.

Keywords: Lorentz transformations; SU$(1,1)$ spinors; Wigner angle; hyperbolic geometry; holonomy angle.

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1. Introduction

In a recent paper [1] the group of proper orthochronous Lorentz transformations that leave one of the spatial coordinates fixed was represented by 2×2 matrices whose entries are double numbers (also known as hyperbolic numbers). This representation is useful if one is interested in Lorentz transformations involving one or two spatial directions only.

As is well known, the composition of two pure boosts in two different directions is equivalent to the composition of a boost and a spatial rotation through an angle known as the Wigner angle (see, *e.g.*, Ref. [2] and the references cited therein). In Ref. [1] the Wigner angle was calculated making use of the representation mentioned above. Actually, the representation obtained in Ref. [1] is one of the group SO$(2,1)$, which can be regarded as a subgroup of the Lorentz group, SO$(3,1)$. In the same manner as the SO(3) transformations can be viewed as the Lorentz transformations maintaining the condition $t = 0$, the SO$(2,1)$ transformations correspond to the Lorentz transformations maintaining, *e.g.*, $z = 0$. Besides the representation given in Ref. [1], the group SO$(2,1)$ can also be represented by complex 2×2 matrices (belonging to the group SU$(1,1)$) and by real 2×2 matrices (belonging to SL$(2,\mathbb{R})$) (see, *e.g.*, Refs. [3,4]).

The existence of the Wigner angle is the basis of the Thomas precession, which is relevant in atomic and nuclear physics, explaining the observed fine-structure intervals and the spin-orbit interaction in atomic nuclei, respectively.

The aim of this paper is to show explicitly that the orthochronous proper Lorentz transformations maintaining $z = 0$ can be represented by (two-component) SU$(1,1)$ spinors or, equivalently, that the inertial frames sharing the origin and the z-axis can be represented by SU$(1,1)$ spinors. We also show that this correspondence can be employed to find the Wigner angle making use of one scalar product between spinors.

The spinor formalism is a very useful tool in the standard general relativity, which deals with four-dimensional curved space-times (see, *e.g.*, Refs. [5, 6]). Similarly, the SU$(1,1)$ spinors employed here are useful in the study of $2+1$ space-times (see, *e.g.*, Ref. [4]), which, even though are not the main objective in the theory of gravitation, are of interest as guides for the more realistic case (see, *e.g.*, Ref. [7]). The group SU$(1,1)$ is also of interest in connection with squeezed light and polarization optics, and the Wigner angle has analogs in those fields (see, *e.g.*, Refs. [8,9]).

In Sec. 2 we summarize some basic facts about the representation of SO$(2,1)$ by SU$(1,1)$ and the SU$(1,1)$ spinors, and we show that there exist two scalar products between SU$(1,1)$ spinors. In Sec. 3 we show that each spinor represents an inertial frame of the restricted class mentioned above and we relate the Wigner angle with one of the scalar products between spinors. As a second application we derive the standard formulas for the relativistic Doppler effect and the aberration of light making use of the fact that a null vector with one of its spatial components equal to zero can be represented by a SU$(1,1)$ spinor. In Sec. 4 the connection with the hyperbolic geometry is briefly discussed.

2. Representation of SO$(2,1)$

The homogeneous Lorentz group can be defined as the set of linear transformations leaving

$$x^2 + y^2 + z^2 - (ct)^2$$

invariant, where (x, y, z, ct) are the coordinates of an event with respect to some inertial frame. Hence, the homogeneous Lorentz transformations that leave, *e.g.*, $z = 0$ invariant, also leave

$$x^2 + y^2 - (ct)^2 \tag{1}$$

invariant. The linear transformations leaving (1) invariant form the group O$(2,1)$.

Apart from the natural representation of O$(2,1)$ by real 3×3 matrices there exist representations of this group, or its subgroups, by 2×2 matrices (see, *e.g.*, Refs. [1,3,4]). In this section we show, in an elementary manner, that some O$(2,1)$ transformations (namely, those with positive determinant belonging to the connected component of the identity) can be represented by complex 2×2 matrices (a detailed discussion can be found in Refs. [3,4]).

As in Refs. [1,10], we start by establishing a one-to-one correspondence between points and certain 2×2 matrices. In the present case we associate the point with (real) coordinates (x, y, ct) with the 2×2 complex matrix

$$P = \begin{pmatrix} ct & -x - \mathrm{i}y \\ x - \mathrm{i}y & -ct \end{pmatrix}. \tag{2}$$

Then, one can readily verify that

$$\operatorname{tr} P = 0, \qquad P^\dagger = \eta P \eta, \tag{3}$$

where

$$\eta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and that any complex 2×2 matrix satisfying Eqs. (3) must be of the form (2). Thus, there is a one-to-one correspondence between points (x, y, ct) and matrices (2), which is employed below.

Since $\det P = x^2 + y^2 - (ct)^2$, and the determinant is invariant under similarity transformations, the transformation $P \mapsto P'$ given by

$$P' = QPQ^{-1}, \tag{4}$$

where P' is the matrix (2) corresponding to (x', y', ct'), represents an $\mathrm{O}(2,1)$ transformation provided that $P'^\dagger = \eta P' \eta$ [see Eqs. (3)], that is, $(Q^{-1})^\dagger P^\dagger Q^\dagger = \eta\,(QPQ^{-1})\,\eta$, or $(Q^\dagger)^{-1} \eta P \eta Q^\dagger = \eta Q P Q^{-1} \eta$, which is satisfied if $\eta Q^\dagger = Q^{-1}\eta$ or, equivalently,

$$Q \eta Q^\dagger = \eta. \tag{5}$$

(Note that the condition $\operatorname{tr} P = 0$ is preserved by any similarity transformation (4).)

Taking the determinant on both sides of Eq. (5), using the fact that $\det Q^\dagger$ is the conjugate of $\det Q$, one finds that $\det Q = \mathrm{e}^{\mathrm{i}\alpha}$, for some $\alpha \in \mathbb{R}$, then the matrix $\tilde{Q} = \mathrm{e}^{-\mathrm{i}\alpha/2} Q$ also satisfies Eq. (5) and $\det \tilde{Q} = 1$. Furthermore, $\tilde{Q} P \tilde{Q}^{-1} = QPQ^{-1}$, which means that Q and $\tilde{Q}$ produce the same $\mathrm{O}(2,1)$ transformation. Hence, we can restrict ourselves to matrices with determinant equal to 1 that satisfy Eq. (5). These matrices form the group $\mathrm{SU}(1,1)$. (The group $\mathrm{SU}(p,q)$ is formed by que $(p+q) \times (p+q)$ complex matrices, Q, with determinant equal to 1 such that $Q\eta Q^\dagger = \eta$, where η is the diagonal matrix with p entries equal to $+1$ and q entries equal to -1.) It may be noticed that if Q belongs to $\mathrm{SU}(1,1)$ then $-Q$ also belongs to this group and from Eq. (4) it follows that Q and $-Q$ produce the same $\mathrm{O}(2,1)$ transformation.

It can be shown that the transformations (4) with $Q \in \mathrm{SU}(1,1)$ correspond to $\mathrm{SO}(2,1)$ transformations (that is, transformations belonging to $\mathrm{O}(2,1)$ with determinant equal to 1) [3,4]. For our purposes it suffices to verify that the transformations (4) with $Q \in \mathrm{SU}(1,1)$ reproduce the pure boosts in arbitrary directions (in the xy-plane) and the rotations in the xy-plane. In fact, a straightforward computation shows that a rotation through an angle ϕ on the xy-plane is given by (4) with Q given by

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \tag{6}$$

or its negative. Similarly, for a pure boost in an arbitrary direction in the xy-plane which makes an angle θ with the x-axis, Q is given by

$$\begin{pmatrix} \cosh w/2 & -\mathrm{e}^{\mathrm{i}\theta} \sinh w/2 \\ -\mathrm{e}^{-\mathrm{i}\theta} \sinh w/2 & \cosh w/2 \end{pmatrix} \tag{7}$$

or its negative, with the relative velocity, v, given by

$$v = c \tanh w. \tag{8}$$

The parameter w is called rapidity. Then, from Eq. (4) it follows that the matrix corresponding to the composition of a pure boost, in the direction making an angle θ with the x-axis, followed by a rotation through the angle ϕ is given by the product of (6) (as the factor on the left) by (7), namely

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \cosh w/2 & -\mathrm{e}^{-\mathrm{i}\phi/2} \mathrm{e}^{\mathrm{i}\theta} \sinh w/2 \\ -\mathrm{e}^{\mathrm{i}\phi/2} \mathrm{e}^{-\mathrm{i}\theta} \sinh w/2 & \mathrm{e}^{\mathrm{i}\phi/2} \cosh w/2 \end{pmatrix}. \tag{9}$$

2.1. Two-component spinors. Scalar products

The 2×2 matrices $Q \in \mathrm{SU}(1,1)$ act on two-component vectors

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad (u, v \in \mathbb{C}),$$

called spinors (or $\mathrm{SU}(1,1)$ spinors), in the simple manner: $\psi \mapsto Q\psi$ (see also Eq. (25) below). As a consequence of (5), the scalar product of the spinors χ and ψ defined by $\chi^\dagger \eta \psi$ is invariant under the action of $\mathrm{SU}(1,1)$. In fact, $\chi'^\dagger \eta \psi' = (Q\chi)^\dagger \eta (Q\psi) = \chi^\dagger (Q^\dagger \eta Q) \psi = \chi^\dagger \eta \psi$, since (5) is equivalent to $Q^\dagger \eta Q = \eta$. (This can be proved in the following way: since η^2 is the unit matrix, Eq. (5) implies that $Q^{-1} = \eta Q^\dagger \eta$, hence $(\eta Q^\dagger \eta) Q$ is the unit matrix and multiplying on the left by η one gets the desired relation.)

There is a second scalar product between spinors that can be constructed by defining first the *mate* of a spinor. A straightforward computation shows that any matrix belonging to $\mathrm{SU}(1,1)$ is of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{10}$$

where $\overline{\alpha}$ is the complex conjugate of α, and α, β are complex numbers such that $|\alpha|^2 - |\beta|^2 = 1$ (*cf.* Ref. [10]). (The complex numbers α and β are analogous to the Cayley–Klein parameters employed in the case of the $\mathrm{SO}(3)$ transformations, see, *e.g.*, Ref. [10] and the references cited therein.) Hence, if $\psi' = Q\psi$, with

$$\psi' = \begin{pmatrix} u' \\ v' \end{pmatrix}, \tag{11}$$

we have, in explicit form,

$$\begin{aligned} u' &= \alpha u + \beta v, \\ v' &= \overline{\beta} u + \overline{\alpha} v, \end{aligned} \tag{12}$$

which is equivalent to

$$\begin{aligned} \overline{v'} &= \alpha \overline{v} + \beta \overline{u} \\ \overline{u'} &= \overline{\beta} \overline{v} + \overline{\alpha}\, \overline{u}. \end{aligned} \tag{13}$$

Comparing (13) with (12) one concludes that the mate of ψ, defined by

$$\widehat{\psi} \equiv \begin{pmatrix} \overline{v} \\ \overline{u} \end{pmatrix}, \tag{14}$$

transforms in the same way as ψ [3], hence, for any pair of spinors χ and ψ we can form another scalar product: $\widehat{\chi}^\dagger \eta \psi$ (also invariant under the action of SU$(1,1)$). (The definition (14) differs from that given in Ref. [4] by a constant factor.) One can readily verify that $\chi^\dagger \eta \psi = \overline{\psi^\dagger \eta \chi}$ and $\widehat{\psi}^\dagger \eta \chi = -\widehat{\chi}^\dagger \eta \psi$.

3. Representation of the Lorentz transformations by spinors

Given a two-component spinor

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix},$$

with $|u|^2 - |v|^2 = 1$ (that is, $\psi^\dagger \eta \psi = 1$), which we shall call a *unit* spinor, there exists a unique $Q \in \mathrm{SU}(1,1)$ such that

$$\psi = Q^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or, equivalently,

$$Q\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{15}$$

In fact, Q is explicitly given by

$$\begin{pmatrix} \overline{u} & -\overline{v} \\ -v & u \end{pmatrix}$$

[see Eq. (10)] and since any $Q \in \mathrm{SU}(1,1)$ gives rise to a SO$(2,1)$ transformation, which represents a Lorentz transformation of a restricted class, it follows that any unit spinor defines a proper orthochronous Lorentz transformation that leaves invariant the z-axis.

Hence, starting with an inertial frame, S_0, any other inertial frame, $\tilde{\mathrm{S}}$, sharing the z-axis with S_0, is represented by a unit spinor (defined up to sign). S_0 itself is represented by

$$\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{16}$$

or its negative (since the Lorentz transformation leading from S_0 to S_0 is the identity) and, according to Eqs. (9) and (15), the unit spinor representing $\tilde{\mathrm{S}}$ is

$$\xi = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\tilde{\phi}/2} \cosh \tilde{w}/2 \\ \mathrm{e}^{\mathrm{i}\tilde{\phi}/2} \mathrm{e}^{-\mathrm{i}\tilde{\theta}} \sinh \tilde{w}/2 \end{pmatrix} \tag{17}$$

or its negative, if $\tilde{\mathrm{S}}$ is moving with respect to S_0 with velocity $c \tanh \tilde{w}$ in the direction making an angle $\tilde{\theta}$ with the x-axis, and the axes $\tilde{x}$ and $\tilde{y}$ are rotated an angle $\tilde{\phi}$ with respect to the axes x and y. The scalar products $\chi^\dagger \eta \xi$ and $\widehat{\chi}^\dagger \eta \xi$ are

$$\chi^\dagger \eta \xi = \mathrm{e}^{\mathrm{i}\tilde{\phi}/2} \cosh \tilde{w}/2 \tag{18}$$

and

$$\widehat{\chi}^\dagger \eta \xi = -\mathrm{e}^{\mathrm{i}\tilde{\phi}/2} \mathrm{e}^{-\mathrm{i}\tilde{\theta}} \sinh \tilde{w}/2. \tag{19}$$

Even though the spinors χ and ξ depend on the choice of the inertial frame S_0, the scalar products (18) and (19), being invariant under the SU$(1,1)$ transformations, only depend on the inertial frames represented by χ and ξ (in the same manner as the position vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$, of two points of the Euclidean space depend on the choice of the origin, but the distance $|\mathbf{r}_2 - \mathbf{r}_1|$ is independent of that choice).

Thus, given two inertial frames S_0 and $\tilde{\mathrm{S}}$ sharing the z-axis, they can be represented by unit spinors χ and ξ, respectively, and, according to Eq. (18), the modulus of the scalar product $\chi^\dagger \eta \xi$ determines the velocity of $\tilde{\mathrm{S}}$ with respect to S_0, while the argument of $\chi^\dagger \eta \xi$ determines the angle between the axes $\tilde{x}, \tilde{y}$ and the axes x, y. The modulus of $\widehat{\chi}^\dagger \eta \xi$ also gives the relative velocity of the two frames, but its argument does not give the angles $\tilde{\phi}$ and $\tilde{\theta}$ separately [see Eq. (19)].

3.1. The Wigner angle

Now, let S and S$'$ be two inertial frames moving with respect to S_0 with velocities v and v' in the xy-plane which make angles θ and θ' with respect to the x-axis (of S_0), respectively, with the spatial axes of S parallel to those of S_0 and, similarly, the spatial axes of S$'$ parallel to those of S_0; in other words, S and S$'$ are related to S_0 by means of pure boosts and these boosts are represented (with respect to S_0) by the unit spinors

$$\psi = \begin{pmatrix} \cosh w/2 \\ \mathrm{e}^{-\mathrm{i}\theta} \sinh w/2 \end{pmatrix} \quad \text{and}$$

$$\psi' = \begin{pmatrix} \cosh w'/2 \\ \mathrm{e}^{-\mathrm{i}\theta'} \sinh w'/2 \end{pmatrix},$$

respectively, with $v = c \tanh w$ and $v' = c \tanh w'$ [see Eq. (17)].

Then, the scalar products $\psi^\dagger \eta \psi'$ and $\widehat{\psi}^\dagger \eta \psi'$ are

$$\begin{aligned} \psi^\dagger \eta \psi' &= \cosh w/2 \cosh w'/2 \\ &\quad - \mathrm{e}^{\mathrm{i}(\theta - \theta')} \sinh w/2 \sinh w'/2 \end{aligned} \tag{20}$$

and

$$\begin{aligned} \widehat{\psi}^\dagger \eta \psi' &= \mathrm{e}^{-\mathrm{i}\theta} \sinh w/2 \cosh w'/2 \\ &\quad - \mathrm{e}^{-\mathrm{i}\theta'} \cosh w/2 \sinh w'/2. \end{aligned} \tag{21}$$

By equating Eqs. (18) and (20), we conclude that, with respect to S, the inertial frame S′ is moving with rapidity $\tilde{w}$, and with its axes rotated by an angle $\tilde{\phi}$ (the Wigner angle), given by

$$\begin{aligned} e^{i\tilde{\phi}/2} \cosh \tilde{w}/2 &= \cosh w/2 \cosh w'/2 \\ &\quad - e^{i(\theta-\theta')} \sinh w/2 \sinh w'/2. \end{aligned} \tag{22}$$

One can readily verify that the real and imaginary parts of this equation are precisely Eqs. (20) and (21) of Ref. [1]. Similarly, the real and imaginary parts of the equation

$$\begin{aligned} -e^{i\tilde{\phi}/2} e^{-i\tilde{\theta}} \sinh \tilde{w}/2 &= e^{-i\theta} \sinh w/2 \cosh w'/2 \\ &\quad - e^{-i\theta'} \cosh w/2 \sinh w'/2, \end{aligned} \tag{23}$$

obtained by equating Eqs. (19) and (21), are precisely Eqs. (22) and (23) of Ref. [1].

3.2. Aberration of light and the relativistic Doppler effect

The aberration of light and the relativistic Doppler effect can be conveniently studied with the aid of the formalism introduced above. An electromagnetic plane wave is characterized by a wave four-vector k^α, which is null. Assuming that $k^3 = 0$, we are left with three possibly nonzero components, k^0, k^1 and k^2, such that $(k^1)^2 + (k^2)^2 - (k^0)^2 = 0$, hence $k^1 = k^0 \cos\alpha$, $k^2 = k^0 \sin\alpha$, where α is the angle between the propagation direction and the x-axis, and in this case the analog of the matrix (2) can be expressed in terms of a $\mathrm{SU}(1,1)$ spinor, κ,

$$\begin{pmatrix} k^0 & -k^1 - ik^2 \\ k^1 - ik^2 & -k^0 \end{pmatrix} = \begin{pmatrix} k^0 & -k^0 e^{i\alpha} \\ k^0 e^{-i\alpha} & -k^0 \end{pmatrix} = \kappa\kappa^\dagger\eta$$

with

$$\kappa \equiv \sqrt{k^0} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}. \tag{24}$$

The two-component spinor κ transforms according to

$$\alpha' = Q\alpha. \tag{25}$$

That is, in the case of the wave four-vector, the transformation rule (4) is a consequence of (25). Hence, for a boost along the x-axis [see Eq. (7)] the transformation (25) takes the form

$$\sqrt{k'^0} \begin{pmatrix} e^{i\alpha'/2} \\ e^{-i\alpha'/2} \end{pmatrix} = \begin{pmatrix} \cosh w/2 & -\sinh w/2 \\ -\sinh w/2 & \cosh w/2 \end{pmatrix} \times \sqrt{k^0} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}$$

which is equivalent to the single equation

$$\sqrt{k'^0} e^{i\alpha'/2} = \sqrt{k^0} \big(e^{i\alpha/2} \cosh w/2 - e^{-i\alpha/2} \sinh w/2 \big) \tag{26}$$

which, separating the modulus and the argument of the complex number on the left-hand side, amounts to

$$k'^0 = k^0 (\cosh w - \sinh w \cos\alpha) = k^0 \big(\gamma - \gamma(v/c) \cos\alpha \big)$$

and

$$\tan \tfrac{1}{2}\alpha' = e^w \tan \tfrac{1}{2}\alpha = \sqrt{\frac{1 + v/c}{1 - v/c}} \tan \tfrac{1}{2}\alpha. \tag{27}$$

It may be remarked that even though Eq. (27) is a well-known result, in the standard procedure, making use of the four-vector transformations, it is somewhat laborious to get it; and even with the two-component $\mathrm{SL}(2,\mathbb{C})$ spinors its derivation is not so straightforward.

4. Geometric interpretation

The spinor

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

defines two vectors, $\mathbf{R}_\psi$ and $\mathbf{M}_\psi$, with components R^1, R^2, R^3 and M^1, M^2, M^3 given by [4]

$$R^j = i\psi^\dagger \eta \tau^j \psi, \qquad M^j = i\hat{\psi}^\dagger \eta \tau^j \psi, \tag{28}$$

$j = 1, 2, 3$, with

$$\tau^1 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^3 \equiv \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{29}$$

that is

$$\begin{aligned} \mathbf{R}_\psi &= (u\overline{v} + v\overline{u}, -iu\overline{v} + iv\overline{u}, |u|^2 + |v|^2), \\ \mathbf{M}_\psi &= (u^2 + v^2, -iu^2 + iv^2, 2uv). \end{aligned} \tag{30}$$

(The matrices τ^j are a basis for the Lie algebra (over $\mathbb{R}$) of the traceless 2×2 complex matrices, A, such that $A^\dagger\eta = -\eta A$, which is the Lie algebra of $\mathrm{SU}(1,1)$. One can readily verify that the matrices (6) and (7) can be expressed in the form $\exp(\frac{1}{2}\phi\,\tau^3)$ and $\exp[\frac{1}{2}w(\sin\theta\,\tau^1 - \cos\theta\,\tau^2)]$, respectively.) The components R^j are real and the M^j are complex. The vectors R^j and M^j are orthogonal, in the sense that $g_{ij}R^iM^j = 0$, where

$$(g_{ij}) = \mathrm{diag}(1, 1, -1),$$

M^j is null, that is $g_{ij}M^iM^j = 0$, and if ψ is a unit spinor then R^i is normalized, $g_{ij}R^iR^j = -1$.

In the case of the unit spinor ξ given by (17), the real vectors $\mathbf{R}_\xi$ and $\mathrm{Re}\,\mathbf{M}_\xi$ are unit vectors along the axes $\tilde{t}$ and $\tilde{x}$ of $\tilde{\mathrm{S}}$ (with respect to the x, y, ct-axes of S_0). Since $g_{ij}R^iR^j = -1$ and $R^3 > 0$, the point (R^1, R^2, R^3) belongs to a sheet of a hyperboloid. With ξ parametrized as in (17), the point (R^1, R^2, R^3) is given by $(\sinh\tilde{w}\cos\tilde{\theta}, \sinh\tilde{w}\sin\tilde{\theta}, \cosh\tilde{w})$ and therefore w and θ can be used as local coordinates on the hypersurface $M \equiv \{(x, y, ct) \in \mathbb{R}^3 \,|\, x^2 + y^2 - (ct)^2 =$

$-1, ct > 0\}$. The (indefinite) metric tensor $g_{ij} \mathrm{d}x^i \mathrm{d}x^j = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 - (c\mathrm{d}t)^2$ induces a (positive definite) metric tensor on M given by

$$(\mathrm{d}w)^2 + \sinh^2 w\, (\mathrm{d}\theta)^2. \tag{31}$$

(This straightforward construction of the metric (31) can be compared with the one presented in Ref. [11].)

The meridians $\theta = \mathrm{const.}$ of M are geodesics of the metric (31). Hence, the distance (on M) between the point with $w = 0$ (corresponding to S_0) and the point $(\sinh \tilde{w} \cos \tilde{\theta}, \sinh \tilde{w} \sin \tilde{\theta}, \cosh \tilde{w})$ of M (corresponding to $\tilde{\mathrm{S}}$), along the geodesic joining these points, is $\tilde{w}$.

Thus, each point of M correspond to the time axis of some inertial frame and the geodesic distance between two points of M is the rapidity of one the frames with respect to the other. (It should be stressed that M with the metric (31) is a maximally symmetric space, with constant curvature, analogous to the standard sphere of $\mathbb{R}^3$; among other things, M is a homogeneous space, which is in agreement with the fact that all the inertial frames are equivalent.)

As in the case of the SU(2) and the SL$(2,\mathbb{C})$ spinors, the SU$(1,1)$ spinors can be represented by flags (see, *e.g.*, Refs. [4, 5]) and in all cases one spinor and its negative define the same flag. For a unit spinor ψ the flagpole is given by the vector $\mathbf{R}_\psi$ and the flag points along $\mathrm{Re}\,\mathbf{M}_\psi$. A straightforward computation shows that, for a unit spinor ξ parametrized in the form (17),

$$\mathrm{Re}\,\mathbf{M}_\xi = \cos(\tilde{\theta} - \tilde{\phi})\,\mathbf{e}_w - \sin(\tilde{\theta} - \tilde{\phi})\,\mathbf{e}_\theta, \tag{32}$$

where $\{\mathbf{e}_w, \mathbf{e}_\theta\}$ is the orthonormal basis of the tangent space of M (at $\mathbf{R}_\xi$) defined by the coordinates (w, θ) of M.

The flag corresponding to the unit spinor χ points along $(1, 0, 0)$ [see Eqs. (30)], which forms an angle $\tilde{\theta}$ with the tangent vector to the geodesic curve $\theta = \tilde{\theta}$, at the point $(0, 0, 1)$ of M, while the flag corresponding to ξ forms an angle $\tilde{\theta} - \tilde{\phi}$ with the tangent vector to this curve, at the point $\mathbf{R}_\xi$ of M [see Eq. (32)]. As is well known, when a vector is translated parallelly to itself along a geodesic, the angle between this vector and the tangent vector to the geodesic does not vary along the geodesic, hence, the angle, $\tilde{\phi}$, between the spatial axes of the inertial frames S_0 and $\tilde{\mathrm{S}}$, is the angle between the flag corresponding to $\tilde{\mathrm{S}}$ and the flag corresponding to S_0 parallelly translated to itself along the geodesic joining the points of M that represent these frames. (An entirely similar result holds in the case of the SU(2) spinors [12].)

In the case of the frames S_0, S and S$'$ considered in Sec. 3.1, we will have three points of M determined by the intersections of the time axes of the frames with M. These three points define a geodesic triangle (constructed joining each pair of points by a geodesic), with the distance between vertices [determined with the metric (31)] being the relative rapidity between the corresponding frames.

When the flag corresponding to S_0 is translated parallelly to itself along the geodesics going from S_0 to S and from S_0 to S$'$ it coincides with the flags corresponding to S and S$'$, respectively (by construction, since S and S$'$ are obtained by pure boosts from S_0), but when this flag is translated parallelly to itself from S to S$'$, the resulting flag forms an angle equal to the Wigner angle with the flag corresponding to S$'$. In differential geometry, this angle is called the holonomy angle of the closed curve (in this case the triangle) and, for a constant curvature hypersurface, such as M, this angle is equal to the curvature of the hypersurface multiplied by the area enclosed by the curve (see also Refs. [2, 11] and the references cited therein).

5. Concluding remarks

As is well known, the entire proper orthochronous Lorentz group can be represented by SL$(2,\mathbb{C})$ matrices (see, *e.g.*, Refs. [5, 6]), which amounts to say that, excluding spatial reflections and time inversions, each inertial frame can be identified with a *pair* of two-component spinors (frequently denoted by o and ι in the Newman–Penrose notation). The alternatives presented in Ref. [1] and this paper constitute a considerable simplification in the restricted case discussed here, where a single two-component spinor is enough.

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