The linear and the angular momentum stored in a distribution of charges in a magnetic field

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We show that it is possible to define, *e.g.*, the z-component of the linear momentum of the system formed by a charged particle and a magnetic field if and only if the magnetic field is invariant under translations along the z-axis. Similarly, it is possible to define the z-component of the angular momentum of the system formed by a charged particle and a magnetic field if and only if the magnetic field is invariant under rotations about the z-axis.

Keywords: Linear momentum; angular momentum; electromagnetic field; symmetries.

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1. Introduction

In the elementary treatment of electromagnetism, the energy stored in a bounded electric charge distribution is defined as the work needed to bring the charges from infinity to their final positions (see, *e.g.*, Refs. [1, 2]). In the computation one makes use of the fact that the electrostatic force is conservative and, therefore, the paths followed by the charges in order to get to their final positions are irrelevant. Hence, there is a well defined energy for a given charge configuration, without having to specify the way in which it was formed.

Imitating this procedure, in this paper we compute the linear and the angular momentum transferred to a point charge that is displaced from, *e.g.*, infinity to some final position in a given static magnetic field $\mathbf{B}$. We find that the component along some axis of the linear momentum transferred to the point charge is independent of the path followed if and only if the magnetic field is invariant under translations along that axis. Similarly, the component along some axis of the angular momentum transferred to the point charge is independent of the path if and only if the magnetic field is invariant under rotations about that axis. This means that there is a well defined value of one component of the stored linear or angular momentum of the system formed by the charged particle and the magnetic field if and only if the magnetic field has the appropriate symmetry.

We also show that, if the magnetic field is invariant under translations along some axis $\mathbf{u}$, then the component along $\mathbf{u}$ of the linear momentum of the system (in cgs units) is given by

$$\mathbf{u}\cdot\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,\mathrm{d}v, \tag{1}$$

where $\mathbf{E}$ is the electric field produced by the point charge and the integral is over all space. In a similar manner we show that if the magnetic field is invariant under rotations about some axis $\mathbf{u}$, then the component along $\mathbf{u}$ of the angular momentum of the system is given by

$$\mathbf{u}\cdot\frac{1}{4\pi c}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})\,\mathrm{d}v. \tag{2}$$

As is well known, by combining the Maxwell equations one finds that it is possible to define a density of linear momentum for the electromagnetic field (see, *e.g.*, Ref. [3]), which is precisely given by $\mathbf{E}\times\mathbf{B}/4\pi c$ [*cf.* Eq. (1)], but this result does not depend on the assumption of some symmetry for $\mathbf{B}$. The difference is related to the fact that in the standard derivation of the density mentioned above one considers the interaction of the electric and magnetic fields with the charges and currents producing them, whereas in the calculation presented here we consider the force on an electric charge in a fixed magnetic field, produced by other (unspecified) sources. The coincidence of the final results when the magnetic field has the appropriate symmetry explains why the computations given in Refs. [4, 5], to solve Feynman's paradox [6], work, since the magnetic field considered in those papers is axially symmetric.

In Sec. 2 we find an expression for the z-component of the linear momentum transferred to a charge in a given magnetic field and we show that if the magnetic field does not depend on z it is possible to define a value for the z-component of the linear momentum of the system formed by the charge and the magnetic field. In Sec. 3 we obtain analogous results for the case of the z-component of the angular momentum, assuming that the magnetic field is invariant under rotations about the z-axis. Throughout this paper we use cgs units.

2. The transferred linear momentum

In this section and the following one we start from the usual expression for the force on a charged particle, with electric charge q, in a magnetic field $\mathbf{B}$:

$$\mathbf{F}=\frac{q}{c}\mathbf{v}\times\mathbf{B}, \tag{3}$$

where c is the speed of light in vacuum and $\mathbf{v}$ is the velocity of the charged particle. According to Newton's second law, the right-hand side of Eq. (3) must be equal to $\mathrm{d}\mathbf{p}/\mathrm{d}t$, where $\mathbf{p}$ is the linear momentum of the particle. Hence, the linear momentum *transferred* to the charge, $\mathbf{p}_\mathrm{t}$, in order to displace it quasistatically must be such that

$$\mathrm{d}\mathbf{p}_\mathrm{t} = -\frac{q}{c}\mathbf{v}\times\mathbf{B}\,\mathrm{d}t = \frac{q}{c}\mathbf{B}\times\mathrm{d}\mathbf{r}$$

and the total linear momentum transferred to the charge when it is displaced along a path C is given by the line integral

$$\Delta\mathbf{p}_\mathrm{t} = \frac{q}{c}\int_C \mathbf{B}\times\mathrm{d}\mathbf{r}.$$

If we want to consider the component of $\Delta\mathbf{p}_\mathrm{t}$ along some direction, without any loss of generality we can take that direction as the z-axis of a set of Cartesian axes. We have

$$\Delta p_{\mathrm{t}z} = \frac{q}{c}\int_C (B_x\mathrm{d}y - B_y\mathrm{d}x). \tag{4}$$

The value of the line integral (4) only depends on the endpoints of the path C if and only if $B_x\mathrm{d}y - B_y\mathrm{d}x + 0\,\mathrm{d}z$ is exact. This means that the crossed partial derivatives of the coefficients must coincide:

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}, \qquad \frac{\partial B_x}{\partial z} = 0, \qquad \frac{\partial B_y}{\partial z} = 0. \tag{5}$$

Since the magnetic field must obey $\nabla\cdot\mathbf{B} = 0$, the first equation (5) amounts to $\partial B_z/\partial z = 0$ and, therefore, Eqs. (5) mean that the Cartesian components of $\mathbf{B}$ do not depend on z, or that $\mathbf{B}$ is invariant under the translations along the z-axis.

Another way of expressing the fact that the differential form $B_x\mathrm{d}y - B_y\mathrm{d}x + 0\,\mathrm{d}z$ is exact is saying that there exists a function Π, defined up to an additive constant, such that $B_x\mathrm{d}y - B_y\mathrm{d}x = \mathrm{d}\Pi$ or, equivalently, that there exists a function Π such that

$$\nabla\Pi = -B_y\hat{x} + B_x\hat{y} = \hat{z}\times\mathbf{B}. \tag{6}$$

When the magnetic field depends on z, the transferred momentum (4) does depend on the path followed and we *cannot* define the "function of state" Π (borrowing the term employed in thermodynamics).

Thus, if the magnetic field is independent of z, the amount of the z-component of linear momentum transferred to the charge during its motion along C only depends on the endpoints of the path, and we can *assign* a value for the z-component of the linear momentum of the system (formed by the magnetic field and the charge) given by $(q/c)\,\Pi(x,y,z)$, if (x,y,z) are the Cartesian coordinates of the position of the charge [see Eq. (4)]. In Refs. [7] and [8], Exercise 1.22, it is shown that for a charged particle in a given magnetic field independent of z, $m\dot{z} + q\Pi/c$ is a constant of motion.

It should be clear that if instead of considering the z-component of the linear momentum of the system, we want to define the component of the linear momentum of the system along the direction of the (constant) unit vector $\mathbf{u}$, the magnetic field must be invariant under translations in the direction of $\mathbf{u}$ and the linear momentum will be defined in terms of a function Π such that $\nabla\Pi = \mathbf{u}\times\mathbf{B}$. The assumed symmetry of $\mathbf{B}$ guarantees the existence of Π.

2.1 Expression in terms of the fields

Denoting by $\mathbf{a}$ the position vector of the charge q, according to Gauss's law, the electric field produced by this charge must satisfy $\nabla\cdot\mathbf{E} = 4\pi q\delta(\mathbf{r}-\mathbf{a})$, hence, if the magnetic field is invariant under translations along the direction of $\mathbf{u}$, the component of the linear momentum of the system along $\mathbf{u}$ is given by

$$\mathbf{u}\cdot\mathbf{p} = \frac{q}{c}\Pi(\mathbf{a}) = \frac{1}{4\pi c}\int(\nabla\cdot\mathbf{E})\,\Pi\,\mathrm{d}v,$$

where Π is a function such that $\nabla\Pi = \mathbf{u}\times\mathbf{B}$ [*cf.* Eq. (6)] and the integration is over all space. Making use of the Gauss theorem we have

$$\mathbf{u}\cdot\mathbf{p} = -\frac{1}{4\pi c}\int\mathbf{E}\cdot\nabla\Pi\,\mathrm{d}v = -\frac{1}{4\pi c}\int\mathbf{E}\cdot\mathbf{u}\times\mathbf{B}\,\mathrm{d}v$$
$$= \mathbf{u}\cdot\frac{1}{4\pi c}\int\mathbf{E}\times\mathbf{B}\,\mathrm{d}v. \tag{7}$$

In this way, the component of the linear momentum along $\mathbf{u}$ is expressed in terms of the electric and magnetic fields alone. It may be noticed that the right-hand side of Eq. (7) contains the integral of the density of linear momentum $\mathbf{E}\times\mathbf{B}/4\pi c$ mentioned at the Introduction.

3. The transferred angular momentum

Making use of Eq. (3), we now calculate the angular momentum transferred to a point charge in order to displace it quasistatically along a path C in a given magnetic field $\mathbf{B}$

$$\Delta\mathbf{L}_\mathrm{t} = \frac{q}{c}\int_C \mathbf{r}\times(\mathbf{B}\times\mathrm{d}\mathbf{r}) = \frac{q}{c}\int_C (\mathbf{r}\cdot\mathrm{d}\mathbf{r})\mathbf{B} - (\mathbf{r}\cdot\mathbf{B})\mathrm{d}\mathbf{r}.$$

Without any loss of generality, we consider the z-component of the transferred angular momentum but in order to simplify the interpretation it will be convenient to make use of the circular cylindrical coordinates (ρ,ϕ,z). Then $\mathrm{d}\mathbf{r} = \mathrm{d}\rho\,\hat{\rho} + \rho\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{z}$, where $\{\hat{\rho},\hat{\phi},\hat{z}\}$ is the orthonormal basis defined by the cylindrical coordinates, $\mathbf{r} = \rho\,\hat{\rho} + z\,\hat{z}$, and

$$\Delta L_{\mathrm{t}z} = \frac{q}{c}\int_C \left[(\rho\mathrm{d}\rho + z\mathrm{d}z)B_z - (\rho B_\rho + zB_z)\mathrm{d}z\right]$$
$$= \frac{q}{c}\int_C \rho\,(B_z\mathrm{d}\rho - B_\rho\mathrm{d}z). \tag{8}$$

The value of the line integral (8) depends only on the endpoints of the path if and only if $\rho B_z\mathrm{d}\rho - \rho B_\rho\mathrm{d}z + 0\,\mathrm{d}\phi$ is exact, which amounts to the conditions

$$\frac{\partial(\rho B_z)}{\partial z} = -\frac{\partial(\rho B_\rho)}{\partial\rho},$$
$$\frac{\partial(\rho B_z)}{\partial\phi} = 0, \qquad \frac{\partial(\rho B_\rho)}{\partial\phi} = 0. \tag{9}$$

Using the fact that

$$0 = \nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z},$$

we see that Eqs. (9) are equivalent to saying that the components B_ρ, B_ϕ, B_z do not depend on ϕ. In other words, the magnetic field is invariant under rotations about the z-axis.

The symmetry conditions (9) are equivalent to the existence of a function Λ, defined up to an additive constant, such that $\mathrm{d}\Lambda = \rho B_\rho \mathrm{d}\rho - \rho B_z \mathrm{d}z$ or that

$$\nabla\Lambda = \rho B_z \hat{\rho} - \rho B_\rho \hat{z} = (\hat{z} \times \mathbf{r}) \times \mathbf{B}. \tag{10}$$

Then, the z-component of the angular momentum of the system can be *defined* as $(q/c)\Lambda$, with the function Λ evaluated at the position of the charge. (Note that the functions Π and Λ depend only on the magnetic field $\mathbf{B}$.) In Refs. [7], [8], Example 1.21, and [9] it is shown that for a charged particle in a given magnetic field invariant under rotations about the z-axis, $m(x\dot{y} - y\dot{x}) + q\Lambda/c$ is a constant of motion.

3.1 Expression in terms of the fields

Using the same notation as in Sec. 2.1, if the magnetic field is invariant under rotations about the z-axis, the z-component of the angular momentum of the system can be expressed as

$$\hat{z} \cdot \mathbf{L} = \frac{q}{c}\Lambda(\mathbf{a}) = \frac{1}{4\pi c}\int (\nabla \cdot \mathbf{E})\,\Lambda\,\mathrm{d}v,$$

with Λ defined by Eq. (10) and the integration is over all space. Making use of the Gauss theorem we have

$$\begin{aligned} \hat{z} \cdot \mathbf{L} &= -\frac{1}{4\pi c}\int \mathbf{E} \cdot \nabla\Lambda\,\mathrm{d}v \\ &= -\frac{1}{4\pi c}\int \mathbf{E} \cdot [(\hat{z} \times \mathbf{r}) \times \mathbf{B}]\,\mathrm{d}v \\ &= \hat{z} \cdot \frac{1}{4\pi c}\int \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\,\mathrm{d}v. \end{aligned} \tag{11}$$

It should be clear that a similar expression would be applicable if instead of $\hat{z}$ we take any other constant vector, provided that the magnetic field is invariant under rotations about that constant vector. The right-hand side of Eq. (11) contains the integral of the density $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$, which is usually interpreted as a density of angular momentum for the electromagnetic field.

3. Final remarks

As pointed out at the Introduction, for the system formed by a point charge and a static magnetic field, the z- component of the linear or angular momentum of the combined electromagnetic field computed making use of the density of linear momentum $\mathbf{E} \times \mathbf{B}/4\pi c$ coincides with the z-component of the stored linear or angular momentum, provided that the magnetic field is invariant under translations along the z-axis or rotations about the z-axis, respectively.

In all the calculations presented above we have considered a single test charge in a given magnetic field. Owing to the additivity of the linear and the angular momenta, the results obtained above can be applied to a collection of test charges if the interaction among them is neglected.

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