Stable identification of sources located on the cerebral cortex from EEG over the scalp

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In this work, we present a stable algorithm for the inverse problem of identifying cortical sources from electroencephalographic measurements on the scalp. This inverse problem is ill-posed due to the numerical instability it presents, *i.e.*, small changes in the measurements can produce large variations in the location of the sources. A boundary value problem is used to find correlations between the sources and measurements. In the case in which the head is modeled using two concentric spheres, we use spherical harmonics to find the solution to the inverse source cortical problem. To handle the numerical instability of this problem, we use the Tikhonov regularization method and a cut-off of the harmonic expansion series. From numerical tests, we found these parameters with which we get good approximations. Finally, we illustrate the algorithm proposed with synthetic examples.

1. Introduction

Electroencephalography is a well-known, non-invasive technique to investigate the brain, based on recording electrodes located on the scalp of its electrical potential (voltage). The EEG is generated by the activity of big conglomerates of neurons that work simultaneously, called bioelectrical sources, which can be located in the cerebral cortex or the brain's subcortical layers. The electrical activity of neurons is due to the ionic currents. One of the main applications of electroencephalography is to study evoked potentials, for determining zones of bioelectrical activity associated with a specific stimulus, either for studying cognitive processes or for diagnosing and detecting epileptic foci (in both cases, located on the cerebral cortex or subcortical layers). Another application of the EEG is related to determining sleep anomalies. One crucial point is that epileptic foci can be related to edemas, tumors, and calcifications in the brain. In this work, we consider the inverse electroencephalographic problem (EIP), which consists of determining, from electroencephalographic measurements on the scalp, to identify the bioelectrical sources that generate such measurements. In this context, the sources are large clusters of neurons that work in a coupled manner to generate potentials that can be recorded on the scalp employing an electroencephalograph [1,2]. The sources can be located in the cerebral cortex or the cerebral volume. The EIP is an ill-posed problem in Hadamard's sense [3], which is due to:

1. Different sources can produce the same measurement.

2. Presents a numerical instability, *i.e.*, small changes in the measurement can produce large variations in the location of the source.

Regularization methods are used to handle this instability, [3] that allow finding stable approximate solutions from the measurement with error. For the study of the EIP, we will first analyze the so-called direct problem, which consists of determining the measurement when the source is known. In the direct problem, unlike the inverse problem, if two sources are close, the respective measurements will also be close.

As we commented above, the bioelectrical sources can be located in the cerebral cortex or the cerebral volume (subcor-

tical). Since the problem is linear regarding the sources, we can separate the study of the inverse problem in cortical o subcortical sources. More precisely, if we have two sources, one cortical and one subcortical, the EEG produced by the two sources is the sum of the EEG produced for each source. The mentioned linearity corresponds with the superposition principle. For this work, we will consider the case of sources on the cerebral cortex and disregard sub-cortical sources' activity. Moreover, different works study inverse problems in the case of volumetric sources [4–7]. It was demonstrated in Ref. [8] that the EIP has a unique solution. Thus, the ill-posedness of the EIP is associated with numerical instability.

This work presents a stable algorithm to recover cortical sources from EEG measurements when two concentric spheres model the head. In this case, we can use the expansion of the source and the measurement in spherical harmonics. This simplified model has been widely used for studying the inverse source problem since it gives enough information about the brain's electrical activity [9–13]. There are numerically comprehensive direct modeling schemes, including realistic geometries and electrical conductivities, even electrically anisotropic tissue. These models have different advantages and disadvantages regarding speed, accuracy, and interpretability of results [12]. To handle the numerical instability, we use two regularization parameters, namely,

1. The Tikhonov regularization parameter.

2. A truncation of the expansion series.

Both parameters were chosen by numerical tests.

We show the feasibility of the algorithm with numerical examples.

The work is divided as follows: Section 2 shows the boundary value problem that allows the operational formulation of the EIP; Section 3 shows the solution of the forward problem and the Tikhonov regularized solution of the inverse problem, which is used to the cut-off of the expansion series; Section 4 shows numerical examples of the proposed algorithm. Finally, In Sec. 5, the conclusions are shown.

2. Cortical electroencephalographic boundary problem

The analysis of the inverse source cortical problem is carried out by means of the following boundary problem [11]:

$$\Delta u_1 = 0 \quad \text{in} \quad \Omega_1, \tag{1}$$

$$\Delta u_2 = 0 \quad \text{in} \quad \Omega_2, \tag{2}$$

$$u_1 = u_2 \quad \text{on} \quad S_1, \tag{3}$$

$$\sigma_1 \frac{\partial u_1}{\partial \mathbf{n}_1} = \sigma_2 \frac{\partial u_2}{\partial \mathbf{n}_2} + g \quad \text{on} \quad S_1, \tag{4}$$

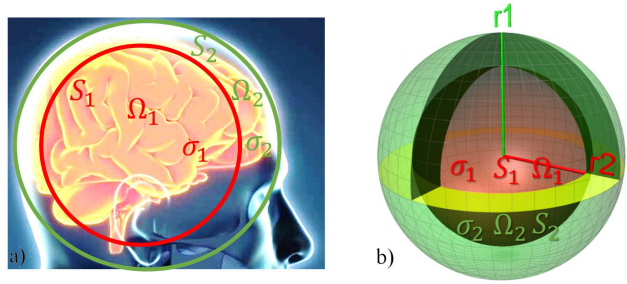$$\frac{\partial u_2}{\partial \mathbf{n}_2} = 0 \quad \text{on} \quad S_2, \tag{5}$$

FIGURE 1. Schematic representation of the head in two conductive layers. The cerebral cortex is represented by S_1, which can be considered as an interface between the brain and the rest of the head.

where $\Omega = \Omega_1 \cup S_1 \cup \Omega_2 \subset \mathbb{R}^n$, with $n = 2$ or 3, is a region sufficiently smooth. The positive constants σ_1 and σ_2 are the conductivities of the regions Ω_1 and Ω_2, respectively (see Fig. 4). Function g represents the source defined on the interface S_1, while $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outward normal vectors on the boundary of Ω_1 and Ω_2, respectively. We denote by u the solution of the problem (1)-(5) in Ω and define $u_i = u|_{\Omega_i}$, $i = 1, 2$. Δ represents to the Laplace operator, which is also denoted by ∇^2. The boundary condition (3) corresponds to the continuity of the potential, and condition (4) corresponds to the jump of the normal flux which is equal to the function g, while condition (5) indicates that the conductivity in the exterior region $\bar{\Omega}^C$ is zero (in some applications it corresponds to the conductivity of the air). Ω_1 represents the brain, Ω_2 to the rest of the layers that make up the head (intracranial fluid, skull, scalp), S_1 represents the surface of the cerebral cortex, S_2 to the surface of the scalp (see Fig. 2). The function g represents the activity of sources located on the cerebral cortex. We will call the contour problem (1)-(5) the Electroencephalographic Boundary Value Problem and denote it by (EBP).
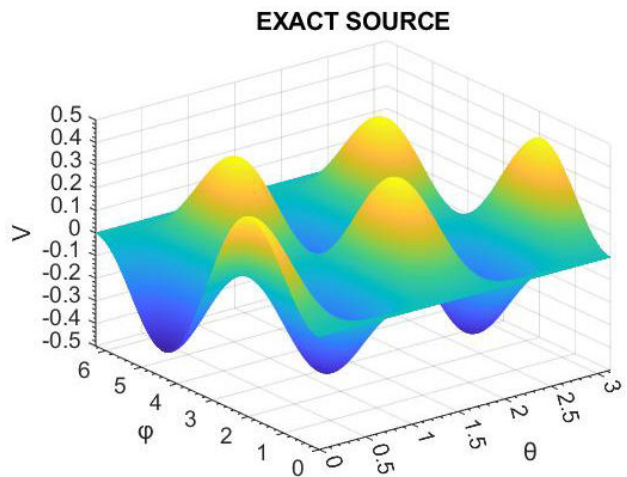

FIGURE 2. Exact source $g(\theta, \varphi) = R_1^3 cos(\varphi)\, sen^2(\varphi) \cos(2\theta)$.

2.1. Solubility analysis of EBVP

A necessary and sufficient condition for the classical solution of the EBVP is given by [8]:

$$\int_{S_1} g\, ds = 0. \tag{6}$$

Definition 1. *Given g on S_1, the Electroencephalographic Forward Problem (EFP) consists of finding the measurement $V = u|_{S_2}$, where u is the solution of the EBVP.*

Definition 2. *Given a function V defined on S_2, the Electroencephalographic Inverse Problem (EIP) consists of determining a source g defined on S_1 such that the solution u of the EFP corresponding to g, satisfies that $u|_{S_2} = V$.*

2.2. The weak solution

2.2.1. Weak solution of the EBVP

We consider the following spaces:

$$\mathcal{U} = \left\{ g \in L_2(S_1) : \int_{S_1} g\, dS_1 = 0 \right\} = L_2(S_1)/\mathbb{R},$$

$$\mathcal{V} = \left\{ u \in H^1(\Omega) : \int_{\Omega} u\, d\Omega = 0 \right\} = H^1(\Omega)/\mathbb{R},$$

$$\mathcal{W} = \left\{ v \in L_2(S_2) : \int_{S_2} v\, dS_2 = 0 \right\} = L_2(S_2)/\mathbb{R}.$$

Definition 3. *Given $g \in \mathcal{U}$, a function $u \in H^1(\Omega)$, is called a weak solution of the problem (1)-(5)) if it satisfies*

$$\sigma_1 \int_{\Omega_1} \nabla u_1 . \nabla v\, d\Omega + \sigma_2 \int_{\Omega_2} \nabla u_2 . \nabla v\, d\Omega$$

$$= \int_{S_1} gv\, ds, \quad \forall\, v \in H^1(\Omega). \tag{7}$$

The condition (6) is also necessary and sufficient for the existence of the weak solution of the EBVP [11]. There is a unique weak solution $u \in \mathcal{U}$ that satisfies

$$\|u\|_{H^1(\Omega)} \leq C\|g\|_{L_2(S_1)}, \tag{8}$$

where the constant C does not depend of g [11].

We want to emphasize that the compatibility condition (6) is necessary and sufficient for the existence and uniqueness of the both classical and weak solutions [8].

2.3. Inverse problem. Operational formulation

We can define the following operator $T : \mathcal{U} \to \mathcal{V}$, defined by $T(g) = u$, which is continuous due to (8). The composition of the operator T, which is continuous, and the compact trace operator $Tr : \mathcal{V} \to \mathcal{W}$, defines the compact operator $A : \mathcal{U} \to \mathcal{W}$, given by $A = Tr \circ T$, *i.e.* $A(g) = u|_{S_2}$, which is injective. Using this operator we will study the problem of identifying sources defined on S_1.

The inverse operator A^{-1} is not continuous and it leads to numerical instability, which is the cause of the ill-posedness of the problem. To handle the numerical instability, we will use the Tikhonov regularization method. Since $Im(A)$ is dense in $\mathcal{W}$, we can apply the Tikhonov regularization [3].

The EIP can be expressed in terms of the operator A in the the following form:

Given a function $V \in \mathcal{W}$ find a source $g \in \mathcal{U}$ such that $A(g) = V$.

Note that the equation $A(g) = V$ does not have solution if $V \notin Im(A)$. We can use Tikhonov regularization since $Im(A)$ is dense in $\mathcal{W}$ [8, 14].

3. EIP Solution

3.1. Solution of the EFP

The boundary value problem (1)-(5) allows to get correlations between EEG and sources. Through this model, we can get a theoretical (exact or ideal) EEG containing the main characteristics to distinguish a real EEG. The boundary value problem (1)-(5) allows to get correlations between EEG and sources. Through this model, we can get a theoretical (exact or ideal) EEG containing the main characteristics to distinguish a real EEG. To achieve this, we solve the so-called forward or direct problem, which consists of determining the exact EEG when the source is known, *i.e.*, when we know the domain and correspondence rule of the function with which the source is represented. To solve the direct problem, as a first step, we find the solution of the EBVP. Then we restrict this solution to the region's boundary (more precisely, we find the trace of the solution to the region's boundary).

To find the solution to the problem, it is proposed that the function $g \in \mathcal{U}$ is given in the form

$$g(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_{nm} Y_{nm}(\theta, \varphi), \tag{9}$$

where the Fourier coefficients g_{nm} of the source g are given by

$$g_{nm} = \int_{S_1} g(\theta, \varphi) Y_{nm}(\theta, \varphi) dS_1. \tag{10}$$

The solution of the EBVP (1)-(5) is sought in the form:

$$u_1 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm}^1 r^n Y_{nm}, \tag{11}$$

and

$$u_2 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (A_{nm}^2 r^n + B_{nm}^2 r^{-1-n}) Y_{nm}, \tag{12}$$

where the coefficients A_{nm} and B_{nm} of the potentials are unknown and must be determined from the boundary conditions of the problem. From the condition $u_1 = u_2$, on S_1, it is found that

$$A_{nm}^1 R_1^n = A_{nm}^2 R_1^n + B_{nm}^2 R_1^{-1-n}. \quad (13)$$

From the condition $\sigma_1(\partial u_1/\partial \mathbf{n}_1) = \sigma_2(\partial u_2/\partial \mathbf{n}_2) + g$ on S_1, it is obtained:

$$\sigma_1 A_{nm}^1 n R_1^{n-1} = \sigma_2 [A_{nm}^2 n R_1^n$$
$$+ B_{nm}^2 (n-1) R_1^{-2-n}] + g_{nm}. \quad (14)$$

From the boundary condition $\sigma_2(\partial u_2/\partial \mathbf{n}_2) = 0$, it is found:

$$\sigma_2 [A_{nm}^2 n R_2^n + B_{nm}^2 (n-1) R_2^{-2-n}] = 0. \quad (15)$$

By subtracting A_{nm}^2 from the Eq. (15) we obtain:

$$A_{nm}^2 = B_{nm}^2 (n+1) R_2^{-2-n} / n R_2^{n-1}$$
$$= (n+1/n) B_{nm}^2 R_2^{-2n-1}. \quad (16)$$

Multiplying by R_1 the Eq. (16), we get:

$$\sigma_1 A_{nm}^1 n R_1^n = \sigma_2 [A_{nm}^2 n R_1^n$$
$$+ B_{nm}^2 (-n-1) R_1^{-1-n}] + g_{nm} R_1, \quad (17)$$

from where

$$A_{nm}^1 = \frac{[(n+1)R_1^{2n+1} + n R_2^{2n+1}]}{n R_1^{n-1} H_n} g_{nm}. \quad (18)$$

Multiplying the Eq. (14) by $n\sigma_1$ yields:

$$n\sigma_1 A_{nm}^1 R_1^n = n\sigma_1 [A_{nm}^2 R_1^n + B_{nm}^2 R_1^{-1-n}]. \quad (19)$$

From (18) and (19), we have the following equation

$$n(\sigma_1 - \sigma_2) A_{nm}^1 R_1^n + [n\sigma_1 + (n+1)\sigma_2]$$
$$\times B_{nm}^2 R_1^{-1-n} = g_{nm} R_1. \quad (20)$$

Substituting A_{nm}^2 given by (16), we obtain

$$B_{nm}^2 = \frac{R_1^{n+2} R_2^{2n+1}}{H_n} g_{nm}, \quad (21)$$

where $H_n = (n+1)(\sigma_1-\sigma_2)R_2^{2n+1} + [n\sigma_1 + (n+1)\sigma_2]R_2^{n+2}$. Substituting B_{nm}^2 on Eq. (16) is obtained

$$A_{nm}^2 = \frac{R_1^{n+2}}{n H_n} g_{nm}. \quad (22)$$

Therefore, the solution of the forward problem for $g \in \mathcal{U}$, is given by

$$A(g) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \frac{(2n+1)R_1^{n+2}R_2^n}{nE_n + F_n} \right\} g_{nm} Y_{nm}, \quad (23)$$

where $E_n = (n+1)(\sigma_1-\sigma_2)R_1^{2n+1}$, $F_n = [n\sigma_1 + (n+1)\sigma_2]R_2^{n+2}$. Recall that the exact (ideal) measurement V (EEG) is obtained through the relation $V(\theta, \varphi) = A(g)(\theta, \varphi)$.

3.2. Solution of the EIP

From the expression (22) para $V = u_2|_{S_2}$, we have that the Fourier coefficients of V are given by

$$V_{nm} = \frac{[(2n+1)R_1^{n+2}R_2^n]}{nE_n + F_n} g_{nm}, \quad (24)$$

from where the coefficients of the source g are given by

$$g_{nm} = \frac{[nE_n + F_n]}{(2n+1)R_1^{n+2}R_2^n} V_{nm}. \quad (25)$$

Recall that the harmonics $Y_{nm}(\theta, \varphi)$ form an orthogonal basis for the harmonic functions orthogonal to constants, a space arising from the compatibility condition of the function g.

3.3. Solution of the IEP using the Tikhonov regularization method and a cut-off

We consider the equation $A(g) = V$. To handle the numerical instability of the equation, the Tikhonov regularization method is used, which consists of adding a penalization term to the least square problem $J(g) = (1/2)\|A(g) - V\|_{L_2(S_2)}^2$, *i.e.*, the method minimizes the Tikhonov functional

$$J_\alpha(g) = \frac{1}{2}\|A(g) - V\|_{L_2(S_2)}^2 + \frac{\alpha}{2}\|g\|_{L_2(S_1)}^2, \quad (26)$$

where $\alpha(\delta) > 0$ and can be chosen using the L-curve method [15] or the Morozov discrepancy principle [3].

Now, we consider the equation $A(g) = V_\delta$, where $\|V - V_\delta\|_{L_2(S_2)}^2$, and the functional $J_\alpha(g) = (1/2)\|A(g) - V_\delta\|_{L_2(S_2)}^2 + (\alpha/2)\|g\|_{L_2(S_1)}^2$. Let $g_{\alpha(\delta)} = R_\alpha V_\delta < \delta$ be an approximation of g where $R_\alpha = (A^*A + \alpha I)^{-1}A^*V_\delta$ is the operator associated with the Tikhonov regularization strategy, which has the following properties described in the following theorems [3].

Theorem 1. $R_\alpha : \mathcal{W} \to \mathcal{U}$ *is boundedly invertible, and* $g_{\alpha(\delta)} = R_\alpha V_\delta$ *is the unique solution of the normal equation* $(A^*A + \alpha I)g_{\alpha(\delta)} = A^*V_\delta$. *Furthermore,* $\|R_\alpha\| \le 1/\sqrt{\alpha}$.

Theorem 2. *Any choice* $\alpha(\delta)$ *such that* $\lim_{\delta \to 0} \alpha(\delta) = 0$ *and* $\lim_{\delta \to 0} \delta^2/\alpha(\delta) = 0$ *ensures that*

$$\sup\left\{\|R_\alpha V_\delta - A^{-1}V\|_{\mathcal{U}} : V_\delta \in \mathcal{W}, \|V - V_\delta\|_{\mathcal{W}} \le \delta\right\} \to 0$$

when $\delta \to 0$. *In this case,* R_α *is called 'admisible' and*

$$\lim_{\delta \to 0} g_{\alpha(\delta)} = g \quad \text{in} \quad \mathcal{U}. \quad (27)$$

The minimum of the functional (26), with V_δ instead V, is solution of the normal equations [3]:

$$(A^*A + \alpha I) g^{\alpha(\delta)} = A^*V_\delta, \quad (28)$$

where A^* represents the adjoint operator, which is defined through the adjoint boundary value problem

$$\Delta w_1 = 0 \quad \text{in} \quad \Omega_1, \tag{29}$$

$$\Delta w_2 = 0 \quad \text{in} \quad \Omega_2, \tag{30}$$

$$w_1 = w_2 \quad \text{on} \quad S_1, \tag{31}$$

$$\sigma_1 \frac{\partial w_1}{\partial \mathbf{n}_1} = \sigma_2 \frac{\partial w_2}{\partial \mathbf{n}_2} \quad \text{on} \quad S_1, \tag{32}$$

$$\sigma_2 \frac{\partial w_2}{\partial \mathbf{n}_2} = \psi \quad \text{on} \quad S_2, \tag{33}$$

by the correspondence rule $A^*(\psi) = \sigma_1 w_1$ with $\psi \in \mathcal{W}$. If

$$\psi(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_{nm} Y_{nm}(\theta, \varphi), \tag{34}$$

it is found that

$$A^*(\psi) = -\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{(2n+1)R_2^{n+2} r^n}{n G_n}$$

$$\times \psi_{nm} Y_{nm}(\theta, \varphi), \tag{35}$$

where

$$G_n = [(n+1)(\sigma_1 - \sigma_2)] R_1^{2n+1} + [n\sigma_1 + (n+1)\sigma_2] R_2^{2n+1}.$$

After solving the normal equations, we found that the Fourier coefficients of the regularized source $f_{\alpha(\delta)}$ are given by

$$f_{nm}^{\alpha} = \frac{n(2n+1)(2n+3) G_n R_2^{n+2}}{-\sigma_1 K_n + \alpha n^2 (2n+3) [G_n]^2} V_{nm}^{\delta}, \tag{36}$$

where V_{nm}^{δ} are the Fourier coefficients of V_{δ}, and $K_n = (2n+1)^2 R_1^{2n+3} R_2^{2n+2}$. Thus, by (36) we get a stable algorithm to recover the source from the measurement V_{δ}, which is associated to the EEG.

After numerical tests, we will use 10 terms of the expansion series since we get a good approximation of the regularized sources. Thus, we are using N and α as regularization parameters.

4. Numerical examples

In this section, we present numerical examples to illustrate the algorithm given in Eq. (36). In these examples, we proceed as follows:

- The EBVP (1)-(5) is solved for the exact source f.

- The forward problem (the trace of the solution of the EBVP) is solved to get the exact or ideal measurement V. We take $N = 20$ terms in the expansion series.

- To get V_{δ}, we aggregate a random error to the exact measurement V using the rand function of MATLAB to emulate the error in the measurement.

- The regularized source $f_{\alpha(\delta)}$ is obtained using (36) taking $N = 10$ and $\alpha = 10^{-6}$.

- We calculate the relative error given by

$$RE = \frac{\|f - f_{\alpha(\delta)}\|_{L_2(S_1)}}{\|f\|_{L_2(S_1)}}.$$

Example 1. We consider the source $g(x, y, z) = (x^2 - y^2)z$, which in spherical coordinates $g(R_1, \theta, \varphi) = R_1^3 \cos(\varphi) \operatorname{sen}^2(\varphi) \cos(2\theta)$. This function satisfies the compatibility condition (6). Figure 2 shows the graphic of this source, and Fig. 3 shows the same graphic obtained using $N = 20$ terms of the expansion in spherical harmonics. The solution of the forward problem $A(g)$ is shown in Fig. 4, which corresponds to the exact (ideal) measurement taking $N = 20$ in (23). The error measurement, in Fig. 5, was obtained by adding a random error to each coefficient (using
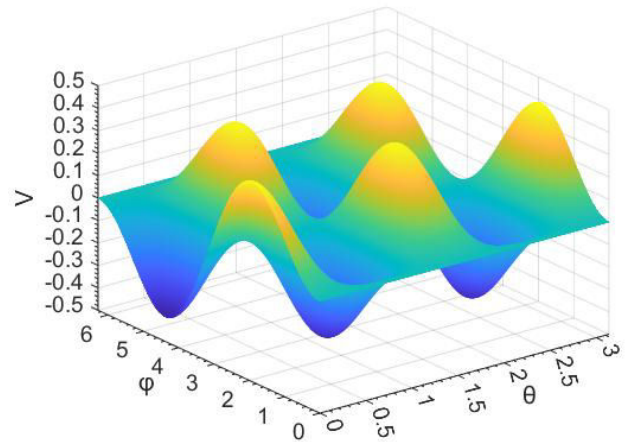

APPROXIMATE SOURCE

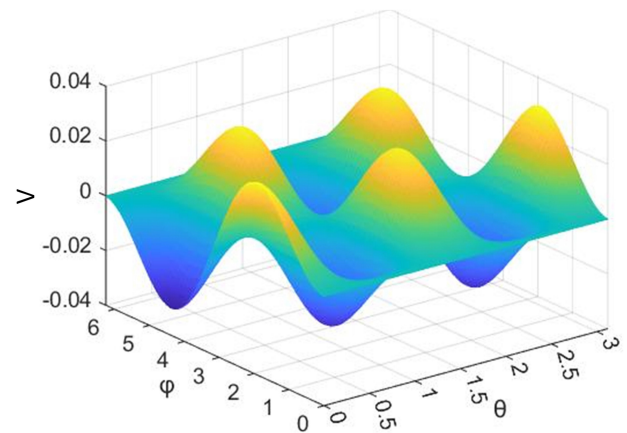FIGURE 3. Approximated source taking 20 terms in Eq. (9).

IDEAL MEASUREMENT

FIGURE 4. Ideal measurement $A(g) = V$ obtained solving the forward problem taking 20 terms in Eq. (23).
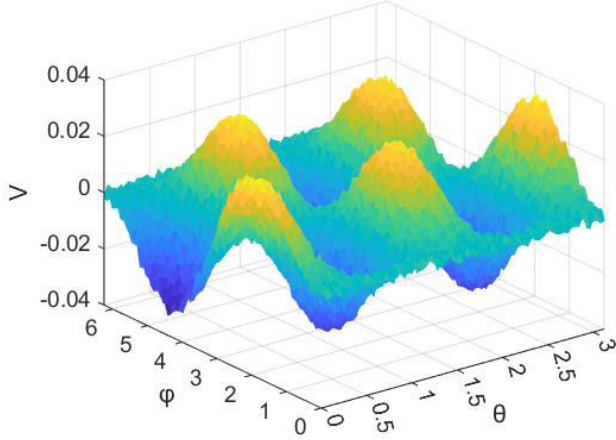
FIGURE 5. Measurement with error obtained adding a random error to the exact measurement $A(g) = V$.
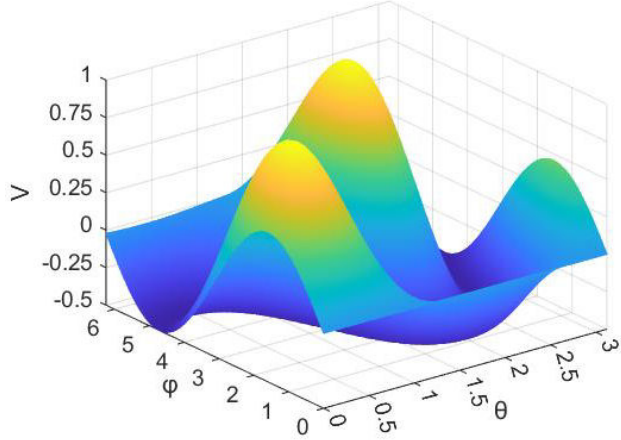

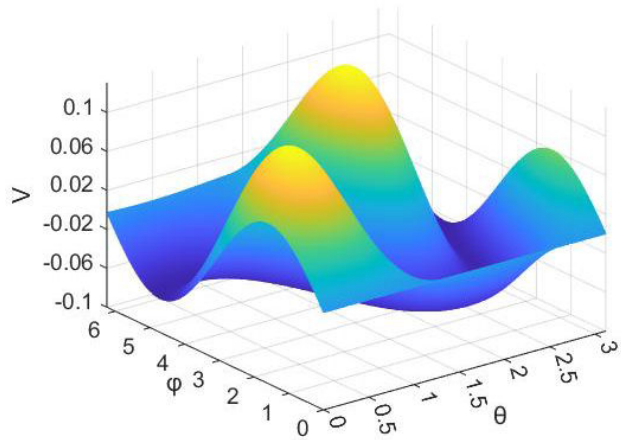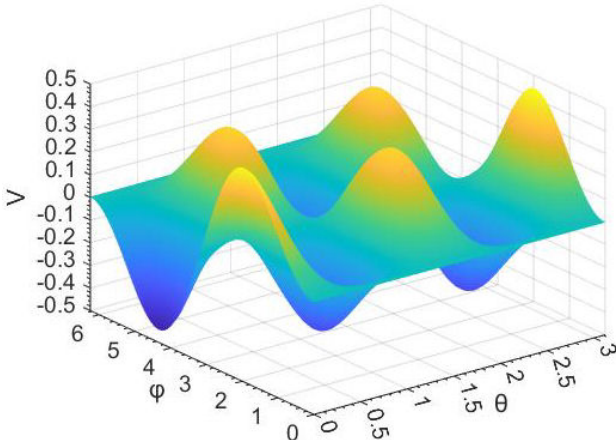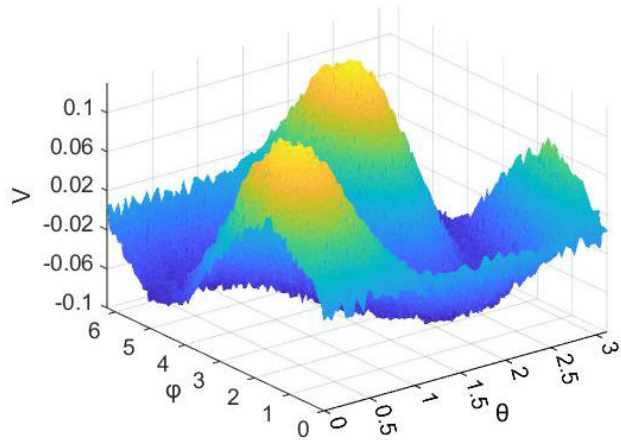
FIGURE 6. Recovered source without regularization ($\alpha = 0$) and $N = 80$ terms.

FIGURE 7. Regularized source obtained taking $\alpha = 10^{-6}$ and $N = 10$ terms in Eq. (36).

FIGURE 8. Exact source $g(x, y, z) = (x^2 - y^2)z$.

FIGURE 9. Ideal measurement $A(g) = V$ obtained solving the forward problem taking 20 terms in Eq. (23).

FIGURE 10. Measurement with error obtained adding a random error to the exact measurement $A(g) = V$.

RECOVERED SOURCE

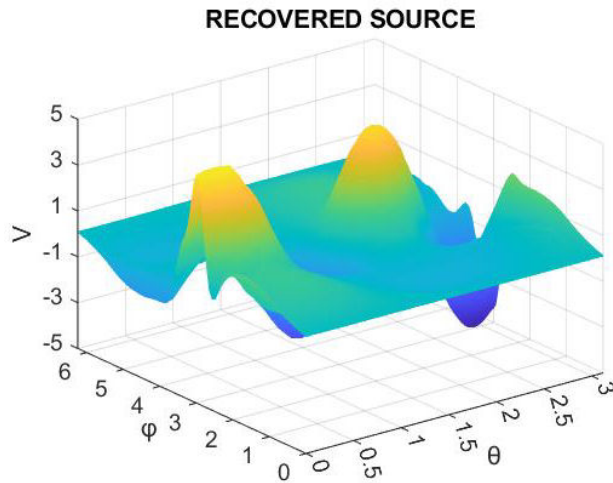FIGURE 11. Recovered source without regularization ($\alpha = 0$) and $N = 80$ terms.
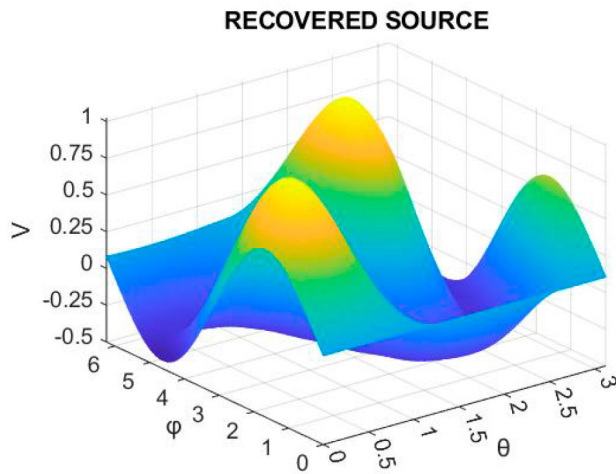
RECOVERED SOURCE

FIGURE 12. Regularized source obtained by taking $N = 10$ and $\alpha = 10^{-6}$.

the rand function of MATLAB), taking $\delta = 0.1$. Figure 6 shows the recovered source without regularization and taking $N = 90$ expansion terms. The relative error between the exact and recovered sources is 9.10. Figure 7 shows the reg-

ularized source with $\alpha = 10^{-6}$ and $N = 10$. The relative error is 0.08.

Example 2. We consider the source $g(x, y) = xz + yz + xy$. This function also satisfies the compatibility condition (6). Figure 8 shows the graphic of the exact source. Figure 9 shows the solution $A(g) = V$ of the forward problem. The error measurement is obtained as in the previous example, and it is shown in Fig. 10. Figure 11 shows the recovered source without regularization ($\alpha = 0$) and $N = 80$ terms. The relative error is 2.03. Figure 12 shows the regularized source obtained by taking $N = 10$ and $\alpha = 10^{-6}$. The relative error is 0.09.

5. Conclusion

We study the inverse electroencephalographic problem for cortical sources (IEP), which consists of finding the cerebral cortex sources from EEG scalp measurements. This problem is ill-posedness since it presents numerical instability. We present an algorithm to solve the IEP, which uses the Tikhonov regularization and a cut-off (truncation) to handle the numerical instability. We found the regularization parameters from the numerical test to get the stable algorithm. The former regularization parameter is the Tikhonov regularization parameter. The latter is the term N in which we truncate the harmonic expansion of the source. From the numerical tests, we found these parameters, which give good approximations of the sources. In the case of complex geometries for the head, also we can use the Tikhonov functional (26) to recover the source. In this case, it is possible to use an iterative method to find the minimum of the mentioned Tikhonov functional. One of the most used is the conjugate gradient method. However, other methods can be used, like the Quasi-Newton methods. Furthermore, we must solve the direct problems in each iteration using a numerical method, like finite differences or the Finite element method. The parameter associated with the truncation gives an idea to choose the subspace obtained in the discretization of the numerical method used for solving the direct problems. Finally, we illustrate the proposed algorithm with synthetics examples that show its feasibility.

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