Landau levels for a Weyl pair in a monolayer medium and thermal quantities

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In this paper, we consider a Weyl pair under the effect of an external uniform magnetic field in a monolayer medium without considering any charge-charge interaction between the particles. Choosing the interaction of the particles with the magnetic field in the symmetric gauge we seek for an analytical solution of the corresponding form of a one-time fully-covariant two-body Dirac equation derived from quantum electrodynamics via the action principle. As it is usual with two-body problems, we separate the relative motion and center of mass motion coordinates. Assuming the center of mass is at rest, we derive a matrix equation in terms of the relative motion coordinates without considering any group theoretical method. This equation gives a wave equation in exactly soluble form and accordingly we obtain the spinor components and complete energy eigen-states (in closed form) for such a spinless composite structure. Our results not only give exact Landau levels for such a Weyl pair in a monolayer medium but also show the considered system behaves as a two-dimensional harmonic oscillator. Furthermore, our findings give exactly the excited states of a Weyl particle under the effect of uniform external magnetic field in a monolayer graphene sheet and there is no imprint to distinguish these modes from each other. This means that the performed experiments based on Landau levels for a monolayer graphene sheet may actually involve many-body effects. Our results provide a suitable basis to analyze the associated thermal quantities and accordingly we discuss the thermal properties by determining free energy, total energy, entropy and specific heat for the composite system in question.

Keywords: Landau levels; graphene; Weyl fermions, charge carriers; many-body system; thermal properties.

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1. Introduction

In the non-relativistic quantum mechanics frame, the dynamics of two-body systems is investigated through one-time equations including inter-particle interaction potentials depending on relative radial coordinate and this is the well-known way to describe the bound states, resonance states as well as the scattering states. These equations include, of course, two-body wave functions. Relativistic quantum mechanics, phenomenologically establishes two-body equations to analyse the dynamics of two interacting particles and these equations consist of free Hamiltonians for each particle besides inter-particle interaction potentials. Even though, one of the main problems is how to choose the interaction potentials, which are chosen phenomenologically. That is, mesonic or electrodynamic (one-boson or one-photon exchange) potentials are preferred in general. Also, two-time problem appears in these phenomenologically constructed two-body equations because each particle feels its proper time. After the Dirac equation was written, the first acceptable attempt to write a two-body Dirac equation was made by Breit [1]. This equation includes two free Dirac Hamiltonians plus an interaction term established by modifying the Darwin potential. However, this equation works properly only in the weak coupling regime due to the retardation effects. That is, it cannot give precise results if the particles have high velocities or the interaction is large range. This was a serious problem that must be overcome. To overcome that another formalism was introduced by Bethe and Salpeter by starting from the Quantum Field Theory [2]. However, this new formalism could only provide approximate solutions for bound states due to the relative time difference between particles. Thus, the required equation including relativistic kinematics had to be exactly soluble in 3+1 dimensions and had to be a one-time fully-covariant two-body equation taking into account the retardation effects and including correct spin algebra. Furthermore, such an equation had to be usable in curved spaces. In Ref. [3], Barut has shown us how it is possible to derive a complete and one time fully-covariant two-body Dirac equation from Quantum Electrodynamics. This equation includes the correct spin algebra spanned by direct (Kronecker) product of Dirac matrices, takes into account the retardation effects, includes the most general electric and magnetic potentials [3] and moreover it can be usable in curved spaces [4]. In 3+1 dimensions, the solution of the Barut's equation requires group theoretical methods to separate radial and angular parts. Briefly, this equation leads to 16 equations and these equations can be reduced into 8 equations thanks to symmetry. However, it results in two second-order differential equations (coupled) or four first-order equations. This means that the solution of this equation for Hydrogen-like systems could be obtained only through a perturbative way [6] and related precise solutions cannot be obtained. In 3+1 dimensions, Moshinsky and Loyola used the Barut's equation to analyze a Dirac pair with Dirac oscillator interaction and applied the obtained perturbative results to estimate mass spectra for composite particles such as mesons and baryons [7].

In 2+1 dimensions, relativistic quantum theory and gravity have gained interest after the seminal papers in the Refs. [8, 9] and discovery of the Banados-Teitelboim-Zanelli black hole [10] and Graphene [11, 35]. Prior to the aforementioned discoveries, it was believed that 2+1 dimensional studies can be useful only for discussing some conceptual issues. Graphene is a two-dimensional (2D) material exhibiting exceptionally high crystal and very high electronic quality [11, 35]. This 2D material is formed by carbon atoms in a honeycomb lattice. Low energy electronic spectrum of the graphene can be described by massless Dirac particles (electron and hole) called as Weyl fermions. Graphene and 2D materials are the premier sources of the latest information on commercial and practical applications of 2D materials. These materials are defined as crystalline materials consisting of single or few-layer atoms, in which the inter-atomic interactions are much stronger than those along the stacking direction. They have unique physical and chemical properties due to their reduced dimensionality and quantum confinement effects [36, 37]. These properties enable particles or quasiparticles such as electrons, excitons and magnons to exhibit exotic behaviors differing from their 3D bulk counterparts stemming from the quantum confinement effect. They have attracted tremendous research interest in recent years because of their potential applications in various fields, such as nanoelectronics, optoelectronics, the quantum Hall effect, phase space representation of Wigner functions, quantum heat engine and excitonic systems (see the following Refs. [38–46]). Thus, investigations based on the dynamics of a Weyl pair exposed to an external magnetic field in a monolayer medium can be very useful for clarifying some points. To do this, the fully-covariant two-body Dirac equation can be very useful to determine precise solutions.

In this paper, we consider a Weyl pair under the effect of an external uniform magnetic field in a homogeneous monolayer medium and try to determine the dynamics of such a pair by solving the corresponding form of the fully-covariant two-body Dirac equation. To do this, we choose the coupling of each particle with the external field in the symmetric gauge which allows us to compare the result with the related relativistic oscillators and arrive at a wave equation for such a spinless static system. We obtain energy eigen-states besides the associated spinor components and then we discuss the results in detail. The form of the obtained non-perturbative energy spectrum allows us to determine the associated thermal quantities and we also discuss the thermal properties by determining free energy, total energy, entropy and specific heat for the system in question.

2. Two-body Dirac equation

In this part, we will introduce the covariant two-body equation and will be interested in the relative motion of a mutually non-interacting fermion-antifermion pair exposed to an external uniform magnetic field, by choosing the interaction of the particles with the external field in symmetric gauge, so that we can obtain precise solutions. Then, for the system in question, we will derive the corresponding form of the covariant two-body Dirac equation and will arrive at a set of coupled equations in matrix form. Here, it is worth mentioning that choosing this gauge allows us also to write the equations in the most symmetric form. The generalized form of this equation can be written as [3–5]

$$\left\{\mathcal{H}_1 \otimes \gamma^{0^{\overline{f}}} + \gamma^{0^{f}} \otimes \mathcal{H}_2\right\} \Psi\left(\mathbf{x}_1, \mathbf{x}_2\right) = 0,$$
$$\mathcal{H}_1 = \left[\gamma^{\mu^{f}} \not{\mathcal{D}}_f + i\mathcal{M}_1 \mathbf{I}_2\right], \quad \mathcal{H}_2 = \left[\gamma^{\mu^{\overline{f}}} \not{\mathcal{D}}_{\overline{f}} + i\mathcal{M}_2 \mathbf{I}_2\right],$$
$$\not{\mathcal{D}}_f = \partial^f_\mu + i\frac{e^f \mathcal{A}^f_\mu}{\hbar\mathcal{V}} - \Gamma^f_\mu, \quad \not{\mathcal{D}}_{\overline{f}} = \partial^{\overline{f}}_\mu + i\frac{e^{\overline{f}} \mathcal{A}^{\overline{f}}_\mu}{\hbar\mathcal{V}} - \Gamma^{\overline{f}}_\mu,$$
$$\mathcal{M}_1 = \frac{m_f \mathcal{V}}{\hbar}, \quad \mathcal{M}_2 = \frac{m_{\overline{f}} \mathcal{V}}{\hbar}, \tag{1}$$

in which f and $\overline{f}$ refer to fermion and antifermion, respectively, γ^μ are the generalized Dirac matrices, $\mathbf{I}_2$ are the two-dimensional identity matrices, $\mathcal{A}_\mu$ are the 3-vector potentials, e stands for the elementary electrical charge of the particles, Γ_μ are the spinorial affine connections, m represents the mass of the particle, $\mathcal{V}$ is the Fermi velocity, $\hbar$ is the reduced Planck constant, Ψ is the bi-spinor depending on both the spacetime position vectors ($\mathbf{x}_f, \mathbf{x}_{\overline{f}}$) of the particles and the symbols $\otimes$ are used to indicate the direct product. Here, we are interested in a fermion-antifermion system in a globally and locally flat monolayer medium that can be represented by the line element: $ds^2 = \mathcal{V}^2 dt^2 - dx^2 - dy^2$ for which the spinorial affine connections do not make any contribution to the dynamics of the particles since they vanish [14]. The generalized Dirac matrices are found through the relation $\gamma^\mu = e^\mu_{(a)} \overline{\gamma}^{(a)}$ where $e^\mu_{(a)}$ are inverse tetrad fields and $\overline{\gamma}^{(a)}$ are flat Dirac matrices that can be chosen by means of Pauli matrices ($\sigma_x, \sigma_y, \sigma_z$) in three dimensions. The flat Dirac matrices, which must be selected to provide the signature $(+, -, -)$ of the given metric, can be chosen as $\overline{\gamma}^0 = \sigma_z$, $\overline{\gamma}^1 = i\sigma_x$ and $\overline{\gamma}^2 = i\sigma_y$ where $i = \sqrt{-1}$ [14]. The tetrad fields can be constructed by using the relation: $g_{\mu\nu} = \mathrm{diag}(\mathcal{V}^2, -1, -1) = e^{(a)}_\mu e^{(b)}_\nu \eta_{(a)(b)}$ where $g_{\mu\nu}$ is the contravariant metric tensor, $e^{(a)}_\mu$ are the tetrad fields and $\eta_{(a)(b)}$ is the flat Minkowski tensor $\eta_{(a)(b)} = \mathrm{diag}(1, -1, -1)$. Thereby, it is possible to chose these fields as $e^{(0)}_0 = \pm\mathcal{V}$, $e^{(1)}_1 = \pm 1$ and $e^{(2)}_2 = \pm 1$. Choosing positive signature, one can determine the inverse tetrad fields as $e^0_{(0)} = \frac{1}{\mathcal{V}}$, $e^1_{(1)} = 1$ and $e^2_{(2)} = 1$ since the tetrad fields must admit the following orthogonality and orthonormality conditions: $e^\mu_{(a)} e^{(a)}_\nu = \delta^\mu_\nu$ and $e^{(a)}_\mu e^\mu_{(b)} = \delta^{(a)}_{(b)}$ where $a, b = 0, 1, 2$ and $\mu, \nu = t, x, y$ [15]. Here, we consider that the particles interact only with the external uniform magnetic field ($\mathcal{A}^{f,\overline{f}}_t = 0$). We can choose the coupling of each particle with the external field in symmetric gauge [16] (see also [13]) as $\mathcal{A}^f_x = -\mathcal{B}_0 y_f/2$,

$\mathcal{A}_y^f = \mathcal{B}_0 x_f/2$, $\mathcal{A}_x^{\overline{f}} = -\mathcal{B}_0 y_{\overline{f}}/2$, $\mathcal{A}_y^{\overline{f}} = \mathcal{B}_0 x_{\overline{f}}/2$ where $\mathcal{B}_0$ is the amplitude of the external magnetic field. Now, we need to separate the center of mass and relative motion coordinates carefully to acquire a set of coupled equations, by means of relative motion coordinates, for such a pair ($e_f = -e_{\overline{f}} = e$ and $m_f = m_{\overline{f}} = m$). This requires to define first the center of mass ($\mathcal{R}$) and relative motion (r) coordinates. We can use the following expressions to acquire a matrix equation in terms of the r coordinates [17]

$$r_\mu = x_\mu^f - x_\mu^{\overline{f}}, \quad \mathcal{R}_\mu = \frac{x_\mu^f}{2} + \frac{x_\mu^{\overline{f}}}{2}, \quad x_\mu^f = \frac{r_\mu}{2} + \mathcal{R}_\mu,$$
$$x_\mu^{\overline{f}} = -\frac{r_\mu}{2} + \mathcal{R}_\mu, \quad \partial_{x_\mu}^f = \partial_{r_\mu} + \frac{\partial_{\mathcal{R}_\mu}}{2},$$
$$\partial_{x_\mu}^{\overline{f}} = -\partial_{r_\mu} + \frac{\partial_{\mathcal{R}_\mu}}{2},$$

which leads $\partial_{x_\mu}^f + \partial_{x_\mu}^{\overline{f}} = \partial_{\mathcal{R}_\mu}$. Let the center of mass locates at $(x = 0, y = 0)$ point of the spatial background and does not carry momentum. This requires that the particles carry opposite momenta with respect to each other and any pairing effect may become important for this static case. The generalized Dirac matrices in Eq. (1) become as $\gamma^{t^{f,\overline{f}}} = \sigma_z/\mathcal{V}$, $\gamma^{x^{f,\overline{f}}} = i\sigma_x$ and $\gamma^{y^{f,\overline{f}}} = i\sigma_y$ since they are independent from the spacetime coordinates even though the temporal parts depend on the velocity $\mathcal{V}$(constant). By assuming the interaction is time-independent we can define the bi-spinor, in terms of the r coordinates, as follows[i] $\Psi = e^{-i(\mathcal{E}/\hbar)t}\tilde{\psi}(\mathbf{r})$, where $\mathcal{E}$ is the energy of the system and $\tilde{\psi}(\mathbf{r})$ represents the spatial part of the spinor. At last, we derive the following matrix equation: $\hat{M}\tilde{\psi} = 0$ in which

$$\hat{M} = \begin{pmatrix} \overset{-}{\not{\mathcal{E}}} & \hat{\mathcal{D}}_- - \mathcal{B}\chi_- & -\hat{\mathcal{D}}_- - \mathcal{B}\chi_- & 0 \\ \hat{\mathcal{D}}_+ + \mathcal{B}\chi_+ & -\not{\mathcal{E}} & 0 & \hat{\mathcal{D}}_- + \mathcal{B}\chi_+ \\ -\hat{\mathcal{D}}_+ + \mathcal{B}\chi_+ & 0 & -\not{\mathcal{E}} & -\hat{\mathcal{D}}_- + \mathcal{B}\chi_- \\ 0 & \hat{\mathcal{D}}_+ - \mathcal{B}\chi_+ & -\hat{\mathcal{D}}_+ - \mathcal{B}\chi_+ & \overset{+}{\not{\mathcal{E}}} \end{pmatrix},$$

$$\tilde{\psi} = (\psi_1\ \psi_2\ \psi_3\ \psi_4)^{\mathbf{T}}, \quad \overset{\pm}{\not{\mathcal{E}}} = \not{\mathcal{E}} \pm \frac{2m\mathcal{V}}{\hbar}, \quad \not{\mathcal{E}} = \frac{\mathcal{E}}{\hbar\mathcal{V}},$$
$$\hat{\mathcal{D}}_\pm = \partial_x \pm i\partial_y, \quad \chi_\pm = x \pm iy, \tag{2}$$

and[ii] $\mathcal{B} = e\mathcal{B}_0/4\hbar\mathcal{V}$. Here, one should notice that each of the spinor components depends on the x, y coordinate pair, as $\psi_j(x, y), (j = 1, 2, 3, 4)$. Thus we need, at least, a symmetry to acquire an analytical solution of this matrix equation. Let us transform the system into polar coordinates (r, ϕ), in terms of the transformed spinor $\tilde{\psi} = \left(\psi_1(r)e^{i(s-1)\phi}\ \psi_2(r)e^{is\phi}\ \psi_3(r)e^{is\phi}\ \psi_4(r)e^{i(s+1)\phi}\right)^{\mathbf{T}}$ [18], through spin (s) raising ($+$) and spin lowering ($-$) operators: $\hat{\mathcal{D}}_\pm = e^{\pm i\phi}\left(\pm\frac{i}{r}\partial_\phi + \partial_r\right)$ [17] besides $\chi_\pm = x \pm iy$. After some arrangements, one can arrive at the following set of equations

$$\not{\mathcal{E}}\vartheta_1(r) + 2\dot{\vartheta}_3(r) - 2\mathcal{B}r\vartheta_4(r) = 0,$$
$$\not{\mathcal{E}}\vartheta_2(r) = 0,$$
$$\not{\mathcal{E}}\vartheta_3(r) - \frac{2}{r}\vartheta_1(r) - 2\dot{\vartheta}_1(r) = 0,$$
$$\not{\mathcal{E}}\vartheta_4(r) - 2\mathcal{B}r\vartheta_1(r) = 0, \tag{3}$$

in which the dot means derivative with respect to the r, for a static spinless composite system consisting of a Weyl ($m = 0$) pair exposed to an external uniform magnetic field in a spatially flat monolayer medium if

$$\vartheta_1(r) = \psi_1(r) + \psi_4(r), \quad \vartheta_2(r) = \psi_1(r) - \psi_4(r),$$
$$\vartheta_3(r) = \psi_2(r) - \psi_3(r), \quad \vartheta_4(r) = \psi_2(r) + \psi_3(r),$$

and only if $\mathcal{E} \neq 0$. That is $\vartheta_2(r) = 0$ if one considers the $\mathcal{E} \neq 0$ case. This cannot appear when $m \neq 0$, of course.

3. Landau levels for a Weyl pair

Here, we try to determine exact Landau levels for a static Weyl pair under the influence of an external uniform magnetic field in a flat monolayer medium. To acquire this, we look for an analytical solution of the set of equations in Eq. (3). For this purpose, we start by considering a dimensionless independent variable, $\xi = \mathcal{B}r^2$ which leads $r = \sqrt{\xi/\mathcal{B}}$. Here, we should notice that $\xi \rightsquigarrow 0$ if $r \rightsquigarrow 0$ and $\xi \rightsquigarrow \infty$ if $r \rightsquigarrow \infty$ provided that $\mathcal{B}_0 \neq 0$. By means of the variable ξ, Eq. (3) leads to the following set of equations

$$\not{\mathcal{E}}\vartheta_1(\xi) + 4\mathcal{B}\sqrt{\frac{\xi}{\mathcal{B}}}\dot{\vartheta}_3(\xi) - 2\mathcal{B}\sqrt{\frac{\xi}{\mathcal{B}}}\vartheta_4(\xi) = 0,$$
$$\not{\mathcal{E}}\vartheta_3(\xi) - \frac{2}{\sqrt{\frac{\xi}{\mathcal{B}}}}\vartheta_1(\xi) - 4\mathcal{B}\sqrt{\frac{\xi}{\mathcal{B}}}\dot{\vartheta}_1(\xi) = 0,$$
$$\not{\mathcal{E}}\vartheta_4(\xi) - 2\mathcal{B}\sqrt{\frac{\xi}{\mathcal{B}}}\vartheta_1(\xi) = 0, \tag{4}$$

one of which is algebraic. In the second and third equation, we can easily see that the $\vartheta_3(\xi)$ and $\vartheta_4(\xi)$ components can be expressed in terms of $\vartheta_1(\xi)$. That is the first equation in Eq. (4) gives a wave equation for the component $\vartheta_1(\xi)$ and this wave equation can be rewritten by considering an ansatz function, $\vartheta_1(\xi) = \vartheta(\xi)/\sqrt{\xi}$, as

$$\ddot{\vartheta}(\xi) + \left(-\frac{1}{4} + \frac{\not{\mathcal{E}}^2}{16\mathcal{B}\xi}\right)\vartheta(\xi) = 0. \tag{5}$$

Solution function of this equation can be expressed in terms of the Kummer Confluent Hypergeometric function [19, 20], ${}_1F_1$, as $\vartheta(\xi) = \mathcal{C}^*\xi e^{-\xi/2}\ {}_1F_1\left([\{-\not{\mathcal{E}}^2 + 16\mathcal{B}\}/16\mathcal{B}], [2], \xi\right)$ where $\mathcal{C}^*$ is a constant. For large values of the argument ξ, a ${}_1F_1([\epsilon], [\delta], \xi)$ function becomes [21]

$${}_1F_1([\epsilon], [\delta], \xi) \approx \frac{\Gamma(\delta)}{\Gamma(\epsilon)}e^{\xi}\xi^{\epsilon-\delta}\left[1 + \mathcal{O}(|\xi|^{-1})\right].$$

This function is divergent when $\xi \rightsquigarrow \infty$. Here, we seek for a regular solution of Eq. (5) and hence we need that $\epsilon = -n$ where $n = 0, 1, 2...$ In our case $-n = (-\not{\xi}^2 + 16\mathcal{B})/16\mathcal{B}$. This condition guarantees the solution function becomes well-behaved when $\xi \rightsquigarrow \infty$. Through this termination, which gives the quantization condition, we acquire the following energy spectrum for the considered system

$$\mathcal{E}_n = \pm 2\frac{\hbar\mathcal{V}}{\ell_{\mathcal{B}}}\sqrt{n+1}, \quad n = 0, 1, 2.., \tag{6}$$

where $\ell_{\mathcal{B}}$ is the magnetic length, $\ell_{\mathcal{B}} = \sqrt{\hbar/e\mathcal{B}_0}$ [22]. Furthermore, we can determine the defined spinor components as follows

$$\begin{aligned}
\theta_{1_n}(\xi) &= \mathcal{C}^* \sqrt{\xi} e^{-\frac{\xi}{2}} {}_1F_1([-n], [2], \xi), \\
\theta_{3_n}(\xi) &= -\mathcal{C}^* \frac{2\sqrt{\xi}e^{-\frac{\xi}{2}}}{\not{\xi}\sqrt{\frac{\xi}{\mathcal{B}}}} n\xi \, {}_1F_1([-n+1], [3], \xi) \\
&\quad - \mathcal{C}^* \frac{2\sqrt{\xi}e^{-\frac{\xi}{2}}}{\not{\xi}\sqrt{\frac{\xi}{\mathcal{B}}}} (\xi - 2) \, {}_1F_1([-n], [2], \xi), \\
\theta_{4_n}(\xi) &= \mathcal{C}^* \frac{2\mathcal{B}}{\not{\xi}} \sqrt{\frac{\xi}{\mathcal{B}}} \sqrt{\xi} e^{-\frac{\xi}{2}} {}_1F_1([-n], [2], \xi).
\end{aligned} \tag{7}$$

From (6), we see that the energy of such a static pair depends on the Fermi velocity ($\mathcal{V} \sim c/300$ [13]), reduced Planck constant ($\hbar$), magnetic length ($\ell_{\mathcal{B}}$) and overtone quantum number n. Furthermore, one should notice that the considered system behaves like two-dimensional relativistic harmonic oscillator and it does not stop oscillating even when $n = 0$ (see Fig. 1). The results show that magnitude of the energy levels ($|\mathcal{E}_n|$) is large if $\ell_{\mathcal{B}} \ll 1$ and $|\mathcal{E}_n|$ increases as $\mathcal{B}_0$ increases for any quantum state (see Fig. 1).

TABLE I. For $e = 1$, $\hbar = 1$ and $c = 1$.

$\mathcal{B}_0$	$\ell_{\mathcal{B}}$
1	1
5	0.447
10	0.316
15	0.258
20	0.223
25	0.2
30	0.182

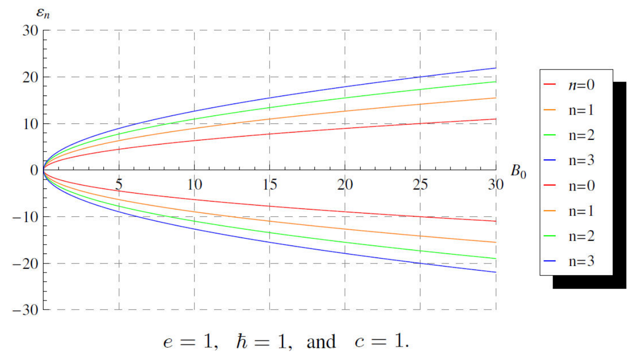

FIGURE 1. Dependence of the energy levels on the amplitude of the external uniform magnetic field (see also [25]).

4. Thermal properties

4.1. Euler-Maclaurin formula

In this section, we calculate the different thermodynamic variables using the standard definition of the partition function Z. In order to obtain more accurate quantities, we shall use the infinite sums of n-contributions of the energy multiplied by the constant β. Therefore, it is convenient to write the different thermodynamic variables in terms of these sums and perform a numerical computation for each variable for a certain range of the temperature T. The partition function is given by

$$Z = \sum e^{-\beta E_n}, \tag{8}$$

here $\beta = 1/kT$, k is the Boltzmann constant. Here, considering only positive energies in calculating, the partition function can be justified as follows: (i) The Dirac equation has an exact Foldy-Wouthuysen transformation and this means that positive and negative energy solutions do not mix. (ii) We assume that the negative energy (antiparticle) as fully occupied: It is correct because all fermions are ordered by the Pauli's principle. Now, to evaluate the partition function, we use the Euler-Maclaurin formula which gives the difference between an integral and a closely related sum. It makes the connection between the sum and the integral explicit for sufficiently smooth functions. In the most general form, it can be written as [26, 27]

$$\begin{aligned}
\sum_{n=a}^{b} f(n) &= \frac{1}{2}\{f(a) + f(b)\} + \int_a^b f(n)\,dn \\
&\quad + \sum_{i=2}^{k} \frac{b_i}{i!}\left\{f^{(i-1)}(b) - f^{(i-1)}(a)\right\} \\
&\quad - \int_a^b \frac{B_k(\{1-t\})}{k!} f^{(k)}(t)\,dt,
\end{aligned} \tag{9}$$

where a and b are arbitrary real numbers with difference $b-a$ being a positive integer number, B_n and b_n are Bernoulli

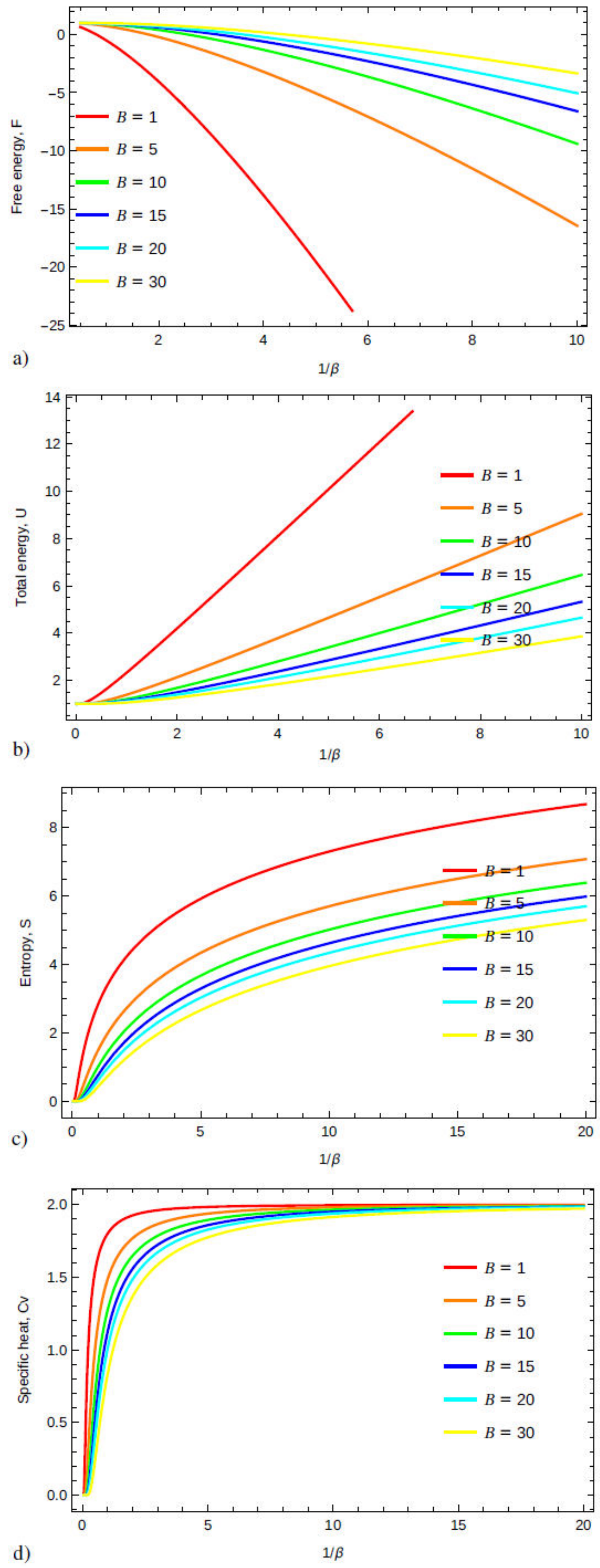

FIGURE 2. Thermal properties as a function of $1/\beta$ for different values of the magnetic length l_B: a) Free energy F. b) Total energy U. c) Entropy S. d) Specific heat C_v.

polynomials and numbers, respectively, and k is any positive integer. The condition we impose on the real function f is that it should have continuous k-th derivative. The symbol $\{x\}$ for a real number x denotes the fractional part of x. Here, the remainder term (error term)

$$R_k = \int_a^b \frac{B_k\left(\{1-t\}\right)}{k!} f^{(k)}\left(t\right) dt, \tag{10}$$

is the most essential in the Euler-Maclaurin equation. If $f\left(x\right)$ and all its derivatives tend to 0 as $x \to \infty$, the formula can be simplified:

$$\sum_{n=0}^{\infty} f\left(n\right) = \frac{f\left(0\right)}{2} + \int_0^{\infty} f\left(n\right) dn - \sum_{i=2}^{k} \frac{b_i f\left(0\right)}{i!} - \int_0^{\infty} \frac{B_k\left(\{1-t\}\right)}{k!} f^{(k)}\left(t\right) dt. \tag{11}$$

The first several Bernoulli numbers are the following:

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_4 = -\frac{1}{30}. \tag{12}$$

The odd terms in the sequence are all 0 except the first one b_1. The Bernoulli polynomials B_n can be defined by a generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n\left(x\right) \frac{t^n}{n!}. \tag{13}$$

The first few Bernoulli polynomials are:

$$\begin{aligned} B_0\left(x\right) &= 1, \\ B_1\left(x\right) &= x - \frac{1}{2}, \\ B_2\left(x\right) &= x^2 - x + \frac{1}{6}, \\ B_3\left(x\right) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \end{aligned} \tag{14}$$

Also, more general, for a positive integer n, we define the periodic Bernoullian function $\bar{B}_n = B_n(\{x\})$ where $\{x\}$ denotes the fractional part of x. We can see that $\bar{B}_n = B_n(\{x\})$ is periodic with period 1 and continuous on $[0,1]$. That means that the fractional parts of the Bernoulli numbers are dense in the interval $[0,1]$ [28]. Following this remark, and as proved by Elliot [29], the final form of the partition function becomes:

$$\sum_{n=0}^{\infty} f\left(n\right) = \frac{f\left(0\right)}{2} + \int_0^{\infty} f\left(n\right) dn - \sum_{i=2}^{k} \frac{b_i f\left(0\right)}{i!} - \int_0^{1} \frac{B_k\left(\{1-t\}\right)}{k!} f^{(k)}\left(t\right) dt. \tag{15}$$

In what follows, all thermodynamic properties of the system in question, such as the free energy, the entropy, total energy and the specific heat, can be obtained through the numerical partition function Z. Looking for simplicity, we will prefer to use the natural units ($\hbar = c = k_B = e = 1$), so that all parameters can be considered as dimensionless.

4.2. Numerical results and discussions

Now, we discuss and comment on our numerical results on the calculation of the thermal quantities obtained via the partition function. We should mention that, in all the figures, we have used adimensional quantities. According to the above considerations, we can define the thermodynamic functions of interest as follows:

$$F = -\frac{\log Z}{\sqrt{\mathcal{B}}\beta}, \quad U = -\frac{1}{\sqrt{\mathcal{B}}}\frac{d\log Z}{d\beta}, \tag{16}$$

and

$$S = \log Z - 2\beta\frac{d\log Z}{d\beta}, \quad C_v = \beta^2\frac{d^2\log Z}{d\beta^2}. \tag{17}$$

The integral appearing in (11) can be calculate as follows:

$$\int_0^\infty e^{-\beta\sqrt{\mathcal{B}}\sqrt{n+1}}dn = \frac{2e^{\beta\left(-\sqrt{\mathcal{B}}\right)}\left(\beta\sqrt{\mathcal{B}}+1\right)}{\beta^2\mathcal{B}}. \tag{18}$$

After fixed $k = 4$, the explicit form of the partition function (Eq. (15)) is given by

$$\begin{aligned} Z(\beta,\mathcal{B}) = {} & e^{-\beta\sqrt{\mathcal{B}}} + \frac{e^{-\sqrt{2\mathcal{B}}\beta}}{46080\beta^2\mathcal{B}} \\ & \times\left\{92160\left(\sqrt{2\mathcal{B}}\beta+1\right) + 23040\mathcal{B}\beta^2\right\} \\ & + \frac{e^{-\sqrt{2\mathcal{B}}\beta}}{46080\beta^2\mathcal{B}}\Big\{-6\mathcal{B}^2\beta^4 \\ & + 957\sqrt{2}\sqrt{\mathcal{B}}^{3/2}\beta^3 - 2\sqrt{2}\sqrt{\mathcal{B}}^{5/2}\beta^5\Big\}. \end{aligned}$$

With the aid of the partition function Z the thermal properties of the considered system can be found easily. These thermodynamic functions are represented according to the inverse temperature β and for different values of the magnetic field $\mathcal{B}$. Thus, we have chosen $\mathcal{B} = 1, 5, 10, 15, 20, 25, 30$. The dimensionless variable $\bar{\beta} = 2\mathcal{V}\hbar\beta/l_{\mathcal{B}} = 2\mathcal{V}\hbar/l_{\mathcal{B}}k_BT$ can help us to define the characteristic temperature T_0 [30], in IS(international system), with the following expression:

$$T_0 = \frac{2\mathcal{V}\hbar}{l_{\mathcal{B}}k_BT}. \tag{19}$$

This temperature is similar to the Debye temperature in the solid state [30–32]. It also depends inversely on the intensity of the magnet field. Table II provides some values for this temperature in SI for the case of graphene. One has massless particles moving through the honeycomb lattice with a velocity $\mathcal{V} = 1.1\times10^6$ m/s the so-called Fermi velocity [31, 33].

TABLE II. Some values of the characteristic temperature T_0.

$\mathcal{B}$(Tesla)	$\ell_{\mathcal{B}}$($\times 10^{-8}$ meter)	T_0(K)
1	2.56	655
5	1.14	1465
10	0.81	2071
15	0.67	2537
20	0.58	2929
25	0.51	3275
30	0.47	3587

The obtained results are illustrated in the Fig. 2. According to this Figure, the following may be observed:

- The effect of the magnetic field is observed in all thermal quantities. This dependency is inversely with the field.
- Entropy and specific heat curves tend to zero at low temperatures.
- Comparing with the existing studies, this remark is due to the adding the reminder term in the Euler-Mclaurin formula which has been dropped in these studies (see for example [30, 34]). This term has the role the avoid the divergence in the partition function and consequently all thermal quantities of our problem.
- Now, at very high temperatures, the specific heat curves converge to 2. This convergence depends inversely on the applied magnetic field $\mathcal{B}$. The convergence to this point is faster in the lower region of the magnetic field $\mathcal{B}$ than in higher values of it.

5. Summary and results

In this manuscript, we have studied the dynamics of a Weyl pair (mutually non-interacting) exposed to an external uniform magnetic field in a monolayer medium. To do this, we have used the fully-covariant two-body Dirac equation derived from Quantum Electrodynamics via the action principle. First of all, by choosing the interaction of the particles with the external uniform magnetic field in the symmetric gauge, we have written the corresponding form of this one-time two-body equation for a general fermion-antifermion pair. Afterwards, we have separated the center of mass motion coordinates and relative motion coordinates as is usual with two-body problems. By assuming the center of mass is at rest at the spatial origin, we have arrived at a matrix equation consisting of four first-order equations (coupled) in terms of the relative motion coordinates. We have transformed the background into the polar space so that we can exploit the angular symmetry. Then, we have reduced the obtained matrix equation resulting in three equations, one of which is algebraic, for a such a spinless composite system formed by a Weyl pair. These equations allow us to derive a wave equation in exactly soluble form. Solution function of this equation can be expressed in terms of the Kummer Confluent Hypergeometric function. Accordingly we have obtained the energy spectrum (see Eq. (6)) in closed-form besides the defined spinor components (see Eq. (7)). Equation (6) has shown that energy of such a pair depends on the Fermi velocity ($\mathcal{V}$), magnitude of the elementary electrical charge (e), amplitude of the external uniform magnetic field ($\mathcal{B}_0$) besides the reduced Planck constant $\hbar$. Our results have shown that such a static pair behaves as two-dimensional harmonic oscillator (this can be seen by taking $\mathcal{B}_0 = \omega\hbar c/2e$, where the ω is the oscillator frequency [23]) and it does not stop oscillating even when the system reaches the ground state ($n = 0$). The obtained energy ($\mathcal{E}_n$) spectrum can be expressed in terms of the magnetic length, $\ell_{\mathcal{B}} = \sqrt{\hbar/e\mathcal{B}_0}$, as $\mathcal{E}_n = \pm\hbar\mathcal{V}/\ell_{\mathcal{B}}\sqrt{4(n+1)}$ and $|\mathcal{E}_n|$ can be very large when $\ell_{\mathcal{B}} \ll 1$ whether $n = 0$ or not. Such a system may appear in a monolayer graphene sheet under the effect of an external uniform magnetic field. Landau levels for a Weyl particle in a monolayer medium was obtained as $E_n = \pm\hbar\mathcal{V}/\ell_{\mathcal{B}}\sqrt{2|n'|}$ where $|n^{'}| = 0, 1, 2..$ [13](see also [24]). Our results, in principle, seem to be as an excited state of the related one-body state and, at first look, we cannot see any imprint to distinguish these modes from each other. Thus, some observations based on Landau levels for a single-layer graphene may include two-body effects (see [25]), at least similar to the one studied here. Finally, we have calculated all the thermal quantities such as free energy, total energy, entropy and specific heat for the considered composite structure via the partition function through the obtained non-perturbative energy expression and the partition function was derived by a method based on the Euler-Maclaurin formula. As a consequence, we observe that

- The third law of thermodynamics of entropy and specific heat
$$\lim_{T\longrightarrow 0} C_V = 0,\ \lim_{T\longrightarrow 0} S = 0,$$
is well fulfilled.
- These thermal properties depend inversely with the magnetic field.
- In higher temperatures, all the curves of specific heat tend towards 2. When the magnetic field increases, this convergence goes towards this limit very slowly.

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Conflict of Interest Statement

No conflict of interest has been declared by the authors.

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i. Note that the Eq. (1) leads $[\sigma_z \otimes \sigma_z](\partial_0^f + \partial_0^{\overline{f}})$ term. Thus, the energy of such a static pair is determined according to the ∂_{R_t}.

ii. **T** means transpose of the spinor.

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