Multiple soliton and traveling wave solutions of the negative-order-KdV-CBS model

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The (3+1)-dimensional new negative-order-KdV-CBS model is investigated in this study. The suggested model combines the Korteweg-de Vries (KdV) and Calogero-Bogoyavlenskii-Schiff (CBS) equations. This research provides multiple soliton solutions and traveling wave solutions for the KdV-CBS model. Multiple exp-function methods have been used for extracting soliton solutions. For this aim, the extended sinh-Gordon equation expansion approach was selected to get traveling wave solutions. The findings are graphically examined by selecting appropriate values for arbitrary parameters.

1. Introduction

Nonlinear partial differential equations can mathematically reflect a number of complicated processes in applied sciences and different fields of engineering, namely fluid dynamics, optics, plasma physics, elastic media, quantum field theory and optical fiber communication. These exact solutions are crucial for understanding various dynamical processes and real-world occurrences in a variety of scientific fields. For this reason, these days this area is becoming increasingly popular. Numerous techniques have been developed in the current context to locate these equations' analytical solutions, including the modified simple equation technique [1], Hirota bilinear method [2], the extended unified method [3], extended transformed rational function procedure [4], the generalized projective Riccati equations procedure [5], auto-Bäcklund transformations [6], Lie-Bäcklund symmetries [7], auxiliary equation method [8], Nuccis reduction approach [9], the generalized Kudryashov technique [10, 11], and so on. In this work, a (3+1)-dimensional new negative-order-KdV-CBS model [13] is taken as

$$p_{xt} + p_{xxxy} + 4p_x p_{xy} + 2p_{xx}p_y$$
$$+ \lambda p_{xx} + \mu p_{xy} + v p_{xz} = 0, \tag{1}$$

where μ, λ, and v are constants. Eq. (1) is the result of combining the KdV equation with the Calogero-Bogoyavlenskii-Schiff (CBS) equation. For $v = 0$ and $\mu = 0$, Eq. (1) becomes the negative-order KdV equation. For $\lambda = 0$ and $v = 0$ Eq. (1) becomes the negative-order CBS equation.

The considered equation is studied in this paper via two efficient analytical techniques: Multiple exp-function procedure [12, 13] and extended sinh-Gordon equation expansion procedure. The detailed description of the proposed approaches is given in Sec. 2. Section 3 is devoted to provide one, two and three soliton solutions of the proposed equation using multiple exp-function method. Section 4 constructs new traveling wave solutions using the expanded sinh-Gordon equation expansion (shGEE) approach. By choosing specific values of random parameters, the found results are also introduced graphically. Section 5 provides some final notes at the end of the work.

2. Proposed analytical techniques

The algorithm of suggested analytical techniques is given in the following subsections.

2.1. Review of multiple exp-function method

Take into consideration the following PDE

$$A(t, x, y, z, p_t, p_x, p_y, p_z...) = 0. \qquad (2)$$

This method is valid for both higher order PDEs and for system of PDEs. The algorithm of this method is based on four steps.

Step 1: In step 1, the following auxiliary equations are defined as

$$\gamma_{i,t} = \Omega_i \gamma_i, \text{ and } \gamma_{i,x} = K_i \gamma_i, \qquad (3)$$

where Ω_i, K_i, $1 \le i \le n$, respectively denotes the angular frequency and wave number. The exponential solutions for Eq. (3) are obtained as

$$\gamma_i = C_i e^{K_i x - \Omega_i t}, \qquad (4)$$

where $C_i, 1 \le i \le n$ are arbitrary constants.

Step 2: In step 2, the original PDE is transformed into a new PDE by assuming a rational function solution involving γ_i, $1 \le i \le n$, that has the following form

$$p(x, t) = \frac{P(\gamma_1, \gamma_2, ..., \gamma_n)}{Q(\gamma_1, \gamma_2, ..., \gamma_n)}, \qquad (5)$$

$$P = \sum_{b,c=1}^{a} \sum_{d,e=0}^{A} e_{ab,de} \gamma_a^d \gamma_b^e,$$

$$Q = \sum_{b,c=1}^{a} \sum_{d,e=0}^{A} f_{ab,de} \gamma_a^d \gamma_b^e, \qquad (6)$$

where $e_{ab,de}$ and $f_{ab,de}$ are the unknowns to be evaluated. Putting Eq. (5) into Eq. (2), this leads to new rational function equation in γ_i, $1 \le i \le n$ as

$$W(x, t, \gamma_1, \gamma_2, ..., \gamma_n) = 0. \qquad (7)$$

Step 3: By equalizing the coefficients of various powers of γ_i, an algebraic equation system is retrieved in the unknown $e_{ab,de}$ and $f_{ab,de}$, K_i and Ω_i. Upon solving the obtained system, we are able to determine polynomials P, Q and γ_i, $1 \le i \le n$. Finally, multiple soliton solutions $p(x, t)$ of the given PDE are obtained with the following form

$$p(x, t)$$
$$= \frac{P(C_1 e^{K_1 x - \Omega_1 t}, C_2 e^{K_2 x - \Omega_2 t}, ..., C_n e^{K_n x - \Omega_n t})}{Q(C_1 e^{K_1 x - \Omega_1 t}, C_2 e^{K_2 x - \Omega_2 t}, ..., C_n e^{K_n x - \Omega_n t})}. \qquad (8)$$

2.2. Review of the extended sinh-Gordon equation expansion technique

Take into consideration the PDE as given in Eq. (2). The following expression

$$p(x, t) = h(\zeta), \qquad \zeta = \beta x + \epsilon y - \nu t, \qquad (9)$$

is used for transforming Eq. (2) into an ODE as

$$F(h, h', h'', ...) = 0, \qquad (10)$$

where ordinary derivatives with respect to ζ are indicated by " $'$ ". F is a polynomial of h and its derivatives. Examine the formal solutions of Eq. (10) as follows:

$$h(\omega) = \sum_{i=1}^{n} \cosh^{i-1}(\omega)$$
$$\times [b_i \sinh(\omega) + a_i \cosh(\omega)]^i + a_0, \qquad (11)$$

where $\omega(\zeta)$ satisfies [14].

$$\omega' = \sqrt{c + d \sinh^2(\omega)}. \qquad (12)$$

The following cases arise after substituting different values of parameters c and d in Eq. (12).

Case 1: Simplified version of sinh-Gordon equation is obtained after taking $c = 0$ and $d = 1$ in Eq. (12)

$$\omega' = \sinh(\omega). \qquad (13)$$

Equation (13) [14] possesses the following solutions

$$\sinh(\omega) = \pm \iota \operatorname{sech}(\zeta), \quad \cosh(\omega) = \pm \tanh(\zeta), \qquad (14)$$

and

$$\sinh(\omega) = \pm \operatorname{csch}(\zeta), \quad \cosh(\omega) = \pm \coth(\zeta), \qquad (15)$$

where $\iota = \sqrt{-1}$.

The solutions of Eq. (11) along with Eq. (13), have the following forms

$$h(\zeta) = \sum_{i=1}^{n} (-\tanh(\zeta))^{i-1}$$
$$\times [\pm \iota b_i \sec h(\zeta) \pm a_i \tanh(\zeta)]^i + a_0 \qquad (16)$$

and

$$h(\zeta) = \sum_{i=1}^{n} (-\coth(\zeta))^{i-1}$$
$$\times [\pm b_i \csc h(\zeta) \pm a_i \coth(\zeta)]^i + a_0. \qquad (17)$$

Case 2: After taking $c = 1$ and $d = 1$, Eq. (12) becomes

$$\omega' = \cosh(\omega). \qquad (18)$$

The sinh-Gordon equation has a compressed form as well. Eq. (18) possesses the following solutions

$$\sinh(\omega) = \tan(\zeta), \quad \cosh(\omega) = \pm \sec(\zeta), \qquad (19)$$

$$\sinh(\omega) = -\cot(\zeta), \quad \cosh(\omega) = \pm \csc(\zeta). \qquad (20)$$

The solutions of Eq. (11) along with Eq. (18), have the following forms

$$h(\zeta) = \sum_{i=1}^{n} (\pm \sec(\zeta))^{i-1}$$
$$\times [b_i \tan(\zeta) \pm a_i \sec(\zeta)]^i + a_0, \qquad (21)$$

$$h(\zeta) = \sum_{i=1}^{n} (\pm \csc(\zeta))^{i-1}$$
$$\times [-b_i \cot(\zeta) \pm a_i \csc(\zeta)]^i + a_0. \qquad (22)$$

The balancing number n is calculated by making balance between the highest order nonlinear term and highest order derivative term. A nonlinear algebraic system is determined by surrogating the value of n in Eq. (11) and using it along with Eq. (12). Then by collecting the coefficients of $\sinh(w)^j \cosh(w)^i$, ($i = 0, 1, 2, \ldots, j = 0, 1$), equal to zero and solving the given system the values of a_i, b_i, ν, ϵ and β are obtained. Finally plugging these values into Eq. (16) and Eq. (17) the required solutions are obtained. Similarly, Case 2 can be proceeded.

3. Formulation of the soliton solutions using proposed techniques

The soliton solutions of the proposed model have been obtained in the following subsections using multiple wave solutions approach and extended shGEEM.

3.1. Applications of multiple exp-function technique

Consider the (3+1)-dimensional new negative-order-KdV-CBS model [13] as

$$p_{xt} + p_{xxxy} + 4p_x p_{xy} + 2p_{xx}p_y$$
$$+ \lambda p_{xx} + \mu p_{xy} + v p_{xz} = 0. \qquad (23)$$

Multiple exp-function method is applied on Eq. (23) to extract soliton solutions.

1. **One-soliton solutions**

The one-soliton solution of Eq. (23) can be assumed as

$$p(x, y, z, t) = \frac{a_0 + a_1 \gamma_1(x, y, z, t)}{1 + b_1 \gamma_1(x, y, z, t)}. \qquad (24)$$

Here a_0, a_1 and b_1 are randomized constants needed to be determined. By inserting Eq. (24) in Eq. (23), a system is retrieved by comparing the coefficients of various powers of γ_1, obtaining the values of the constants as

$$a_0 = \frac{b_0 (-2 b_1 k_1 + a_1)}{b_1},$$
$$\Omega_1 = K_1^2 s_1 + \lambda K_1 + \mu s_1, \ p_1 = 0, \qquad (25)$$

where $\gamma_1 = C_1 e^{K_1 x + s_1 y + p_1 z - \Omega_1 t}$ satisfies the following linear conditions

$$\gamma_{1x} = K_1 \xi_1, \ \gamma_{1t} = -\Omega_1 \xi_1,$$
$$\gamma_{1y} = s_1 \gamma_1, \ \gamma_{1z} = p_1 \gamma_1. \qquad (26)$$

Using Eq. (25) in Eq. (24), we obtain the following one-soliton solution of Eq. (23) as

$$p(x, y, z, t) = \frac{a_1 C_1 b_1 e^{-t(K_1^2 s_1 + \mu s_1 + \lambda K_1) + K_1 x + s_1 y} - 2 b_0 b_1 K_1 + a_1 b_0}{b_1 \left(b_0 + b_1 C_1 e^{-t(K_1^2 s_1 + \mu s_1 + \lambda K_1) + K_1 x + s_1 y} \right)}. \qquad (27)$$

In Fig. 1, the one-soliton solutions for different parametric values are shown.

2. **Two-soliton solutions**

To construct two-soliton solution, the assumed solution of Eq. (23) has the following form

$$p(x, y, z, t) = \frac{2 [K_1 \gamma_1 + K_2 \gamma_2 + a_{12}(K_1 + K_2) \gamma_1 \gamma_2]}{1 + \gamma_1 + \gamma_2 + a_{12} \gamma_1 \gamma_2}, \qquad (28)$$

where a_{12} is unknown to be calculated. By inserting Eq. (28) in Eq. (23), a system is retrieved by comparing the coefficients of γ_1 and γ_2. The constants are evaluated as

$$a_{12} = \frac{K_2^2 + K_1^2 - 2K_1 K_2}{K_1^2 + K_2^2 + 2K_1 K_2}, \qquad \Omega_1 = K_1^2 s_1 + \lambda K_1 + \mu s_1, \qquad \Omega_2 = K_2^2 s_2 + \lambda K_2 + \mu s_2, \qquad (29)$$
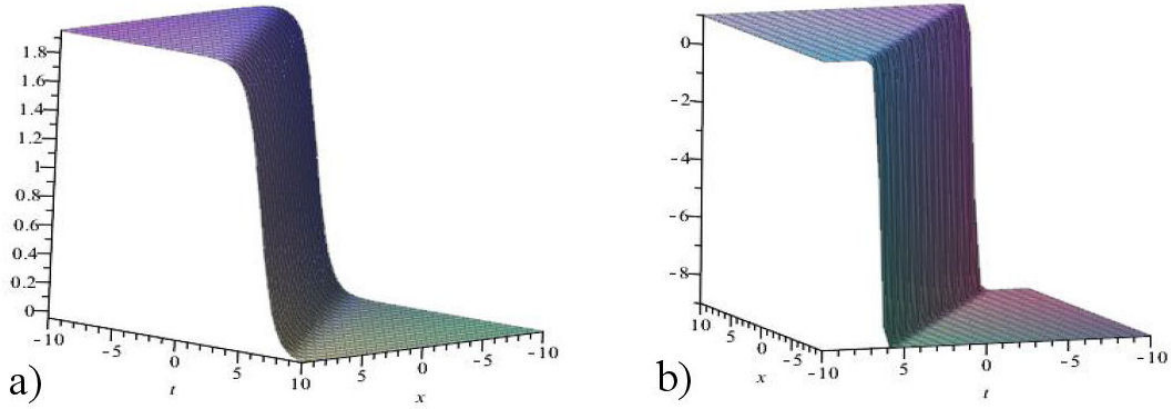
FIGURE 1. a) $b_0 = 11, K_1 = 1, s_1 = 0.1, a_1 = 0.41, b_1 = 0.21, p_1 = 0, \lambda = 1.11, \mu = 1.51, v = 1.06, C_1 = 11, y = 1, z = 1$. b) $b_0 = 1.1, K_1 = 1, s_1 = -0.1, a_1 = 0.1, b_1 = 0.1, p_1 = 0, \lambda = -1, \mu = 1.05, v = 1.06, C_1 = 11, y = 2, z = 1$. Graphical representation of one-soliton.
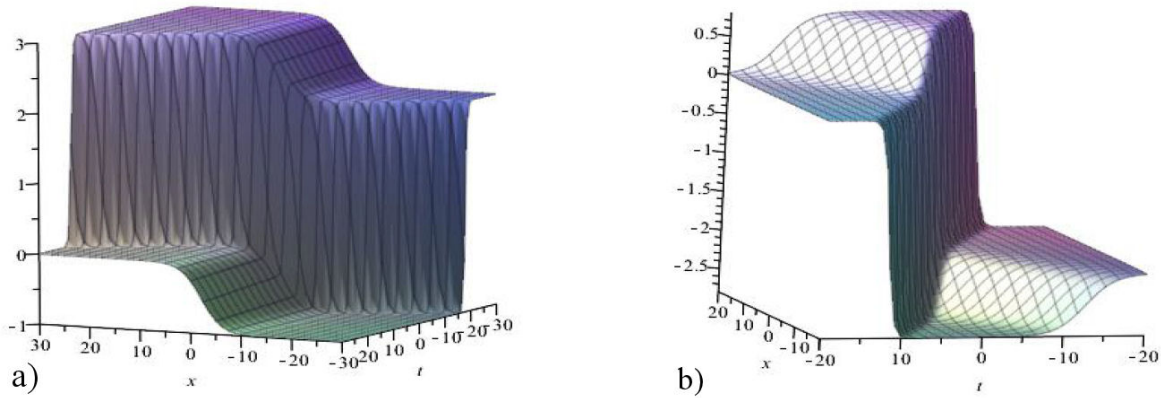

FIGURE 2. a) $C_1 = 0.5, C_2 = 0.5, K_1 = -1.44, K_2 = 0.4, s_1 = 1.4, s_2 = 1.4, p_1 = 0 = p_2, \lambda = 0.75, \mu = 0.5, v = 0.75, y = z = 1$. b) $C_1 = 0.1, C_2 = 0.2, K_1 = 1.5, K_2 = -0.5, s_1 = 1, s_2 = 0.1, p_1 = 0 = p_2, \lambda = 0.1, \mu = 0.1, v = 0.1, y = z = 1$. Graphical representation of two-soliton.

where $\gamma_1 = C_1 e^{K_1 x + s_1 y + p_1 z - \Omega_1 t}$ and $\gamma_2 = C_2 e^{K_2 x + s_2 y + p_2 z - \Omega_2 t}$ satisfies the following linear conditions

$$\gamma_{1x} = K_1 \gamma_1, \quad \gamma_{1t} = -\Omega_1 \gamma_1,$$
$$\gamma_{1y} = s_1 \gamma_1, \quad \gamma_{1z} = p_1 \gamma_1,$$
$$\gamma_{2x} = K_2 \gamma_2, \quad \gamma_{2t} = -\Omega_2 \gamma_2,$$
$$\gamma_{2y} = s_2 \gamma_2, \quad \gamma_{2z} = p_2 \gamma_2. \quad (30)$$

Inserting Eq. (29) in Eq. (28), one may find two-soliton solutions of Eq. (23).

In Fig. 2, the two-soliton solution for different parametric values are shown.

3. Three-soliton Solutions

To construct three-soliton solution, the assumed solution of Eq. (23) can be written as

$$p(x, y, z, t) = \frac{U(\gamma_1, \gamma_2, \gamma_3)}{V(\gamma_1, \gamma_2, \gamma_3)}, \quad (31)$$

where

$$U(\gamma_1, \gamma_2, \gamma_3) = 2\Big[K_1\gamma_1 + K_2\gamma_2 + K_3\gamma_3$$
$$+ a_{12}(K_1 + K_2)\gamma_1\gamma_2 + a_{13}(K_1 + K_3)\gamma_1\gamma_3$$
$$+ a_{23}(K_2 + K_3)\gamma_2\gamma_3$$
$$+ a_{12}a_{13}a_{23}(K_1 + K_2 + K_3)\gamma_1\gamma_2\gamma_3\Big], \quad (32)$$

$$V(\gamma_1, \gamma_2, \gamma_3) = 1 + \gamma_1 + \gamma_2 + \gamma_3 + a_{12}\gamma_1\gamma_2 + a_{13}\gamma_1\gamma_3$$
$$+ a_{23}\gamma_2\gamma_3 + a_{12}a_{13}a_{23}\gamma_1\gamma_2\gamma_3. \quad (33)$$

One can apply the same procedure as elaborated in one-soliton solutions and two-soliton solutions to derive the three-soliton solutions.

3.2. Application of extended shGEEM

In this subsection, mathematical analysis of proposed equation is given and its solutions are constructed along with cases
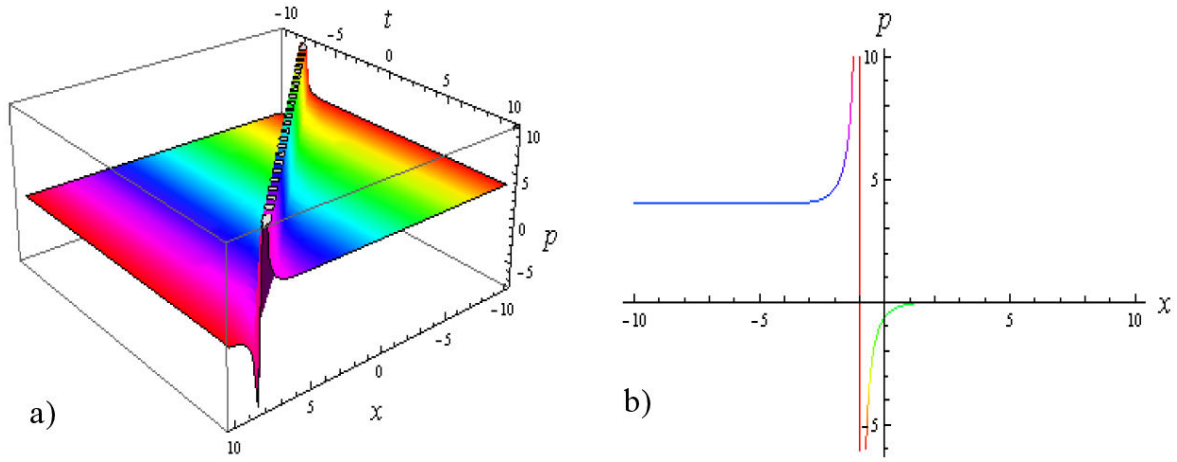
FIGURE 3. a) $a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1$. b) $a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1, t = 1$. Graphical representation of Eq. (39). a) 3D plot and b) 2D plot.
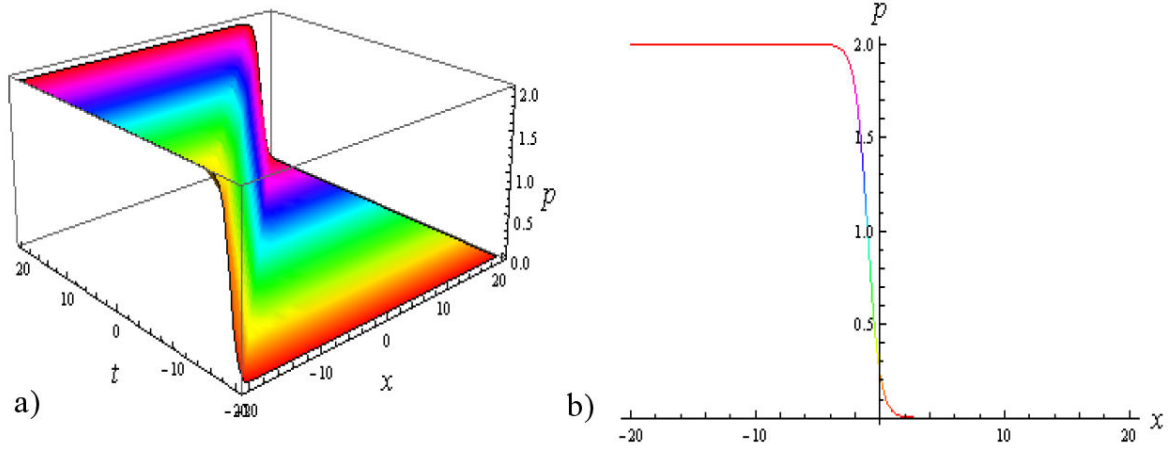

FIGURE 4. a)$a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1$ b)$a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1, t = 1$. Graphical representation of real part of Eq. (40). a) 3D plot and b) 2D plot.

arising in Subsec. 2.2. The traveling wave transformation is taken, as

$$p(x, y, z, t) = h(\zeta), \qquad \zeta = x + by + cz - at, \qquad (34)$$

where a is wave velocity and b and c are random unknowns. By applying the transformation Eq. (34) into Eq. (23), an ODE is obtained, as

$$bh'''' + 6bh'h'' + (\lambda + b\mu + cv - a) h'' = 0. \qquad (35)$$

Case 1: In this case $\omega' = \sinh(\omega)$. According to the extended shGEEM, Eq. (35) has solutions of the form

$$h(\zeta) = \pm \iota b_1 \sec h(\zeta) \pm a_1 \tanh(\zeta) + a_0, \qquad (36)$$

and

$$h(\zeta) = \pm b_1 \csc h(\zeta) \pm a_1 \coth(\zeta) + a_0, \qquad (37)$$

where a_0, a_1 and b_1 are constants needed to be specified. Applying extended shGEEM, the following values of the unknowns are obtained as,
Set 1:

$$a_1 = -2, \quad b_1 = 0, \quad a = 4b + cv + \lambda + b\mu.$$

Set 2:

$$a_1 = -1, \quad b_1 = 1, \quad a = b + cv + \lambda + b\mu.$$
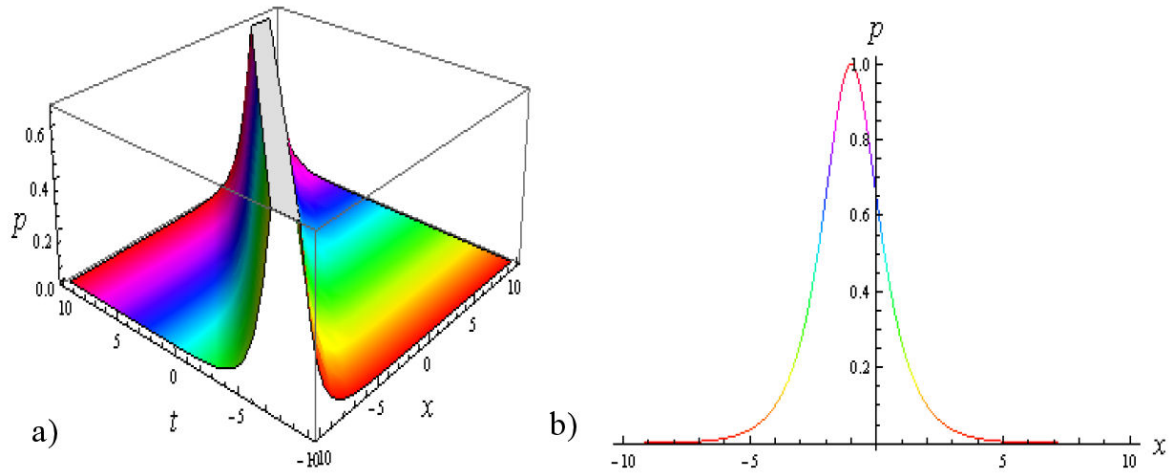
FIGURE 5. a) $a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1$. b) $a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1, t = 1$. Graphical representation of imaginary part of Eq. (40). a) 3D plot and b) 2D plot.
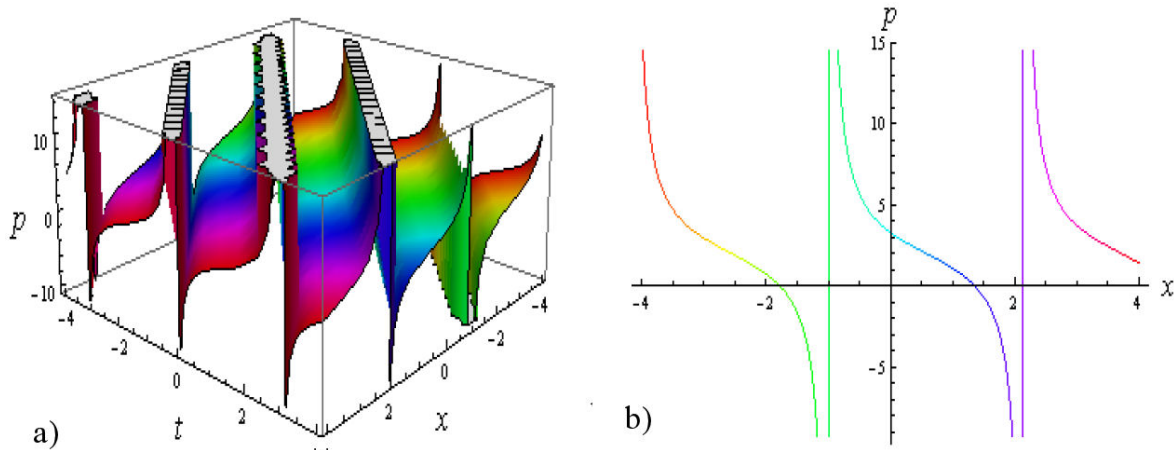

FIGURE 6. a)$a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1$ b)$a_0 = 1, b = 1, c = 1, v = 1, \lambda = 1, \mu = 1, y = z = 1, t = 1$. Graphical representation of Eq. (45). a) 3D plot and b) 2D plot.

Putting the values from **Set 1** into Eq. (36) and Eq. (37), the hyperbolic solutions for the proposed equation are obtained, as

$$p(x, y, z, t) = \mp 2 \tanh(\zeta) + a_0 \tag{38}$$

and

$$p(x, y, z, t) = \mp 2 \coth(\zeta) + a_0, \tag{39}$$

where a_0 is an arbitrary constant and the traveling wave variable ζ is defined as $\zeta = x + by + cz - (4b + cv + \lambda + b\mu)t$.

Putting the values from **Set 2** into Eq. (36) and Eq. (37), the hyperbolic solutions for the proposed equation are obtained, as

$$p(x, y, z, t) = \pm \iota \sec h(\zeta) \mp \tanh(\zeta) + a_0 \tag{40}$$

and

$$p(x, y, z, t) = \pm \csc h(\zeta) \mp \coth(\zeta) + a_0, \tag{41}$$

where a_0 is an arbitrary constant and the traveling wave variable ζ is defined as $\zeta = x + by + cz - (b + cv + \lambda + b\mu)t$.

3.2.1. *Case 2:*

In this particular instance $\omega' = \cosh(\omega)$. According to the extended shGEEM, Eq. (35) has solutions of the form

$$p(x, y, z, t) = b_1 \tan(\zeta) \pm a_1 \sec(\zeta) + a_0 \tag{42}$$

and

$$p(x, y, z, t) = -b_1 \cot(\zeta) \pm a_1 \csc(\zeta) + a_0. \tag{43}$$

Applying extended shGEEM, the following values of the unknowns are obtained as
Set 3:

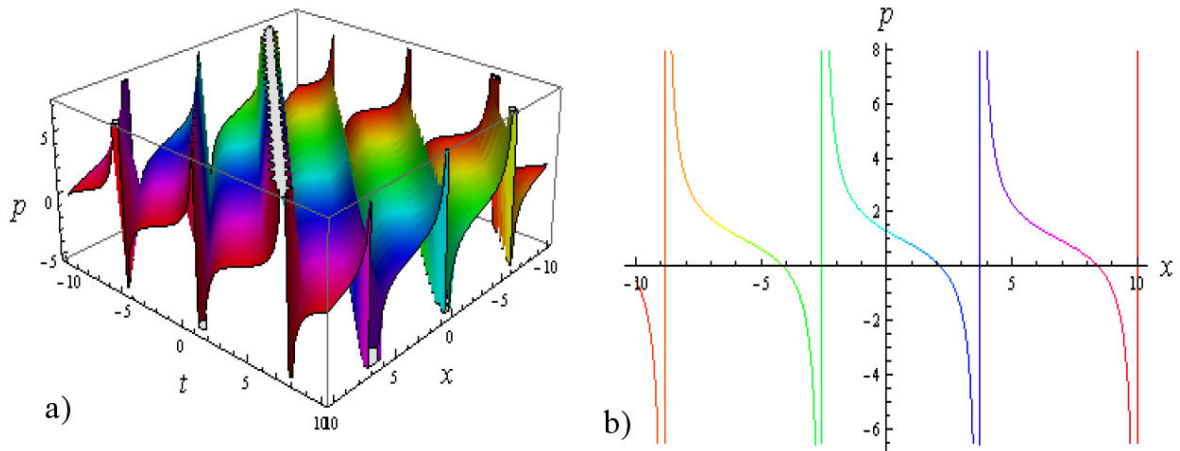$$a_1 = 0, \quad b_1 = -2, \quad a = 4b + cv + \lambda + b\mu.$$

FIGURE 7. a)$a_0 = 1, b = 1.1, c = 1.1, v = 1, \lambda = 1.2, \mu = 1, y = z = 1$. b)$a_0 = 1, b = 1.1, c = 1.1, v = 1, \lambda = 1.2, \mu = 1, y = z = 1, t = 1$. Graphical representation of Eq. (46). a) 3D plot and b) 2D plot.

Set 4:

$$a_1 = 1, \quad b_1 = -1, \quad a = b + cv + \lambda + b\mu.$$

Putting the values from **Set 3** into Eq. (42) and Eq. (43), the trigonometric solutions for the proposed equation are obtained,

$$p(x, y, z, t) = -2\tan(\zeta) + a_0. \tag{44}$$

and

$$p(x, y, z, t) = 2\cot(\zeta) + a_0, \tag{45}$$

where a_0 is an arbitrary constant and the traveling wave variable ζ is defined as $\zeta = x + by + cz - (4b + cv + \lambda + b\mu)t$.

Putting the results from **Set 4** into Eq. (42) and Eq. (43), the trigonometric solutions for the proposed equation are obtained,

$$p(x, y, z, t) = -\tan(\zeta) \pm \sec(\zeta) + a_0, \tag{46}$$

and

$$p(x, y, z, t) = \cot(\zeta) \pm \csc(\zeta) + a_0, \tag{47}$$

where a_0 is an arbitrary constant and the traveling wave variable ζ is defined as $\zeta = x + by + cz - (b + cv + \lambda + b\mu)t$.

4. Results and discussion

This work provides one, two and three soliton solutions and traveling wave solutions of new negative-order-KdV-CBS model. It has been noticed that the graphical illustrations of the obtained solutions highly depend on the selection of specific values for arbitrary parameters. Due to their ability to describe numerous new wave features, the ensuing travelling wave solutions can be fruitful in theoretical examinations of the system under discussion.

5. Conclusion

We analyze the behavior of the novel KdV-CBS model with negative order. We used the multiple exp-function technique and extended sinh-Gordon equation expansion procedure to construct one, two and three soliton solutions and traveling wave solutions of the considered model. Then, for some of the discovered solutions, 3D and 2D graphs with unrestricted parameters were provided. To the best of our knowledge, the findings could aid in deciphering the phenomena that the equation depicted. The discovery of more soliton solutions for this model using comparable integration methods is another future objective.

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