Phase function method for elastic nucleon-nucleon scattering using Hellmann plus coulomb potential

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The phase function method/variable phase approach to potential scattering is exploited to calculate the phase shifts for nucleon-nucleon systems in low and intermediate energy regions by representing the nuclear part of the interaction by the Hellmann potential while the electromagnetic part by the Coulomb one. In addition, the differential and total scattering cross-sections are estimated with our phase parameters. Results reproduced by the concerned potential are in good agreement with the previous works in the literature.

1. Introduction

The physics of low-energy nuclear reactions is an essential element to comprehend the evolution of the Universe. For the theoretical understanding of the nuclear forces and various nuclear properties, nuclear scattering plays a crucial role as the scattered outgoing wave brings out information about the nuclear environment. As the many-body forces acting inside the nucleus are believed to be small, the two-body forces in nucleon-nucleon or nucleon-nucleus interactions play the dominant role. However, the relativistic and non-relativistic bound state difficulties were the main focus of recent studies on the quantum mechanical approach of the Hellmann potential. Recently, a number of scientists have investigated the study of the scattering states of Hellmann issues in an effort to produce fresh findings that will aid in the comprehension of quantum systems. Yazarloo et al. expanded the investigation to include scattering states of the Dirac equation with the Hellmann potential under the spin and pseudospin symmetries [1] in this regard. For the spin and pseudospin symmetries, the Dirac phase shift and normalized wave function were given. Using the PT-/non-PT symmetry and non-Hermitian Hellmann potential, Arda and Sever investigated the approximate non-relativistic bound and scattering states with any ℓ values [2].

The approximate bound state solution of the two-body spinless Salpeter equation for the Hellmann potential was also examined by Arda [3]. Applications of the Hellmann potential can be found in nuclear and high-energy physics. The Meson Field Theory, which describes the meson exchange that results from the interaction between a proton and a neutron, was initially proposed by Yukawa. According to the Yukawa theory, particles with mass parameters in the range between the masses of an electron and a nucleon

are responsible for the existence of the nucleus force. A simple understanding of the binding energy in the atomic nucleus is anticipated to be provided using some parameters of the Yukawa idea. Utilizing the Hellmann potential is another straightforward idea to reproduce effective results for the nucleon-nucleon system. Hellmann potential, the superposition of the Yukawa, and attractive electromagnetic potential $V(r) = -(a/r) + b(e^{-cr}/r)$ was first proposed by Hellmann [4–6]. Here a and b are the strength parameters of the Coulomb and Yukawa potential and c is the screening parameter [5,7] which regulates the shape of the potential. Various authors [8-18] have investigated the Hellman potential in all the limits of quantum mechanics due to its importance in atomic physics. In several relatively recent studies, the quantum mechanical treatment of the Hellmann potential focuses on the relativistic and non-relativistic bound state problems [19-23]. The Hellmann potential model has been treated by several physicists in many of the applications of physics *i.e.* electron-core [24,25], electron-ion inner-shell ionization problem [26, 27], alkali hydride molecules [28], solid-state physics [29, 30].

The aim of this work is to investigate the nucleon-nucleon scattering phase shifts and cross sections for motion in the nuclear Hellmann potential for various partial wave states by exploiting the Phase Function Method (PFM) [31]. In this method, the radial Schrödinger equation is converted into a first-order nonlinear differential equation called the phase equation and is solved numerically from the origin to the asymptotic region to obtain the desired phase shifts. This amounts to separating out the amplitude and the phase part of the wave equation. The wave function's phase shift, for scattering by the potential truncated at a distance r, at each point is represented by the phase function, indicated by the function. A potential that has been completely excised won't un-

dergo any phase change. As a result, the quantity is equal to zero. With the initial condition from the origin to the asymptotic area, the phase function that satisfies a first-order nonlinear differential equation for a local potential is solved to produce the scattering phase shift. As one advances away from the origin during the phase equation's solution, the potential at each point of interaction builds up the phase shift, and it reaches its asymptotic value as soon as one exits the potential's range. In all likelihood, $\delta_\ell(k) = \lim_{r \to \infty} \delta_\ell(k,r)$. In Sec. 2 we present our methodology and Sec. 3 is devoted to the results and discussion. Section 4 is dedicated particularly to cross-section calculations. We present some concluding remarks in Sec. 5.

2. Methodology

Calculation of scattering phase shifts $\delta_\ell(k)$ as a function of the center of mass energy $E_{c.m}(= k^2 > 0$ in the theoretical limit of $\hbar^2/2m = 1$) constitutes one of the core problems of quantum scattering theory. Phase-function method (PFM) [31] is an alternative to the traditional Schrödinger equation approach. This methodology is based on the fact that certain second-order linear homogeneous equations can be reduced to their first-order non-linear counterparts of the Riccati type called Riccati equations. The Riccati equations are satisfied by the phase functions $\delta_\ell(k,r)$ that is having the meaning of the phase shifts at each point of the wave function for scattering by the potential $V(r)$ cut-off at that point. This helps in the investigation of the different regions in the interior of the potential in producing the phase shift [32-39]. The effective Hellmann potential in all partial waves is written as

$$V_H(r) = -\frac{a}{r} + b\left(\frac{e^{-cr}}{r}\right) + \frac{\ell(\ell+1)}{r^2}. \tag{1}$$

All the three parameters a, b and c of the potential have the unit of fm^{-1}. At low energies, the nuclear scattering of two charged hadrons is considered to place under the combined influence of two potentials, one of electromagnetic and the other of nuclear origin. The nuclear Hellmann potential models the two nucleon interactions and the long-range component of the interaction is the electromagnetic potential, which is theoretically extended to infinity. To treat the charged hadron systems one has to add an electromagnetic interaction in addition to the nuclear potential. We consider the electromagnetic potential $V_c(r) = (2k\eta/r)$ as the long-range part of the effective interaction. Here η stands for the Sommerfeld parameter. Even while the pure Coulomb potential is a theoretically limitless long-range interaction, in practice it screens out after a given distance. It is claimed that the conventional phase function method (PFM) [31] must be modified because it does not work well for pure Coulomb and Coulomb-like interactions. In many circumstances where the scattering occurs under the combined impact of Coulomb-like potentials, the use of pure Coulomb interaction may be justified because the Coulomb potential proves to be unimportant after a finite distance. For the sake of simplicity, we

compute the scattering phase shifts for the (p-p) system using the conventional PFM [31] for our model potential in order to assess the viability of our hypothesis. For a local potential $V(r) = V_H(r) + V_c(r)$, $\delta_\ell(k,r)$ satisfies a first-order nonlinear differential equation [31] written as

$$\delta_\ell'(k,r) = -k^{-1}V(r)[\cos\delta_\ell(k,r)\hat{j}_\ell(kr)$$
$$- \sin\delta_\ell(k,r)\hat{\eta}_\ell(kr)]^2, \tag{2}$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati-Bessel functions [40]. The resulting Phase equations for $\ell = 0, 1$ and 2 read as

$$\delta_0'(k,r) = -k^{-1}V(r)[\sin(\delta_0(k,r) + kr)]^2, \tag{3}$$

$$\delta_1'(k,r) = -\frac{V(r)}{k^3r^2}[\sin(\delta_1(k,r) + kr)$$
$$- kr\cos(\delta_1(k,r) + kr)]^2, \tag{4}$$

and

$$\delta_2'(k,r) = -k^{-1}V(r)\left[\left(\frac{3}{k^2r^2} - 1\right)\sin(\delta_2(k,r) + kr)\right.$$
$$\left. - \frac{3}{kr}\cos(\delta_2(k,r) + kr)\right]^2. \tag{5}$$

The quantity k represents the centre of mass momentum and has a relation with the centre of mass energy as $k = \sqrt{2mE}/\hbar$. To obtain the scattering phase shift, the phase equation is solved numerically for the potential under consideration with the condition $\delta_\ell(k,0) = 0$.

3. Results and discussion

We have parameterized the nuclear Hellmann potential to obtain the standard phase parameters [41–43] of different states of the (n-p) and (p-p) systems by solving the differential Eqs. (3)-(5) numerically. The function $\delta_\ell(k,r)$, called the phase function, has at each point the meaning of the phase shift of the wave function for scattering by the potential truncated at a distance r. This implies that while calculating phase accumulation within the range of the interaction, the step size of the variable r becomes crucial. Thus, one has to judiciously optimize the step size to have proper phase parameters. This is one of the limitations of the PFM compared to Jost function approach to the quantum mechanical potential scattering problem. We have auto-adjusted the grid sizes in our numerical routine for different partial wave states to achieve the required phase parameters. We test the numerical accuracy and precision of the approach by using a finite grid representation and determine the low-energy parameters of these potentials. In particular, we take $r_{\max} = 6fm$ and $\Delta r = 0.002, 0.005, 0.001, 0.0005fm$ corresponding to $N = 3 \times 10^3, 1.2 \times 10^3, 6 \times 10^3$ and 1.2×10^4 grid points, respectively, to establish convergence.

The quality of description of standard data on the basis of certain theoretical function or functional of several variables

TABLE I. List of parameters for (n-p) and (p-p) systems.

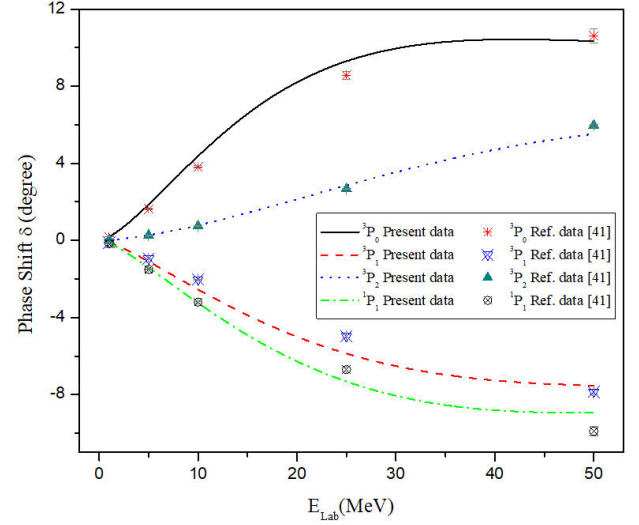| System	States	$a(fm^{-1})$	$b(fm^{-1})$	$c(fm^{-1})$
n-p/p-p	$^1S_0$	0.012	-0.800	0.510
	$^3S_1$	0.014	-1.200	0.390
	$^1P_1$	0.100	-2.100	0.515
	$^3P_0$	0.500	-4.500	0.900
	$^3P_1$	0.200	-2.200	0.600
	$^3P_2$	0.100	-3.000	0.498
	$^1D_2$	0.016	-2.090	0.050
	$^3D_1$	0.017	-1.820	0.010
	$^3D_2$	0.011	-1.980	0.010
	$^3D_3$	0.016	-2.180	0.065

FIGURE 2. (n-p) P-wave scattering phase shifts as a function of laboratory energy.

FIGURE 1. (n-p) S-wave scattering phase shifts as a function of laboratory energy.
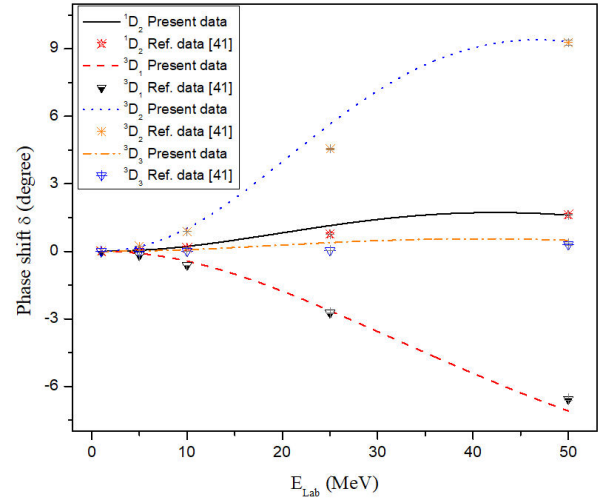
FIGURE 3. (n-p) D-wave scattering phase shifts as a function of laboratory energy.

in Eqs. (3)-(5) are estimated by the χ^2 method. For some defined values of the standard phase shifts, we have obtained the best-fitted parameters of Table I. As the nuclear force is assumed to be charge independent the model parameters for various spin states of (n-p) and (p-p) systems are the same. The parameters for the different states of (n-p) and (p-p) systems are given in Table I.

For the computation of the scattering phase shifts, we have chosen to work with $\hbar^2/m_p = 41.47$ MeV fm^2. The (n-p) and (p-p) scattering phase shifts are presented in Figs. 1-4. From the figures it is noticed that our results for the phase shifts are in conformity with the standard values of Pérez *et al.* [41], Gross and Stadler [42], and Wiringa *et al.* [43]. For $^1S_0$ (n-p) and (p-p) states our results fall below the standard data [41] by one or two degrees in the energy range 5-25 MeV. Looking closely at Fig. 2 it is observed that our results are in reasonable agreement with those of Pérez *et al.* [41] except for $^3P_0$, $^3P_1$ & $^1P_1$ states at energies 25 MeV respectively. The $^1P_1$ phase value falls below the standard one [41] by $2°$ at 50 MeV. For D partial waves our results are

in good qualitative as well as quantitative agreement with Ref. [41] except at 25 MeV for $^3D_2$ state. Over our entire model, potential reproduces correct trends of the (n-p) scattering phases.

Our model potential for (p-p) system reproduces correct phase parameters as shown in Fig. 4 with an exception for $^1S_0$ state where the peak value is less by one degree and at 50 MeV it is $3°$ more than that of Ref. [41]. Except $^1S_0$ state all other states are in quantitative agreement with Pérez *et al.* [41]. With the Hellman model of interaction, the associated potentials for different partial wave states are depicted in Figs. 5-8 in the unit of MeV. The dimension of the potential function $V_H(r)$ in Eq. (1) is in the unit of fm^{-2} but portray them by multiplying with $\hbar^2/m_p = 41.47$ MeV fm^2. It is
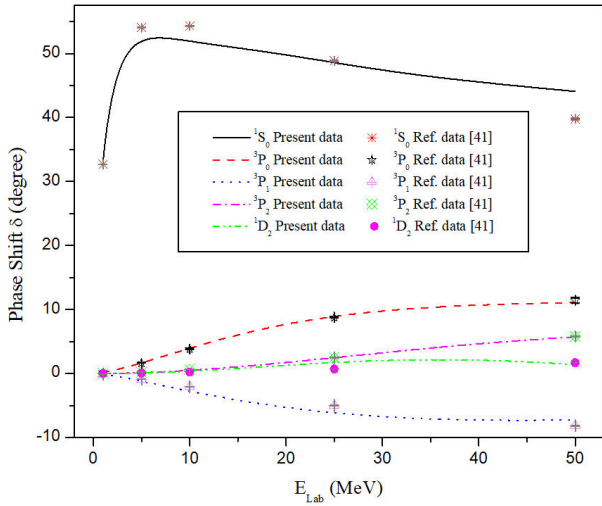
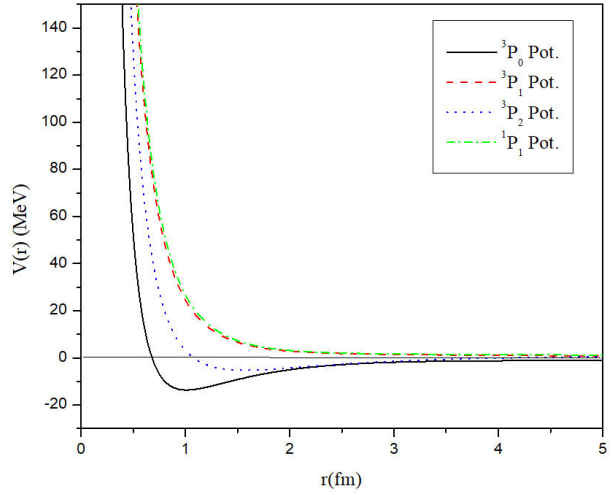FIGURE 4. (p-p) S-, P- and D-wave scattering phase shifts as a function of laboratory energy.

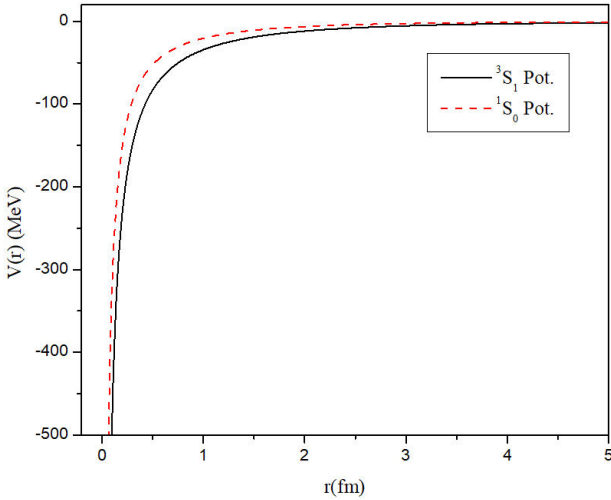FIGURE 6. (n-p) P-wave potentials as a function of r.

FIGURE 5. (n-p) S-wave potentials as a function of r.

FIGURE 7. (n-p) D-wave potentials as a function of r.

observed that repulsive cores develop in the partial wave states other than the S-wave due to the addition of centrifugal barriers.

4. Scattering cross section

Total cross-section $\sigma_t(\alpha)$ with α representing entrance channel is given by $\sigma_t(\alpha) = \sigma_S(\alpha) + \sigma_R(\alpha)$, where $\sigma_S(\alpha)$ represents only elastic scattering cross section and $\sigma_R(\alpha)$ stands for combined inelastic/reaction cross section. However, in low energy collisions, the total cross section emerges mostly from the elastic scattering channel with negligible contribution from the rest of the involved reaction channels [44, 45]. We desire to investigate to what extent our model calculations will be able to yield realistic cross-section data in view of small discrepancies between the results of our phase shift analysis and of other calculations.
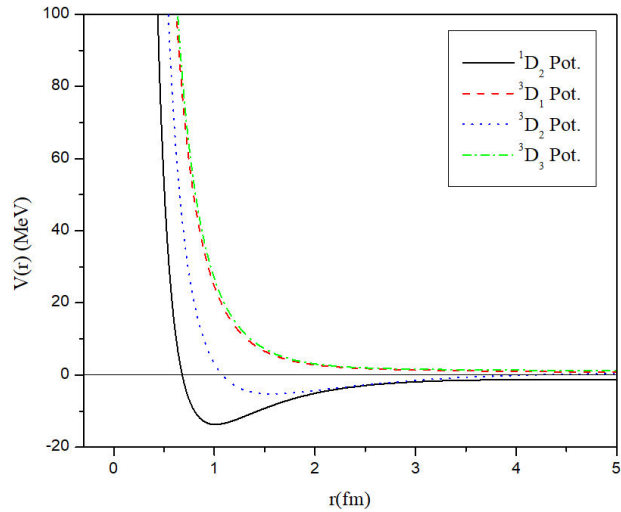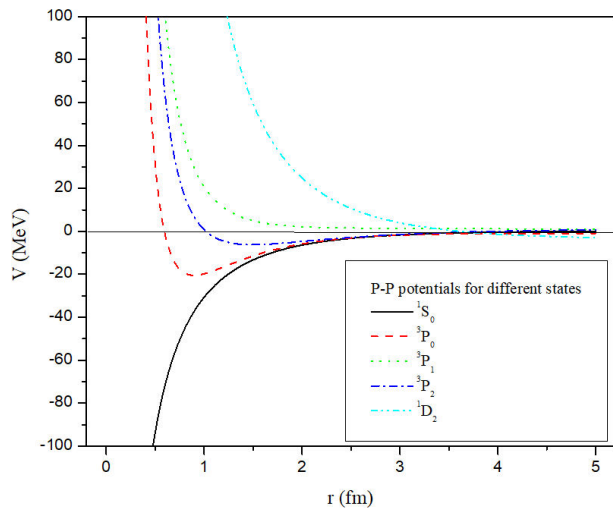

FIGURE 8. (p-p) S-, P- and D-wave potentials as a function of r.

FIGURE 9. Differential p-p scattering cross section as a function of θ (degree) at $E_{Lab} = 2.4$ MeV. Standard theoretical results are from Ref. [50].

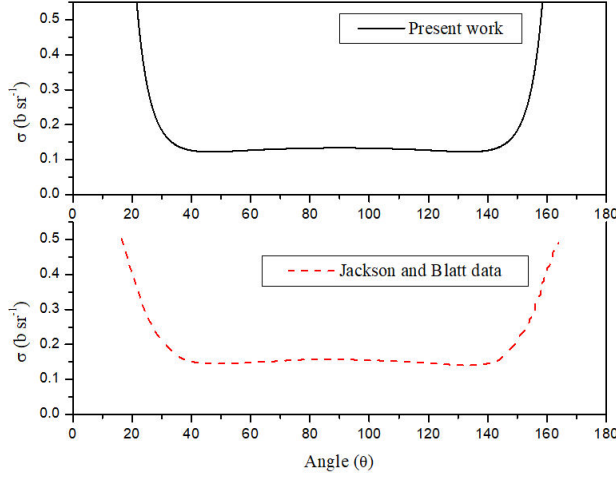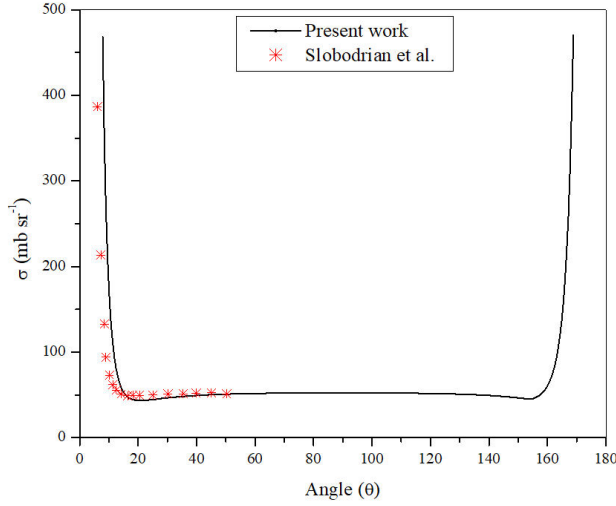FIGURE 10. Differential p-p scattering cross section as a function of θ (degree) at $E_{Lab} = 9.918$ MeV. Standard theoretical results are from Ref. [51].

For Coulomb-distorted nuclear scattering the scattering amplitude is expressed as

$$f_{nC}(\theta) = f_C(\theta) + f_n(\theta), \qquad (6)$$

where

$$f_C(\theta) = -\frac{\eta}{2\chi \sin^2(\theta/2)}$$

$$\times \exp[-i\eta \ln \sin^2(\theta/2) + 2i\sigma_0(\eta)], \qquad (7)$$

and

$$f_n(\theta) = \frac{1}{2i\chi} \sum_{\ell=0}^{\infty} (2\ell + 1) \exp(2i\sigma_\ell(\eta))$$

$$\times P_\ell(\cos\theta)(\exp(2i\delta_\ell^n) - 1). \qquad (8)$$

The quantity δ_ℓ^n is the Coulomb-distorted nuclear phase shift. The negative sign in front of Eq. (7) originates from the fact that the Coulomb force between two protons is repulsive. The Coulomb-distorted nuclear cross-section $\sigma_{nC}(\theta)$ is given by

$$\sigma_{nC}(\theta) = |f_C(\theta) + f_n(\theta)|^2 = |f_{nC}(\theta)|^2. \qquad (9)$$

For identical particles, like (p-p), scattering

$$\sigma(\theta) = |f(\theta) + f(\pi - \theta)|^2. \qquad (10)$$

One may calculate the total scattering cross-section by integrating the differential cross-section $\sigma(\theta)$ over the entire solid angle and the angle integrated cross-section is

$$\sigma_s = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_\ell, \qquad (11)$$
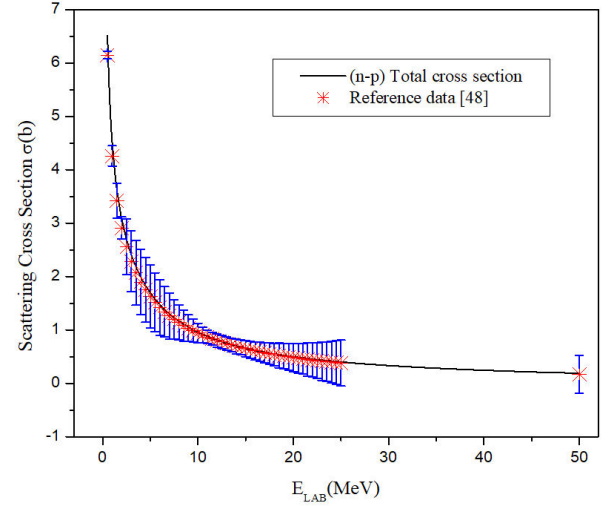

FIGURE 11. (n-p) total scattering cross section as a function of laboratory energy. Standard theoretical results are from Ref. [48].
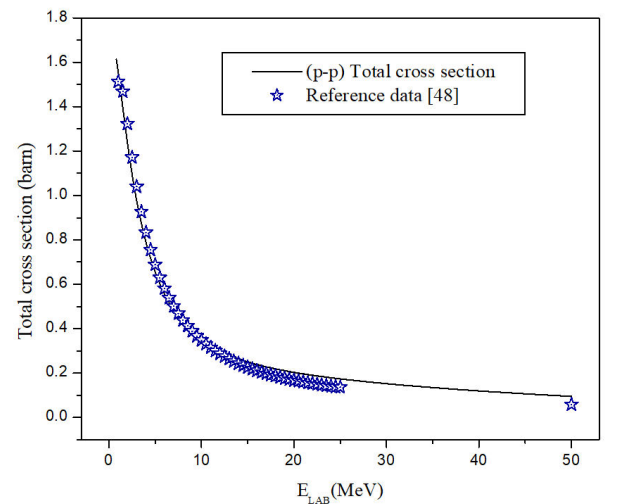

FIGURE 12. (p-p) total scattering cross section as a function of laboratory energy. Standard theoretical results are from Ref. [48].

where δ_ℓ is the total scattering phase shift. Note that this integrated cross-section is sometimes called the total cross-section because it is the total after integration over all angles. The elastic scattering of neutrons by protons has been investigated by a number of researchers [46–49]. In the present text, we calculate differential and total scattering cross section for the (n-p) & (p-p) systems and compare them with the data [47–51] available in the literature by exploiting Eqs. (7)-(11). The (p-p) differential cross sections are portrayed in Figs. 9 and 10 together with those of Refs. [50, 51] over the whole angular range.

The total cross-section calculations for neutron-proton and proton-proton scattering are performed including the contributions of S, P, and D waves. Our cross-section results for the systems under consideration are in excellent agreement with those of Ref. [48]. As represented in Figs. 11 and 12.

5. Conclusions

From the foregoing discussion, it is clear that the Hellmann potential is quite capable of producing the nature of phase shifts of respective states with inconsequential differences in their numerical values. This is due to the fact that unlike atomic cases the nuclear potentials are highly state-dependent and cannot be generated in a proper way from any known interaction. Practically speaking, phase-shift studies of experimental data are frequently used to compare the various models. Compared to n-p scattering observables, particularly the n-p analyzing power, is crucial for validating different aspects of the potential. For incident laboratory energies up to 30 MeV, high-accuracy cross-section and analyzing power data are provided for the hadron-hadron scattering. The simplest situation in which delicate effects like many-body forces or charge-symmetry breaking can be investigated is the nucleon-deuteron system. With the use of a two-body separable potential, nucleon-deuteron elastic scattering has been studied within the Faddeev formalism [52]. It is therefore of great interest to investigate the N-N elastic scattering using a straightforward local potential model. A synchronized description of all obtainable theoretical and experimental data over a large energy range would expose more information on the nucleon-nucleon interaction and possibly on the significance of three-body forces for the nucleon-deuteron system. Having close agreements for the spin-independent elastic channel (n-p) and (p-p) scattering phase parameters with those of Pérez et al. [41], Gross and Stadler [42], Wiringa et al. [43] and cross sections with the results of Arndt et al. [48], Jackson-Blatt [50], Slobodrian et al. [51], etc. it can be concluded that this simple-minded approach of PFM to compute scattering phase shifts and there from the cross sections for motion in the nuclear Hellmann potential will be an interesting gateway to a wide variety of physicists and deserves due attention.

Conflict of interest

The authors declare that they have no conflict of interest in this work.

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