New Eigensolution of the Klein–Gordon and Schrödinger equations for improved modified Yukawa-Kratzer potential and its applications using Bopp's Shift method and standard perturbation theory in the 3D-ERQM and 3D-ENRQM symmetries

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The deformed Klein-Gordonequation has been solved in three-dimensional extended relativistic quantum mechanics (3D-ERQM) symmetries for the improved modified Yukawa-Kratzer potential (IMYKP) model under the influence of the deformation space-space symmetries. The new relativistic energy eigenvalues were calculated using the parametric Bopp's shift method and standard perturbation theory in addition to the approximation scheme suggested by Greene and Aldrich for the inverse square terms. The new relativistic energy eigenvalues of (LiH, HCl, CO and H_2) molecules under the IMYKP model it was shown to be sensitive to the atomic quantum numbers (j, l, s, m), mixed potential depths (V_0, D_e, r_e), the screening parameter's inverse α and noncommutativity parameters (Θ,τ,χ). In addition, we analyzed the nonrelativistic energy values by applying the well-known transmission rules known in the literature. In addition, we studied many special cases useful to researchers in the framework of the new extended symmetries, such as the improved modified Kratzer potential, the improved generalized Kratzer potential, the improved Kratzer potential, the improved modified Kratzer plus screened Coulomb potential, the improved Hellmann potential, the improved Yukawa potential, and improved inversely square Yukawa potential. We noticed that these particular results are identical to our previous work and other known works in the literature. The study is further extended to calculate the mass spectra of mesons of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) within the framework of the IMYKP model in three-dimensional extended non-relativistic quantum mechanics (3D-ENRQM) symmetries.

Keywords: Klein-Gordonequation, modified Yukawa-Kratzer potential; Noncommutative space; Bopp's shift method; star products.

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1. Introduction

The Klein-Gordon and Dirac equations, or relativistic mod equations, can be solved analytically or numerically to offer us a lot of information about a physical systems. Various techniques are available in the literature to obtain their solutions, such as the parametric Nikiforov-Uvarov (pNU) method, the exact quantization rule, the Qiang-Dong proper quantization rule, the path integral method, the asymptotic iteration method (AIM), the factorization method, the Laplace transform approach, the supersymmetric quantum mechanics (SUSYQM), the ansatz method, and the series expansion method. These equations play a vital role in statistical physics, solid-state physics, quantum field theory, atomic and sub-atomic physics, and molecular physics. Various research has been carried out on these equations of many considerable potentials. Parmar and Vinodkumar (2021) studied the combined modified Yukawa and Kratzer potential (MYKP) in the case of the Klein-Gordon equation (KGE) for calculated numerical results of the energy spectrum for CO, H_2, LiH, and HCl molecules by pNU and SUSYQM methods using the Greene-Aldrich approximation to handle $1/r$ and $1/r^2$ terms in the effective potential [1]. Many researchers have dealt with either the modified Yukawa or Kratzer potentials either individually or through the combination of one or both of them with other potentials within the framework of the different fundamental equations. Edet *et al.* analyzed the modified Kratzer potential plus the screened Coulomb potential by the pNU method and obtained the energy spectrum [2]. Ikot *et al.* obtained the exact bound state energy spectrum of the Schrödinger equation with energy-dependent molecular Kratzer potential using AIM [3]. Ikot *et al.* (2019) obtained the energy eigenvalues and the corresponding normalized eigenfunctions of screened Kratzer potential for lithium hydride (LiH) and hydrogen chloride (HCl) diatomic molecules within the framework of non-relativistic quantum mechanics via the pNU method. Ikot *et al.* (2019) obtained the energy eigenvalues and the corresponding normalized eigenfunctions of the screened Kratzer potential for lithium hydride (LiH) and hydrogen chloride (HCl) diatomic molecules within the framework of non-relativistic quantum mechanics via the pNU method [4]. Ahmadov *et al.* [5] studied solutions of KGE with the Manning-Rosen equation plus a class of Yukawa potentials using pNU and SUSYQM methods and presented the energy spectrum for any *l*-state and the corresponding radial wave functions in terms of the hypergeometric functions. Purohit *et al.* (2021) solved the Schrödinger equation in D dimensions and obtained the eigenspectrum of the energy and momentum for time-independent and time-dependent Hulthén-screened cosines Kratzer potentials Using the Qiang-Dong proper quantization rule and the supersymmetric quantum mechanics approach [6]. Recently, Purohit *et al.* (2022) obtained the energy spectrum for MYKP with the magnetic field and Aharanov-Bohm flux field using the pNU approach and SEM in this study [7]. The Kratzer potential (KP), which is known first and foremost by Kratzer itself [8], is used in quantum atomic-molecular physics and has played a vital role in the history of molecular and quantum chemistry [9]. Purohit *et al.* (2020) [10] obtained via the generalized Nikiforov–Uvarov method the approximate bound-state solutions of the D-dimensional KGE for screened cosine Kratzer potential (SCKP) using approximation suggested by Greene-Aldrich. Next year, Purohit *et al.* also investigated SCKP under the influence of the magnetic field and Aharanov–Bohm flux field and obtained energy eigenvalues and wave functions with external fields via the pNU method using the approximation method suggested by Greene-Aldrich for handling centrifugal barriers [11]. On the other hand, Yukawa potential, often called screened Coulomb potential, is a short-range potential that has applications in particle, high-energy, and molecular physics. It is used to study the interaction that occurs between the atoms of diatomic molecules [12–14]. Recently, Purohit *et al.* (2022) used the linear plus modified Yukawa potential as the quark-antiquark interaction potential and obtained the energy eigenvalues and associated wave function by solving the KGE analytically using the pNU method [15]. Based on this motivation, in this work, we study new approximate bound state solutions of the deformed Klein-Gordon equation (DKGE) in three-dimensional extended relativistic quantum mechanics (3D-ERQM) symmetries of a newly proposed combined potential called the improved modified exponential screened plus Yukawa potential (IMYKP) within the framework of parametric Bopp's shift method. This is a new potential model that has not yet been studied to the best of our knowledge in 3D-ERQM symmetries. The main objective of this study is to deepen the study of research [1] performed in the frameworks of symmetries known in quantum mechanics known in the literature and research on new applications of this potential. It is worth noting that the first foundations of quantum mechanics were based on non-commutative axioms for all positions $\widehat{q}_{\mu}^{(s,h,i)} * \widehat{q}_{\nu}^{(s,h,i)} \neq \widehat{q}_{\nu}^{(s,h,i)} * \widehat{q}_{\mu}^{(s,h,i)}$ and momentum $\widehat{\pi}_{\mu}^{(s,h,i)} * \widehat{\pi}_{\nu}^{(s,h,i)} \neq \widehat{\pi}_{\nu}^{(s,h,i)} * \widehat{\pi}_{\mu}^{(s,h,i)}$, individually or in combination $\widehat{q}_{\mu}^{(s,h,i)} * \widehat{\pi}_{\nu}^{(s,h,i)} \neq \widehat{\pi}_{\nu}^{(s,h,i)} * \widehat{q}_{\mu}^{(s,h,i)}$, here $(*)$ stands for the Weyl-Moyal star product. However, the scientific task was not accomplished due to mathematical difficulties. This made the specialized researchers ignore two basic axioms and complete the research path in the last axiom, and this is what is known as the relative and non-relativistic quantum mechanics known in the literature. In recent years, real indicators have appeared, for example, the divergence problem in the standard model (SM), where gravitational forces are not included in addition to many other problems [16–26]. It should be mentioned that before the renormalization approach was created and gained popularity, Heisenberg proposed the idea of extended non-commutativity to the coordinates as a possible treatment for eliminating the limitless numbers of field theories in 1930. Snyder published the first work on QFT's history in 1947 [27], and Connes introduced its geometric analysis in 1991 and 1994 [28, 29] in an effort to standardize QFT. Seiberg and Witten obtain a new version of gauge fields in noncommutative gauge theory [30] by extending earlier ideas on the advent of NC geometry in string theory with a nonzero B-field. The elimination of the observed unwanted divergences or infinities that appear to produce short-range in field theories like gravitational theory by creating new quantum fluctuations is one of the potential objectives of NC deformation of space-space and phase-phase. The noncommutative $\mathbb{R}^4$extended to the supersymmetric field theories for the $U(1)$ vector multiplets and chiral multiplets of the fundamental, anti-fundamental and adjoint representations of the gauge group [31], this shows another confirmation of the validity and effectiveness of the non-commutative theory. It is important to refer to some of our work related to the study of each of the two potentials under study in the framework of the extended Schrödinger equation such as new modified Kratzer-type interactions [32], modified Kratzer potential [33] and the generalized perturbed Yukawa potential [34], DKGE such as a linear combination of Hulthén and Kratzer potentials [35], modified More general exponential screened Coulomb potential plus Yukawa potential [36], generalized modified screened Coulomb plus inversely quadratic Yukawa potential [37], Manning-Rosen plus quadratic Yukawa potential [38], the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials [39], modified equal scalar and vector Manning-Rosen and Yukawa potentials [40] and the Kratzer potential which studied by Darroodi *et al.* [41]. For the deformed Dirac equation, we have studied both modified Yukawa potential [42], improved inversely quadratic Yukawa potential within improved Coulomb-like tensor interaction [43], improved Schiöberg potential within the Yukawa tensor interaction, new modified Yukawa potential [44], new modified Yukawa potential [45], spatially dependent mass for the improved Eckart potential including the improved Yukawa tensor interaction [46]. Furthermore, the modified Kratzer potential was studied in the framework of the Duffin-Kemmer-Petiau (DKP) equation [47]. The following are the vector and scalar IMYKP models that will be used in this study ($V_{yk}(\widehat{r})$ and $S_{yk}(\widehat{r})$) as follows:

$$\begin{pmatrix} V_{yk}(\widehat{r}) \\ S_{yk}(\widehat{r}) \end{pmatrix} = \begin{pmatrix} V_{yk}(r) - \frac{\mathbf{L}\Theta}{2r}\frac{\partial V_{yk}(r)}{\partial r} + O\left(\Theta^2\right) \\ S_{yk}(r) - \frac{\mathbf{L}\Theta}{2r}\frac{\partial S_{yk}(r)}{\partial r} + O\left(\Theta^2\right) \end{pmatrix}, \quad (1)$$

where $(V_{yk}(r), S_{yk}(r))$ are the (vector, scalar) modified Yukawa-Kratzer potential, according to the view of 3D-RQM and 3D-NRQM symmetry, known in the literature [1]:

$$V_{yk}(r) = -\frac{A_1 \exp(-2\alpha r)}{r^2} + \frac{A_2 \exp(-\alpha r)}{r} - A_3 - \left(\frac{A_4}{r} - \frac{A_5}{r^2} - D_e\right), \tag{2.1}$$

and

$$S_{yk}(r) = -\frac{S_1 \exp(-2\alpha r)}{r^2} + \frac{S_2 \exp(-\alpha r)}{r} - S_3 - \left(\frac{S_4}{r} - \frac{S_5}{r^2} - D_e\right), \tag{2.2}$$

where potential strength parameters $A_1 \equiv A_3 \equiv V_0$, $A_2 \equiv 2V_0$, $A_4 \equiv 2D_e r_e$, $A_5 \equiv D_e r_e^2$ and α is the screening parameters, D_e is the dissociation energy, r_e is the equilibrium bond length, ($\widehat{r}$ and r) are the interatomic distances in the extended QM and usual QM symmetries, respectively. The coupling $\mathbf{L\Theta}$ is the scalar product of the usual components of the angular momentum operator $\mathbf{L}(L_x, L_y, L_z)$ and the infinitesimal non-commutativity vector $\mathbf{\Theta}(\theta_{12}, \theta_{23}, \theta_{13})/2$. In the case of G_{NC}, the noncentral generators can be suitably realized as self-adjoint differential operators ($\widehat{q}_\mu^{(s,h,i)}$, $\widehat{\pi}_\nu^{(s,h,i)}$) appear in n three varieties. The first is the canonical structure (CS) variety, the second is the Lie structure (LS) variety, and the last corresponds to the quantum plane (QP) variety in the representations of Schrödinger, Heisenberg, and interaction pictures, satisfying a deformed algebra of the form (for simplicity, we have used the natural units $\hbar = c = 1$) [25, 48–55]:

$$\underbrace{\left[x_\mu^{(s,h,i)}, p_\nu^{(s,h,i)}\right] = i\delta_{\mu\nu}}_{\text{In-3D-QM symmetry}} \Longrightarrow \underbrace{\left[\widehat{q}_\mu^{(s,h,i)} \stackrel{*}{,} \widehat{\pi}_\nu^{(s,h,i)}\right] = i\hbar_{eff}\delta_{\mu\nu}}_{\text{In-3D-EQM symmetry}}, \tag{3}$$

and

$$\left[x_\mu^{(s,h,i)}, x_\nu^{(s,h,i)}\right] = 0 \Longrightarrow \left[\widehat{q}_\mu^{(s,h,i)} \stackrel{*}{,} \widehat{q}_\nu^{(s,h,i)}\right] = \begin{cases} i\epsilon_{\mu\nu}\theta : \text{ For CS variety,} \\ ih_{\mu\nu}^{\alpha}\widehat{q}_\alpha^{(s,h,i)} : \text{ For LS variety,} \\ iG_{\mu\nu}^{\alpha\beta}\Lambda_{\alpha\beta} : \text{For QP variety.} \end{cases} \tag{4}$$

with $\Lambda_{\alpha\beta} \equiv \widehat{q}_\alpha^{(s,h,i)}\widehat{q}_\beta^{(s,h,i)}$, here $\widehat{q}_\mu^{(s,h,i)}$ can be equal one of ($\widehat{x}_\mu^s \vee \widehat{x}_\mu^h \vee \widehat{x}_\mu^i$) and $\widehat{\pi}_\mu^{(s,h,i)}$ can be equal one of ($\widehat{p}_\mu^s \vee \widehat{p}_\mu^h \vee \widehat{p}_\mu^i$) which are the generalized coordinates and the corresponding generalizing momentums, respectively, in 3D-EQM symmetries, while the corresponding coordinates $x_\mu^{(s,h,i)}(x_\mu^s \vee x_\mu^h \vee x_\mu^i)$ and $p_\mu^{(s,h,i)}(p_\mu^s \vee p_\mu^h \vee p_\mu^i)$ in the usual 3D-QM symmetries, respectively, ($\vee \equiv$ *or)*. Additionally, the uncertainty relation (in LHS of the below equation) that corresponds to the LHS of Eq. (3), will become in 3D-EQM symmetries as follows:

$$\left|\Delta x_\mu^{(s,h,i)} \Delta p_\nu^{(s,h,i)}\right| \geqslant \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta \widehat{q}_\mu^{(s,h,i)} \Delta \widehat{\pi}_\nu^{(s,h,i)}\right| \geqslant \hbar_{eff}\delta_{\mu\nu}/2. \tag{5}$$

On the other hand, we notice that the RHS of Eq. (4) generates a novel uncertainty relation:

$$\left|\Delta \widehat{q}_\mu^{(s,h,i)} \Delta \widehat{q}_\nu^{(s,h,i)}\right| \geqslant \begin{cases} \frac{\theta|\epsilon_{\mu\nu}|}{2} \text{For CS variety,} \\ \frac{h_{\mu\nu}}{2} \text{ For LS variety,} \\ \frac{G_{\mu\nu}}{2} \text{ For QP variety.} \end{cases} \tag{6}$$

here $h_{\mu\nu}$ and $G_{\mu\nu}$ are equal to the average values:

$$\begin{cases} h_{\mu\nu} = \left|\left\langle \sum_{\alpha}^{3} \left(f_{\mu\nu}^{\alpha} \widehat{x}_\alpha^{(s,h,i)}\right)\right\rangle\right| \\ G_{\mu\nu} = \left|\left\langle \sum_{\alpha,\beta}^{3} \left(G_{\mu\nu}^{\alpha\beta} \widehat{x}_\alpha^{(s,h,i)} \widehat{x}_\beta^{(s,h,i)}\right)\right\rangle\right| \end{cases}.$$

The new subdivided three-uncertainties relations in Eq. (6) have no comparison in the existing literature. Under the Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame, Eqs. (3) and (4) are covariant equations (have the same behavior as $\widehat{q}_\mu^{(s,h,i)}$). We are expanded the modified equal-time noncommutative canonical commutation relations (METNCCCRs) to include both Heisenberg and interaction pictures in both 3D-ERQM and 3D-ENRQM symmetries. Here $\delta_{\mu\nu}$ is the Kronecker symbol, $(\mu, \nu = 1, 2, 3)$, $\theta_{\mu\nu}$ is antisymmetric real constant (3×3) matrices with the dimensionality (length)^2 parameterizing the deformation of space-space, $\epsilon_{\mu\nu}$ is an antisymmetric tensor operator describing the non-commutativity of space-time ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\epsilon_{\epsilon\epsilon} = 0$) and $\theta \in \mathbb{R}$ is the noncommutative parameter, $\hbar_{eff} \cong \hbar$ is the effective Planck constant. The new deformed scalar product $h(x) * g(x)$ is defined by the Weyl-Moyal $*$-product in three different ways [56–61]:

$$(h * g)(x) = \exp\left(i\epsilon^{\mu\nu}\theta\partial_\mu^x\partial_\nu^x\right)(hg)(x) \approx (hg)(x) - \frac{i\epsilon^{\mu\nu}\theta}{2}\partial_\mu^x h \partial_\nu^x g\rfloor_{x^\mu = x^\nu} + O\left(\theta^2\right) \tag{7}$$

The second component in Eq. (7) provides a physical representation of the consequences of space-space non-commutativity. The outline of the paper is as follows: Section 2 presents an overview of the 3D-KGE under the modified Yukawa-Kratzer potential models. Section 3 is devoted to investigating the 3D-DKGE using the well-known Bopp's shift method to obtain the effective potential for the IMYKP model. Furthermore, using standard perturbation theory, we find the expectation values of the radial terms $(1/(1-z)^4$, $z^2/(1-z)^3$, $z^2/(1-z)^4$, $z/(1-z)^2$, $z/(1-z)^3$ and $1/(1-z)^3)$ to calculate the corrected relativistic energy generated by the effect of the perturbed effective potential $\Xi_{pert}^{yk}(r)$ of the IMYKP model, and we derive the global corrected energies for bosonic particles and antiparticles whose spin quantum number has an integer value. Section 4 is reserved to analyze the non-relativistic limit, in 3D-ENRQM symmetries for the IMYKP model. In the next section, we will discuss the most important special cases in the relativistic and non-relativistic cases, which are useful for specialists and readers alike. Section 6 is reserved for the study of spin-averaged mass spectra of HLM under IMKSCP models in 3D-ENRQM symmetries. Finally, we present our conclusion in Sec. 7.

2. An overview of KGE under the modified Yukawa-Kratzer potential model in 3D-RQM symmetry

In order to construct a physical model describing a physical system that interacted with the improved modified Yukawa-Kratzer potential model in the 3D-ERQM symmetries, it is useful to recall the eigenvalues and the corresponding eigenfunctions under the influence of this system within the framework of relativistic quantum mechanics (RQM) known in the literature. In this case, the system is governed by the following Klein-Gordon equation:

$$\left(\nabla^2+(E_{nl}-V_{yk}(r))^2-(M_{yk}+S_{yk}(r))^2\right)\Psi_{nl}(r,\theta,\varphi)=0. \tag{8}$$

The vector potential $V_{yk}(r)$ and space-time scalar potential $S_{yk}(r)$ are produced from the four-vector linear momentum operator A^{μ} ($V_{yk}(r)$, $\mathbf{A}=\mathbf{0}$) and the reduced mass M_{yk} of (LiH, HCl, CO and, H_2) molecules. While E_{nl} is the relativistic eigenvalues, (n,l) represent the principal and spin-orbit coupling terms. Since the modified Yukawa-Kratzer potential model has spherical symmetry, it allows the wave function solution $\Psi_{nl}(r,\theta,\varphi)$ of the known form $(u_{nl}(r)/r)Y_m^l(\theta,\varphi)$ while $Y_m^l(\theta,\varphi)$ is spherical harmonics and m is the projections on the Oz-axis. The radial component $u_{nl}(r)$ satisfies the differential equation as follows [1]:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-M^2-\left(\begin{array}{c}\left(\frac{V_{yk}(r)}{2}\right)^2-\left(\frac{S_{yk}(r)}{2}\right)^2\\-(E_{nl}V_{yk}(r)+2MS_{yk}(r))\end{array}\right)-\frac{l(l+1)}{r^2}\right)u_{nl}(r)=0. \tag{9}$$

Parmar and Vinodkumar used the Alhaidari *et al.* [62] scheme to write the radial part of KGE in Eq. (9), by restyling the vector and scalar potentials $(V_{yk}(r),S_{yk}(r))\rightarrow(V_{yk}(r)/2,S_{yk}(r)/2)$ under the non-relativistic limit. Using $V_{yk}(r)$ from Eq. (2) with $V_{yk}(r)=S_{yk}(r)$ in Eq. (9), we obtain the following

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-M^2-\Xi_{yk}(r)-\frac{l(l+1)}{r^2}\right)u_{nl}(r)=0, \tag{10}$$

with

$$\Xi_{yk}(r)=(E_{nl}+M)\left(A_1\frac{\exp(-2\alpha r)}{r^2}-A_2\frac{\exp(-\alpha r)}{r}+A_3+\frac{A_4}{r}-\frac{A_5}{r^2}-D_e\right). \tag{11}$$

The author of Ref. [1] used the supersymmetry quantum mechanics and factorization methods to obtain the expression of $u_{nl}(r)$ as a function of Gauss's Hypergeometric function ${}_2F_1(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z)$ in usual 3D-RQM symmetries as,

$$\Psi_{nl}(r,\theta,\varphi)=C_{nl}^m\frac{z^{\omega_{nl}}}{r}(1-z)^{X_{nl}}\ {}_2F_1(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z)\,Y_m^l(\theta,\varphi), \tag{12}$$

with

$$\left\{\begin{array}{c}\omega_{nl}=\sqrt{\epsilon_{nl}^2+l(l+1)+(E_{nl}+M)A_5-\frac{(E_{nl}+M)A_4}{\alpha}}\\X_{nl}=\frac{1}{2}+\sqrt{(E_{nl}+M)(A_5-A_1)+(l+1/2)^2}\end{array}\right., \tag{13}$$

where z and ϵ_{nl}^2 equal $(\exp(-\alpha r))$ and $\left(-[(E_{nl}+M)(E_{nl}-M+A_3-D_e)]/\alpha^2\right)$, respectively, while N_{nl} is the normalization constant:

$$C_{nl}^n = \frac{n!\Gamma(2\omega_{nl}+1)}{\Gamma(n+2\omega_{nl}+1)}\sqrt{\frac{\alpha n!\,(2X_{nl}+1)\,\Gamma(n+2\omega_{nl}+2X_{nl}+1)}{2^{2\omega_{nl}+2X_{nl}-1}\Gamma(2\omega_{nl}+n+1)\,\Gamma(2X_{nl}+n+1)}}. \tag{14}$$

The corresponding relativistic energy eigenvalues for the modified Yukawa-Kratzer potential model for (LiH, HCl, CO and, H_2) molecules in 3D-space, obtained the equation of energy [1]:

$$\begin{aligned} E_{nl}^2 - M^2 &= (E_{nl}+M)\left(D_e - A_3 + \alpha^2 A_5 - \alpha A_4\right) + \alpha^2 l(l+1) \\ &\quad - \alpha^2\left[\frac{\frac{(E_{nl}+M)}{\alpha^2}\left(\alpha^2(A_5+A_1)+\alpha(A_2-A_1)+l(l+1)\right)}{2(n+X_{nl})} + \frac{n+X_{nl}}{2}\right]^2. \end{aligned} \tag{15}$$

3. The new solutions of DKGE under the IMYKP models in 3D-ERQM symmetries:

3.1. Review of BS method

Let us begin in this subsection by finding the physical form of IMYKP models in three-dimensional extended relativistic quantum mechanics (3D-ERQM) symmetries. Our objective is achieved by applying the new principles that we have seen in the introduction (Eqs. (3), (4), and (7)), summarized in the new relationships of the modified equal-time noncommutative canonical commutation relations (METNCCCRs) and the notion of the Weyl-Moyal star product. These data allow us to rewrite the usual radial KG equations in Eq. (10) in the 3D-ERQM symmetries as follows:

$$\left(\frac{d^2}{dr^2} + E_{nl}^2 - m_0^2 - \Xi_{yk}(r) - \frac{l(l+1)}{r^2}\right) * u_{nl}(r) = 0. \tag{16.1}$$

There are two approaches to including non-commutativity in quantum field theory: The first method is represented by rewriting the various NC physical fields $\left(\widehat{\Psi}_{nl}, \widehat{\Phi}_{nl}, \widehat{e}_{\mu}^a, ...\right)$ in terms of their corresponding fields $\left(\Psi_{nl}, \Phi_{nl}, e_{\mu}^a, ...\right)$ in the known quantum space in the literature, in proportion to the non-commutative parameters $\mathbf{\Theta}\,(\theta_{12}, \theta_{23}, \theta_{13})/2$, which is similar to the Taylor development [63–71] while the second method depends on reformulating the non-commutative operator $(\widehat{q}, \widehat{\pi})$ with its view of the quantum operators $(\widehat{x}, \widehat{p})$ known in the literature and the properties of space associated with the non-commutative parameters $\mathbf{\Theta}\,(\theta_{12}, \theta_{23}, \theta_{13})/2$. It is normal for the physical results to be identical when using either of them. It is known to specialized researchers that F. Bopp had proposed new quantization rules

$$(x,p) \to \left(\widehat{q} = x - \frac{i}{2}\partial_p, \widehat{\pi} = p + \frac{i}{2}\partial_x\right),$$

instead of the usual correspondence

$$(x,p) \to \left(\widehat{q} = x, \widehat{q} = p + \frac{i}{2}\partial_x\right).$$

This procedure is called Bopp's shifts method (BSM) [72–77]. This quantization procedure is called Bopp quantization [76]. The Weyl-Moyal star product $h(x,p)*g(x,p)$ induces BSM in the respect that it is replaced by $h(x-\frac{i}{2}\partial_p, p+\frac{i}{2}\partial_x)*g(x,p)$ [69]. This, allows us to obtain

$$\begin{cases} \Xi_{yk}(r) * u_{nl}(r) = \Xi_{yk}(r)(\widehat{r})\, u_{nl}(r) \\ \frac{l(l+1)}{r^2} * u_{nl}(r) = \frac{l(l+1)}{\widehat{r}^2} u_{nl}(r) \end{cases}. \tag{16.2}$$

The Bopp's shift method has achieved great success when applied by specialized researchers to the four basic equations correspond to the relativistic deformed Schrödinger equation (see, *e.g.*, [32–34,77–82]) and the other three relativistic equations represented by the deformed Klein-Gordon equation (see, *e.g.*, [83–88]), deformed Dirac equation (see, *e.g.*, [47, 93, 94]) and the deformed Duffin-Kemmer-Petiau equation (see, *e.g.*, [86–89]). It is worth motioning that that Bopp's shift method permutes us to rewrite Eq. (16.1) without star product to the simplest form:

$$\left(\frac{d^2}{dr^2} + E_{nl}^2 - m_0^2 - \Xi_{yk}(\widehat{r}) - \frac{l(l+1)}{\widehat{r}^2}\right) u_{nl}(r) = 0. \tag{17}$$

The modified algebraic structure of covariant canonical commutation relations with the notion of the Weyl-Moyal star product in Eqs. (3) and (4), which become new METNCCCRs with ordinary known products in literature, are as follows (see, *e.g.*, [72–77]):

$$\left[\widehat{q}_{\mu}^{(s,h,i)},\widehat{\pi}_{\nu}^{(s,h,i)}\right]=i\hbar_{eff}\delta_{\mu\nu}\text{ and }\left[\widehat{q}_{\mu}^{(s,h,i)},\widehat{q}_{\nu}^{(s,h,i)}\right]=i\theta_{\mu\nu}. \tag{18}$$

In 3D-ERQM symmetries, one possible way of implementing the algebra defined by Eq. (18) is to construct the noncommutative set of variables ($\widehat{q}_{\mu}^{(s,h,i)}$, $\widehat{\pi}_{\nu}^{(s,h,i)}$) from the corresponding commutative variables ($x_{\mu}^{(s,h,i)}$, $p_{\nu}^{(s,h,i)}$) employing of linear transformations. This can be generally done by using the Seiberg-Witten map, given by [72–77]:

$$\begin{pmatrix}\widehat{q}_{\mu}^{(s,h,i)}\\ \widehat{\pi}_{\mu}^{(s,h,i)}\end{pmatrix}=\begin{pmatrix}x_{\mu}^{(s,h,i)}-\sum\limits_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_{\nu}^{(s,h,i)}+O\left(\theta^{2}\right)\\ p_{\mu}^{(s,h,i)}+O\left(\theta^{2}\right)\end{pmatrix}. \tag{19}$$

This allows us to find the operator $\widehat{r}^2$, in the 3D-ERQM symmetries, equal to [80–85]:

$$\widehat{r}^{2}=r^{2}-\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{20}$$

The Taylor expansion of effective potential $\Xi_{yk}\left(\widehat{r}\right)$ can be expressed as in the 3D-ERQM symmetries, as:

$$\Xi_{yk}\left(\widehat{r}\right)=\Xi_{yk}\left(r\right)-\frac{1}{2r}\frac{\partial\Xi\left(r\right)}{\partial r}\mathbf{L}\Theta+O\left(\Theta^{2}\right), \tag{21}$$

and

$$\frac{l\left(l+1\right)}{\widehat{r}^{2}}=\frac{l\left(l+1\right)}{r^{2}}+\frac{l\left(l+1\right)}{r^{4}}\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{22}$$

Substituting Eqs. (20) and (21) into Eq. (17), we obtain the following as the radial Schrödinger equation:

$$\left(\frac{d^{2}}{dr^{2}}+E_{nl}^{2}-m_{0}^{2}-\Xi_{yk}\left(r\right)-\Xi_{yk}^{pert}\left(r\right)\right)u_{nl}\left(r\right)=0, \tag{23}$$

with

$$\Xi_{yk}^{pert}\left(r\right)=\frac{l\left(l+1\right)}{r^{4}}\mathbf{L}\Theta-\frac{1}{2r}\frac{\partial\Xi\left(r\right)}{\partial r}\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{24}$$

If we compare Eq. (23) and Eq. (10), we observe an additive potential $\Xi_{yk}^{pert}\left(r\right)$ dependent on new radial terms, which is coupled with the coupling $\mathbf{L\Theta}$ that explains the interaction of the physical features of the system with the topological deformations of space-time. From a physical point of view, this means that the spontaneously generated term $\Xi_{yk}^{pert}\left(r\right)$ as a result of the topological properties of deformation space-space can be considered very small compared to the fundamental term

$$\begin{aligned}\Xi_{yk}^{pert}\left(r\right)=\Bigg(&-\frac{l\left(l+1\right)-\left(E_{nl}+M\right)A_{5}}{r^{4}}-\left(E_{nl}+M\right)-\alpha A_{1}\frac{\exp\left(-2\alpha r\right)}{r^{3}}\\ &-A_{1}\frac{\exp\left(-2\alpha r\right)}{r^{4}}+\alpha A_{2}\frac{\exp\left(-\alpha r\right)}{2r^{2}}+A_{2}\frac{\exp\left(-\alpha r\right)}{2r^{3}}-\frac{A_{4}}{2r^{3}}\Bigg)\mathbf{L}\Theta+O\left(\Theta^{2}\right).\end{aligned} \tag{25}$$

Furthermore, by using the unit step function (also known as the viside step function $\theta\left(x\right)$ or simply the theta function), it is possible to rewrite the global induced potential $\Xi_{t\text{-}yk}^{pert}\left(r\right)$ for bosonic particle (positive energy) and bosonic antiparticle (negative energy) in 3D-DKG symmetries as:

$$\Xi_{t\text{-}yk}^{pert}\left(r\right)=\Xi_{yk}^{pert}\left(r\right)\theta\left(\left|E_{nc}^{yk}\right|\right)-\Xi_{yk}^{pert}\left(r\right)\theta\left(-\left|E_{nc}^{yk}\right|\right)=\begin{cases}\Xi_{yk}^{pert}\left(r\right)\text{ for bosonic particles}\\ -\Xi_{yk}^{pert}\left(r\right)\text{ for bosonic antiparticles}\end{cases}, \tag{26}$$

where the step function $\theta\left(x\right)$ is given by:

$$\theta\left(x\right)=\begin{cases}1\text{ for }x\geq0\\ 0\text{ for otherwise}\end{cases}, \tag{27}$$

Eq. (23) cannot be solved analytically for any state $l \neq 0$ because of the centrifugal term and the studied potential itself. In fact, since the effective perturbative potential $\Xi_{yk}^{pert}(r)$ given in Eq. (25) has a strong singularity at $r \to 0$, we need to use the technique of the Greene and Aldrich approximation scheme [95] and applied by Parmar and Vinodkumar [1]. The radial part of the three-dimensional deformed Klein-Gordon equation (3D-DKGE) with the IMYKP model contains the centrifugal terms $l(l+1)/r^2$ and $l(l+1)/r^4$ since we assume $l \neq 0$. However, the IMYKP model is a kind of potential that cannot be solved exactly when the centrifugal term is taken into account unless $l = 0$ is assumed. The conventional approximation used in this paper:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1-\exp\left(-\alpha r\right)\right)^2} = \frac{\alpha^2}{\left(1-z\right)^2} \Leftrightarrow \frac{1}{r} \approx \frac{\alpha}{1-\exp\left(-\alpha r\right)} = \frac{\alpha}{1-z}. \tag{28}$$

This allows us, after direct calculations, to find the following results:

$$\begin{cases} \frac{1}{r^4} \approx \frac{\alpha^4}{(1-z)^4}, \frac{\exp(-2\alpha r)}{r^3} \approx \frac{\alpha^3 z^2}{(1-z)^3} \\ \frac{\exp(-2\alpha r)}{r^4} \approx \frac{\alpha^4 z^2}{(1-z)^4}, \frac{\exp(-\alpha r)}{r^2} \approx \frac{\alpha^2 z}{(1-z)^2} \\ \frac{\exp(-\alpha r)}{r^3} \approx \frac{\alpha^3 z}{(1-z)^3} \text{ and } \frac{1}{r^3} \approx \frac{\alpha^3}{(1-z)^3} \end{cases}. \tag{29}$$

This gives the perturbative effective potential as follows:

$$\Xi_{yk}^{pert}(z) = \left(\frac{K_1}{(1-z)^4} + \frac{K_2 z^2}{(1-z)^3} + \frac{K_3 z^2}{(1-z)^4} + \frac{K_4 z}{(1-z)^2} + \frac{K_5 z}{(1-z)^3} + \frac{K_6}{(1-z)^3}\right) \mathbf{L}\Theta + O\left(\Theta^2\right), \tag{30}$$

with

$$\begin{cases} K_1 = \alpha^4\left[\left(E_{nl}+M\right)A_5 + \alpha^4 l\left(l+1\right)\right] \\ K_2 = K_3 = \left(E_{nl}+M\right)\alpha^4 A_1 \\ K_4 = K_5 = -\left(E_{nl}+M\right)\alpha^3 A_2/2 \\ K_6 = \left(E_{nl}+M\right)\alpha^3 A_4/2 \end{cases}. \tag{31}$$

The IMYKP model is extended by including new radial terms $1/(1-z)^4$, $z^2/(1-z)^3$, $z^2/(1-z)^4$, $z/(1-z)^2$, $z/(1-z)^3$ and $1/(1-z)^3$ to become the IMYKP model in 3D-ERQM symmetries. The new additive part $\Xi_{yk}^{pert}(r)$ is also proportional to the infinitesimal coupling $\mathbf{L\Theta}$, this is logical from a physical point of view, because it explains the interaction between the physical properties of the studied potential $\mathbf{L}$ and the topological properties resulting from the deformation of space-space $\mathbf{\Theta}$. This allows us to consider the additive effective potential as a perturbation potential compared with the main potential $\Xi_{yk}(r)$ (parent potential operator) in the symmetries of 3D-ERQM symmetries, that is, the inequality $\Xi_{yk}^{pert}(r) \ll \Xi_{yk}(r)$ has been achieved. That is all the physical justification for applying the time-independent perturbation theory that can be satisfied. This allows us to give a complete prescription for determining the energy level of the generalized $(n,l,m)^{th}$ excited states.

3.2. The expectation values under the IMYKP models in the 3D-ERQM symmetries

In this subsection, we want to apply the perturbative theory, in the case of 3D-ERQM symmetries, to find the 6-expectation values $T_{(nlm)}^{yk-\mu}$, $T_{(nlm)}^{yk-2}$, $T_{(nlm)}^{yk-3}$, $T_{(nlm)}^{yk-4}$, $T_{(nlm)}^{yk-5}$ and $T_{(nlm)}^{yk-6}$ which are equal, respectively $\left\langle 1/(1-z)^4\right\rangle_{(nlm)}^{yk}$, $\left\langle z^2/(1-z)^3\right\rangle_{(nlm)}^{yk}$, $\left\langle z^2/(1-z)^4\right\rangle_{(nlm)}^{yk}$, $\left\langle z/(1-z)^2\right\rangle_{(nlm)}^{yk}$, $\left\langle z/(1-z)^3\right\rangle_{(nlm)}^{yk}$ and $\left\langle 1/(1-z)^3\right\rangle_{(nlm)}^{yk}$ for bosonic particles taking into account the unperturbed wave function $\Psi_{nl}(r,\theta,\varphi)$ which we have seen previously in Eq. (12). After straightforward calculations, we obtain the expectation values $T_{(nlm)}^{yk-1}$, $T_{(nlm)}^{yk-2}$, $T_{(nlm)}^{yk-3}$, $T_{(nlm)}^{yk-4}$, $T_{(nlm)}^{yk-5}$ and $T_{(nlm)}^{yk-6}$ by applying the standard perturbation theory in first-order as follows:

$$T_{(nlm)}^{yk-1} = C_{nl}^{n2}\int_0^{+\infty} z^{2\omega_{nl}}\left(1-z\right)^{2X_{nl}-4}\left[\,{}_2F_1\left(-n, n+2\omega_{nl}+2X_{nl}, 2X_{nl}+1; z\right)\right]^2 dr, \tag{32.1}$$

$$T_{(nlm)}^{yk-2} = C_{nl}^{n2}\int_0^{+\infty} z^{2\omega_{nl}+2}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_2F_1\left(-n, n+2\omega_{nl}+2X_{nl}, 2X_{nl}+1; z\right)\right]^2 dr, \tag{32.2}$$

$$T_{(nlm)}^{yk-3}=C_{nl}^{n2}\int_{0}^{+\infty} z^{2\omega_{nl}+2}\left(1-z\right)^{2X_{nl}-4}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z\right)\right]^{2}dr, \tag{32.3}$$

$$T_{(nlm)}^{yk-4}=C_{nl}^{n2}\int_{0}^{+\infty} z^{2\omega_{nl}+1}\left(1-z\right)^{2X_{nl}-2}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z\right)\right]^{2}dr, \tag{32.4}$$

$$T_{(nlm)}^{yk-5}=C_{nl}^{n2}\int_{0}^{+\infty} z^{2\omega_{nl}+1}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z\right)\right]^{2}dr, \tag{32.5}$$

and

$$T_{(nlm)}^{yk-6}=C_{nl}^{n2}\int_{0}^{+\infty} z^{2\omega_{nl}}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl},2X_{nl}+1;z\right)\right]^{2}dr. \tag{32.6}$$

We have used useful abbreviations $\left\langle \widehat{X}\right\rangle_{(nlm)}^{sp-yk}=\left\langle n,l,m\left|\widehat{X}\right|n,l,m\right\rangle$ to avoid the extra burden of writing, $\widehat{X}$ equal $1/(1-z)^{4}$, $z^{2}/(1-z)^{3}$, $z^{2}/(1-z)^{4}$, $z/(1-z)^{2}$, $z/(1-z)^{3}$ and $1/(1-z)^{3}$. We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number ($n=0,1,...$) and then generalize the result to the general $(n,l,m)^{th}$ excited state or we use the method proposed by Tas *et al.* [96] and applied by Ahmadov *et al.* [97], to obtain the general excited state directly. Introducing the change of variable $z=\exp(-\alpha r)$. This maps the region $0\leqslant r\lessdot\infty$ to $0\leqslant z\leq 1$ and allows us to obtain $dr=-\frac{dz}{\alpha z}$, and transform Eqs. $(32,i=\overline{1,6})$ into the following form:

$$T_{(nlm)}^{yk-1}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}-1}\left(1-z\right)^{2X_{nl}-4}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz, \tag{33.1}$$

$$T_{(nlm)}^{yk-2}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}+2-1}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz, \tag{33.2}$$

$$T_{(nlm)}^{yk-3}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}+2-1}\left(1-z\right)^{2X_{nl}-4}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz, \tag{33.3}$$

$$T_{(nlm)}^{yk-4}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}+1-1}\left(1-z\right)^{2X_{nl}-2}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz, \tag{33.4}$$

$$T_{(nlm)}^{yk-5}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}+1-1}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz, \tag{33.5}$$

and

$$T_{(nlm)}^{yk-6}=\frac{C_{nl}^{n2}}{\alpha}\int_{0}^{+1} z^{2\omega_{nl}-1}\left(1-z\right)^{2X_{nl}-3}\left[\,{}_{2}F_{1}\left(-n,n+2\omega_{nl}+2X_{nl}+2-2,2X_{nl}+1;z\right)\right]^{2}dz. \tag{33.6}$$

We calculate the integrals in Eqs. $(33,i=\overline{1,6})$ with the help of the special integral formula [98]:

$$\int_{0}^{+1} z^{2\lambda-1}\left(1-z\right)^{\mathbf{2}(\nu+1)}\left[{}_{2}F_{1}\left(-n,n+2\left(\nu+\lambda+1\right);2\lambda+1;z\right)\right]^{2}dz$$
$$=\frac{n!\left(n+\nu+1\right)\Gamma\left(2\lambda\right)\Gamma\left(n+2\left(\nu+1\right)\right)\Gamma\left(2\lambda+1\right)}{\left(n+\nu+1+\lambda\right)\Gamma\left(n+2\lambda+1\right)\Gamma\left(n+2\lambda+2\left(\nu+1\right)\right)}, \tag{34}$$

here $\Gamma(\xi)$ denoting the usual Gamma function. By identifying Eqs. $(33, i=\overline{1,6})$ with the integrals in Eqs. (34), we obtain the following results:

$$T_{(nlm)}^{yk-1} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-2)\,\Gamma(2\omega_{nl})\,\Gamma(n+2X_{nl}-4)\,\Gamma(2\omega_{nl}+\mathbf{1})}{(n+D_{nl}/2-2)\,\Gamma(n+2\omega_{nl}+1)\,\Gamma(n+D_{nl}-4)}, \tag{35.1}$$

$$T_{(nlm)}^{yk-2} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-3/2)\,\Gamma(2\omega_{nl}+2)\,\Gamma(n+2X_{nl}-3)\,\Gamma(2\omega_{nl}+3)}{(n+D_{nl}/2-1/2)\,\Gamma(n+2\omega_{nl}+3)\,\Gamma(n+D_{nl}-1)}, \tag{35.2}$$

$$T_{(nlm)}^{yk-3} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-2)\,\Gamma(2\omega_{nl}+2)\,\Gamma(n+2X_{nl}-4)\,\Gamma(2\omega_{nl}+3)}{(n+D_{nl}/2-1)\,\Gamma(n+2\omega_{nl}+3)\,\Gamma(n+D_{nl}-2)}, \tag{35.3}$$

$$T_{(nlm)}^{yk-4} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-1)\,\Gamma(2\omega_{nl}+1)\,\Gamma(n+2X_{nl}-2)\,\Gamma(2\omega_{nl}+2)}{(n+D_{nl}/2-1/2)\,\Gamma(n+2\omega_{nl}+2)\,\Gamma(n+D_{nl}-1)}, \tag{35.4}$$

$$T_{(nlm)}^{yk-5} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-3/2)\,\Gamma(2\omega_{nl}+1)\,\Gamma(n+2X_{nl}-3)\,\Gamma(2\omega_{nl}+2)}{(n+D_{nl}/2-1)\,\Gamma(n+2\omega_{nl}+2)\,\Gamma(n+D_{nl}-2)}, \tag{35.5}$$

and

$$T_{(nlm)}^{yk-6} = \frac{C_{nl}^{n2}}{\alpha} \frac{n!\,(n+X_{nl}-3/2)\,\Gamma(2\omega_{nl})\,\Gamma(n+2X_{nl}-3)\,\Gamma(2\omega_{nl}+1)}{(n+D_{nl}/2-3/2)\,\Gamma(n+2\omega_{nl}+1)\,\Gamma(n+D_{nl}-3)}, \tag{35.6}$$

with $D_{nl} = 2\omega_{nl} + 2X_{nl}$.

3.3. The corrected energy for the IMYKP models in 3D-ERQM symmetries

What draws attention here is the application of our physical method resulting from the principle of superposition for the purpose of determining the total values of the relativistic energy, in 3D-RNCQM symmetries. The global effective potential $\Xi_{sk}^{eff}(r)$ which is the sum of ($\Xi_{sk}(r) + l(l+1)/r^2 + \Xi_{sk}^{pert}(r)$) is responsible for the production of total relativistic energy within the framework of extended quantum mechanics symmetries under improved modified Yukawa-Kratzer potential. Naturally, the effective potential ($\Xi_{sk}(r) + l(l+1)/r^2$) is responsible for the relativistic energy known in the literature under the modified Yukawa-Kratzer potential that we have seen in Eq. (14), which is dominant in the absence of space-space deformation. Whereas the spontaneously generated potential $\Xi_{sk}^{pert}(r)$ due to space-space deformation will play the role of the corrected energy. Considering that the NC parameter Θ is arbitrary, it can be dealt with physically. Firstly, the influence of the perturbed spin-orbit can be generated from effective perturbed potential $\Xi_{sk}^{pert}(r)$ corresponding to the bosonic particle and antiparticle with spin-s such as LiH, HCl, CO and, H_2. We obtain the perturbed spin-orbit effective potential by replacing the coupling of the angular momentum $\mathbf{L}$ operator and the NC vector Θ with the new equivalent coupling:

$$\mathbf{L\Theta} \rightarrow \Theta\mathbf{LS} \text{ with } \Theta = \sqrt{\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2}.$$

This degree of freedom results from the arbitrary nature of the infinitesimal NC vector Θ. We have oriented the spin-s of the (LiH, HCl, CO and, H_2) molecules to become parallels to the vector Θ which interacted with the IMYKP model. Additionally, we use the following transformation which is well known in 3D-RQM symmetries:

$$\Theta\mathbf{LS} \rightarrow \Theta\mathbf{G}^2 \text{ with } \mathbf{G}^2 = \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right).$$

In 3D-ERQM symmetry, the operators ($\widehat{\mathbf{H}}_{nc}^{yk}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, and the eigenvalues of the operator $\mathbf{G}^2$ are equal to the values:

$$F(j,l,s) = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right], \tag{36}$$

with $|l-s| \leq j \leq |l+s|$ for (LiH, HCl, CO and, H_2) molecules. As a direct consequence, the square partially corrected energies $\Delta E_{yk}^{so2}(n, V_0, \alpha, D_e, r_e, \Theta, j, l, s) \equiv \Delta E_{yk}^{so2}$ due to the perturbed effective potential $\Xi_{yk}^{pert}(r)$ produced for the $(n,l,m)^{th}$ excited state, in 3D-ERQM symmetries as follows:

$$\Delta E_{yk}^{so2} = \Theta F(j,l,s)\ \langle M\rangle_{(nlm)}^{yk}(n, V_0, \alpha, D_e, r_e). \tag{37}$$

The global expectation values $\langle M\rangle_{(nlm)}^{yk}(n,V_0,\alpha,D_{e,}r_e)$ for (LiH, HCl, CO, and, H_2) molecules, which were created from the effect of the IMYKP model, are determined from the following expression:

$$\langle M\rangle_{(nlm)}^{yk}(n,V_0,\alpha,D_{e,}r_e)=\sum_{\mu=1}^{6}K_\mu T_{(nlm)}^{yk-\mu}(n,V_0,\alpha,D_{e,}r_e)\,. \tag{38}$$

The second principal physical contribution is due to the effect of the magnetic perturbative effective potential, which generates the perturbed potential $\Xi_{yk}^{pert}(r)$ under the IMYKP model in the 3D-ERQM symmetries. These effective potentials are achieved when we replace both

$$\mathbf{L\Theta}\rightarrow\tau\overrightarrow{L}\overrightarrow{\aleph}\text{ with }\aleph=\aleph\mathbf{e}_z, \tag{39}$$

with the additive physical condition

$$[\mathbf{\Theta}]=[\tau]\,[\aleph]\equiv(\text{length})^2,$$

here ($\aleph$ and τ) are the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal non-commutativity parameter. This choice, which comes from the fact that the vector $\mathbf{\Theta}$ is arbitrary, or that the magnetic field is directed according to the (Oz) axis serves to simplify quantitative calculations without affecting the nature of the physical point of view, we also need to apply:

$$\langle n',l',m'\,|L_z|\,n,l,m\rangle=m\delta_{m'm}\delta_{l'l}\delta_{n'n}, \tag{40}$$

with $-|l|\leq m\leq+|l|$ for the (LiH, HCl, CO and, H_2) molecules. All of these data allow for the discovery of the new square improved energy shift $\Delta E_{yk}^{mg2}(n,V_0,\alpha,D_{e,}r_e,\,\tau,\,m)$ for (LiH, HCl, CO and, H_2) molecules due to the perturbed Zeeman effect created by the influence of the IMYKP model for the $(n,l,m)^{th}$ excited state in 3D-ERQM symmetries as follows:

$$\Delta E_{yk}^{mg2}(n,V_0,\alpha,D_{e,}r_e,\tau,m)=\tau\aleph(\sum_{\mu=1}^{6}K_\mu T_{(nlm)}^{yk-\mu}(n,V_0,\alpha,D_{e,}r_e))m, \tag{41}$$

After we have completed the first and second stages of the self-production of energy, we are going to discover another very important case under the IMYKP model in 3D-ERQM symmetries. This physical new phenomenon is produced automatically from the influence of perturbed effective potential $\Xi_{yk}^{pert}(r)$ which we have seen in Eq. (30). We consider the bosonic particle (or antiparticle) undergoing rotation with angular velocity Ω. The features of this subjective phenomenon are determined through the replace the arbitrary vector $\mathbf{\Theta}$ with $\chi\Omega$. Allowing us to replace the coupling $\mathbf{L}\Theta$ with $\chi\mathbf{L}\Omega$, as follows:

$$\mathbf{L}\Theta\;\rightarrow\chi\mathbf{L\Omega}. \tag{42}$$

Here χ is just an infinitesimal real proportional constant. The effective potentials $W_{pert}^{yk-rot}(z)$, which induced the rotational movements of the bosonic particles, can be expressed as follows:

$$W_{pert}^{yk-rot}(r)=\chi\left(\frac{K_1}{(1-z)^4}+\frac{K_2z^2}{(1-z)^3}+\frac{K_3z^2}{(1-z)^4}+\frac{K_4z}{(1-z)^2}+\frac{K_5z}{(1-z)^3}+\frac{K_6}{(1-z)^3}\right)\mathbf{L\Omega}+O\left(\Theta^2\right). \tag{43}$$

We chose a rotational velocity Ω parallel to the (Oz) axis ($\Omega=\mathbf{\Omega e}_z$) to simplify the calculations, this, of course, does not change the physical characteristics of the examined problem as much as it simplifies the calculations. The pertubed generated spin-orbit coupling is then transformed into new physical phenomena as follows:

$$\mathbf{L\Omega}\rightarrow\chi\Omega L_z. \tag{44}$$

All of this data allow for the discovery of the new corrected square improved energy ΔE_{yk}^{rot2} (n,V_0, α, D_e, r_e, χ, m) of the (LiH, HCl, CO and, H_2) molecules due to the perturbed effective potential $W_{pert}^{yk-rot}(z)$ which is generated automatically by the influence of the IMYKP model for the $(n,l,m)^{th}$ excited state in 3D-ERQM symmetries as follows:

$$\Delta E_{yk}^{rot2}=\chi\Omega\left(\sum_{\mu=1}^{6}K_\mu T_{(nlm)}^{yk-\mu}(n,V_0,\alpha,D_{e,}r_e)\right)m. \tag{45}$$

It's worth noting that the authors of ref. [99] were studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and 3-dimensional space at zero temperature, but in this case, the rotational term was added to the Hamiltonian operator manually, whereas, in our study, the rotation operator $W_{pert}^{yk-rot}(z)\,\mathbf{L}\Omega$ appears automatically due to the effect of the deformation of space-space under the IMYKP model. The eigenvalues of the operations $\mathbf{G}^2$ for a bosonic particle and antiparticle (negative energy) with spin $s = (1, 2..)$ are equal $[j(j+1) - l(l+1) - s(s+1)]/2$, the possible values of j are $\{|l-s|, |l-s|+1, ..., |l+s|\}$. In the symmetries of the 3D-ERQM symmetries, the total relativistic improved energy E_{nc}^{yk} $(n, V_0, \alpha, \Theta, \tau, \chi, j, l, s, m)$ for the (LiH, HCl, CO and, H_2) diatomic molecules with IMYKP model, corresponding to the generalized $(n,l,m)^{th}$ excited states are expressed as:

$$E_{nc}^{yk}(n, V_0, \alpha, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) = E_{nl} + [\langle M\rangle_{(nlm)}^{yk}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2}, \tag{46}$$

where E_{nl} are usual relativistic energies under the IMYKP model obtained from equations of energy in Eq. (15). It should be noted that the corrected relativistic energy in Eq. (46) can be generalized to include negative energy (the bosonic antiparticle) and the positive relativistic energy (the bosonic particle) as follows:

$$E_{t-nc}^{yk-b} = \begin{cases} E_{nl} \; + [\langle M\rangle_{(nlm)}^{yk}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2} \\ \qquad\qquad \text{for bosonic particle} \\ \\ E_{nl} \; - [\langle M\rangle_{(nlm)}^{yk}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2} \\ \qquad\qquad \text{for bosonic antiparticle} \end{cases}, \tag{47}$$

which can be written explicitly using the step $\theta\left(\left|E_{nc}^{yk}\right|\right)$ function as:

$$E_{t-nc}^{yk-b} = \left|E_{nc}^{yk}\right|\theta\left(\left|E_{nc}^{yk}\right|\right) - \left|E_{nc}^{yk-s}\right|\theta\left(-\left|E_{nc}^{yk}\right|\right). \tag{48}$$

It is important to point out that because we have only used corrections of the first order of infinitesimal noncommutative parameters (Θ, τ, χ), perturbation theory cannot be used to find corrections of the second order $\left(\Theta^2, \tau^2, \chi^2\right)$.

4. The SE with IMYKP models in 3D-ENRQM symmetries

The main purpose of this section is to analyze the non-relativistic limit, in three-dimensional extended non-relativistic QM (3D-ENRQM) symmetries, for the IMYKP model. Two steps must be applied. The first one corresponds to the non-relativistic limit, in usual three-dimensional non-relativistic QM (3D-NRQM) symmetries. This is achieved by transferring the following values ($E_{nl} + \mu$ and $E_{nl} - \mu$), by (2μ and E_{nl}^{nr}), respectively. This step was studied by Parmar and Vinodkumar [1] as:

$$\begin{aligned} E_{nl}^{nr}(n, V_0, \alpha, D_e, r_e) = {} & \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 \\ & - \frac{\alpha^2}{2\mu}\left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5 + A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2}\right]^2, \end{aligned} \tag{49}$$

with

$$X_{nl}^{nr} = \frac{1}{2} + \sqrt{2\mu\,(A_5 - A_1) + (l + 1/2)^2}. \tag{50}$$

Now, under the non-relativistic limit, the relativistic expectation values $T_{(nlm)}^{yk-\mu}$ reduce to the new corresponding non-relativistic expectation values $R_{(nlm)}^{yk-\mu}$, $\left(\mu = \overline{1,6}\right)$ as:

$$R_{(nlm)}^{yk-1} = \frac{C_{nl}^{n2}}{\alpha}\frac{n!\,(n + X_{nl}^{nr} - 2)\,\Gamma\,(2\omega_{nl}^{nr})\,\Gamma\,(n+2X_{nl}^{nr} - 4)\,\Gamma\,(\omega_{nl}^{nr}+\mathbf{1})}{(n + D_{nl}^{nr}/2 - 2)\,\Gamma\,(n + 2\omega_{nl}^{nr} + 1)\,\Gamma\,(n+D_{nl}^{nr} - 4)}, \tag{51.1}$$

$$R_{(nlm)}^{yk-2} = \frac{C_{nl}^{n2}}{\alpha}\frac{n!\,(n + X_{nl}^{nr} - 3/2)\,\Gamma\,(2\omega_{nl}^{nr} + 2)\,\Gamma\,(n+2X_{nl}^{nr} - 3)\,\Gamma\,(2\omega_{nl}^{nr} + 3)}{(n + D_{nl}^{nr}/2 - 1/2)\,\Gamma\,(n + 2\omega_{nl} + 3)\,\Gamma\,(n+D_{nl}^{nr} - 1)}, \tag{51.2}$$

$$R_{(nlm)}^{yk-3} = \frac{C_{nl}^{n2}}{\alpha}\frac{n!\,(n + X_{nl}^{nr} - 2)\,\Gamma\,(2\omega_{nl} + 2)\,\Gamma\,(n+2X_{nl}^{nr} - 4)\,\Gamma\,(2\omega_{nl} + 3)}{(n + D_{nl}^{nr}/2 - 1)\,\Gamma\,(n + 2\omega_{nl}^{nr} + 3)\,\Gamma\,(n+D_{nl}^{nr} - 2)}, \tag{51.3}$$

$$R_{(nlm)}^{yk-4} = \frac{C_{nl}^{n2}}{\alpha}\frac{n!\,(n + X_{nl}^{nr} - 1)\,\Gamma\,(2\omega_{nl}^{nr} + 1)\,\Gamma\,(n+2X_{nl}^{nr} - 2)\,\Gamma\,(2\omega_{nl}^{nr} + 2)}{(n + D_{nl}^{nr}/2 - 1/2)\,\Gamma\,(n + 2\omega_{nl}^{nr}+2)\,\Gamma\,(n+D_{nl}^{nr} - 1)}, \tag{51.4}$$

$$R_{(nlm)}^{yk-5} = \frac{C_{nl}^{n2}}{\alpha}\frac{n!\,(n + X_{nl}^{nr} - 3/2)\,\Gamma\,(2\omega_{nl}^{nr} + 1)\,\Gamma\,(n+2X_{nl}^{nr} - 3)\,\Gamma\,(2\omega_{nl}^{nr} + 2)}{(n + D_{nl}^{nr}/2 - 1)\,\Gamma\,(n + 2\omega_{nl}^{nr} + 2)\,\Gamma\,(n+D_{nl}^{nr} - 2)}, \tag{51.5}$$

and

$$R_{(nlm)}^{yk-6}=\frac{C_{nl}^{n2}}{\alpha}\frac{n!\left(n+X_{nl}^{nr}-3/2\right)\Gamma\left(2\omega_{nl}^{nr}\right)\Gamma\left(n+2X_{nl}^{nr}-3\right)\Gamma\left(2\omega_{nl}^{nr}+1\right)}{\left(n+D_{nl}^{nr}/2-3/2\right)\Gamma\left(n+2\omega_{nl}^{nr}+1\right)\Gamma\left(n+D_{nl}^{nr}-3\right)}, \tag{51.6}$$

with

$$\begin{cases} D_{nl}^{nr}=2\omega_{nl}^{nr}+2X_{nl}^{nr}, \\ \omega_{nl}^{nr}=\sqrt{l\left(l+1\right)+2\mu A_5-\frac{2\mu A_4}{\alpha}-\frac{2\mu\left(E_{nl}^{nr}+A_3-D_e\right)}{\alpha^2}}, \\ C_{nl}^{nr}=\frac{n!\Gamma\left(2\omega_{nl}^{nr}+1\right)}{\Gamma\left(n+2\omega_{nl}^{nr}+1\right)}\sqrt{\frac{\alpha n!\left(2X_{nl}^{nr}+1\right)\Gamma\left(n+2\omega_{nl}^{nr}+2X_{nl}^{nr}+1\right)}{2^{2\omega_{nl}^{nr}+2X_{nl}^{nr}-1}\Gamma\left(2\omega_{nl}^{nr}+n+1\right)\Gamma\left(2X_{nl}^{nr}+n+1\right)}}. \end{cases} \tag{52}$$

while the relativistic factors $K_\alpha\ \left(\alpha=\overline{1,6}\right)$ in Eq. (31) are reduced to the corresponding non-relativistic factors $K_\alpha^{nr}\ \left(\alpha=\overline{1,6}\right)$ as follows:

$$\begin{cases} K_1^{nr}=-\alpha^4\left[A_5+\alpha^4 l\left(l+1\right)/2\mu\right] \\ K_2^{nr}=K_3^{nr}=-\alpha^4 A_1 \\ K_4^{nr}=K_5^{nr}=\alpha^3 A_2/2 \\ K_6^{nr}=-\alpha^3 A_4/2 \end{cases}. \tag{53}$$

As a direct consequence, the new non-relativistic improved energy E_{nc-nl}^{nr-yk} of the excited state $(n,l,m)^{th}$ in 3D-ENRQM symmetries under the IMYKP model equals the non-relativistic energy E_{nl}^{nr} in Eq. (49) under the MYKP model, as a main part, plus non-relativistic correction which is generated with the effect of deformation space-space, as the perturbed part, as follows:

$$\begin{aligned} E_{nc-nl}^{nr-yk}(n,V_0,\alpha,D_e,r_e,\Theta,\tau,\chi,j,l,s,m)&=\frac{\alpha^2 l\left(l+1\right)}{2\mu}+D_e-A_3+\alpha^2A_5 \\ &-\alpha A_4-\frac{\alpha^2}{2\mu}\left[\frac{l\left(l+1\right)\frac{2\mu}{\alpha^2}\left(\alpha^2\left(A_5+A_1\right)+\alpha\left(A_2-A_4\right)\right)}{2\left(n+X_{nl}^{nr}\right)}+\frac{n+X_{nl}^{nr}}{2}\right]^2 \\ &+\langle M\rangle_{(nlm)}^{yk-nr}\left(n,V_0,\alpha,D_e,r_e\right)\left(\left(\tau\aleph+\chi\Omega\right)m+\Theta F\left(j,l,s\right)\right), \end{aligned} \tag{54}$$

where $\langle M\rangle_{(nlm)}^{yk-nr}\left(n,V_0,\alpha,D_e,r_e\right)$ is the non-relativistic expectation values which determined from the projection relation:

$$\langle M\rangle_{(nlm)}^{yk-nr}\left(n,V_0,\alpha,D_e,r_e\right)=\sum_{\mu=1}^{6}K_\mu^{nr}R_{(nlm)}^{yk-\mu}\left(n,V_0,\alpha,D_e,r_e\right). \tag{55}$$

5. Study of important particular cases in 3D-ERQM and 3D-ENRQM symmetries

We will look at some specific examples involving the new bound state energy eigenvalues in Eqs. (46) and (54) in this section. By adjusting relevant parameters of the IMYKP model in 3D-ERQM and three-dimensional extended non-relativistic quantum mechanics (3D-ENRQM) symmetries, we could derive some specific potentials useful for other physical systems for much concern the specialist reaches. It should be noted that these special cases were treated within the framework of relativistic and non-relativistic quantum mechanics known in the literature in ref. [1], and we are now in the process of generalizing them to include extended relativistic and non-relativistic quantum mechanics symmetries.

(1) If we choose, $V_0=0$, we obtain improved modified Kratzer potential (IMKP), and $\alpha\to 0$, from Eqs. (46) and (54), we deduced eigenvalues correspond to IMKP for 3D-ERQM and 3D-ENRQM symmetries as [33]:

$$E_{nc}^{k}(n,D_e,r_e,\Theta,\tau,\chi,j,l,s,m)=E_{nl}^{r-k}+\left[\langle M\rangle_{(nlm)}^{k}\left(\left(\tau\aleph+\chi\Omega\right)m+\Theta F\left(j,l,s\right)\right)\right]^{1/2}, \tag{56}$$

and

$$\begin{aligned} E_{nc-nl}^{nr-k}(n,D_e,r_e,\Theta,\tau,\chi,j,l,s,m)&=D_e-\frac{2\mu D_e^2 r_e^2}{\left(n+1/2+\sqrt{2\mu D_e r_e^2+\left(l+1/2\right)^2}\right)^2} \\ &+\langle M\rangle_{(nlm)}^{nr-k}\left(\left(\tau\aleph+\chi\Omega\right)m+\Theta F\left(j,l,s\right)\right), \end{aligned} \tag{57}$$

where the relativistic eigenvalues E_{nl}^{k} in 3D-RQM symmetries are obtained from Refs. [1,2]:

$$E_{nl}^{k2} - M^2 = \left(E_{nl}^{k} + M\right) D_e - \frac{\left(E_{nl}^{k} + \mu\right) D_e^2 r_e^2}{\left(n + 1/2 + \sqrt{\left(E_{nl}^{k} + \mu\right) D_e^2 r_e^2 + (l+1/2)^2}\right)^2}, \tag{58}$$

while the new relativistic and non-relativistic expectations values $\langle M \rangle_{(nlm)}^{k}$ and $\langle M \rangle_{(nlm)}^{nr-k}$ can be determined from the following limits:

$$\begin{cases} \langle M \rangle_{(nlm)}^{k} = \lim\limits_{(V_0,\alpha)\to(0,0)} \langle M \rangle_{(nlm)}^{yk} \\ \langle M \rangle_{(nlm)}^{nr-k} = \lim\limits_{(V_0,\alpha)\to(0,0)} \langle M \rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{59}$$

(2) If we choose, $A_1 = A_2 = 0$ and $A_3 = -\lambda$, we obtain improved generalized Kratzer potential (IGKP), and $\alpha \to 0$, from Eqs. (46) and (54), we deduced eigenvalues correspond to IGKP for 3D-ERQM and 3D-ENRQM symmetries as:

$$E_{nc}^{gk}(n, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) = E_{nl}^{r-gk} + \left[\langle M \rangle_{(nlm)}^{gk} \left((\tau \aleph + \chi \Omega) m + \Theta F(j,l,s)\right)\right]^{1/2}, \tag{60}$$

and

$$E_{nc-nl}^{nr-gk}(n, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) = D_e + \lambda - \frac{2\mu D_e^2 r_e^2}{\left(n + 1/2 + \sqrt{2\mu D_e r_e^2 + (l+1/2)^2}\right)^2} + \langle M \rangle_{(nlm)}^{nr-gk} \left((\tau \aleph + \chi \Omega) m + \Theta F(j,l,s)\right), \tag{61}$$

where the relativistic eigenvalues E_{nl}^{gk} in 3D-RQM symmetries obtained form [1,100]

$$E_{nl}^{gk2} - M^2 = \left(E_{nl}^{gk} + M\right) (D_e + \lambda) - \frac{\left(E_{nl}^{gk} + \mu\right) D_e^2 r_e^2}{\left(n + 1/2 + \sqrt{\left(E_{nl}^{gk} + \mu\right) D_e^2 r_e^2 + (l+1/2)^2}\right)^2}, \tag{62}$$

while the new relativistic and non-relativistic expectations values $\langle M \rangle_{(nlm)}^{gk}$ and $\langle M \rangle_{(nlm)}^{nr-gk}$ can be determined from the following limits:

$$\begin{cases} \langle M \rangle_{(nlm)}^{gk} = \lim\limits_{(A_1,A_2,\alpha,A_3)\to(0,0,0,-\lambda)} \langle M \rangle_{(nlm)}^{yk} \\ \langle M \rangle_{(nlm)}^{nr-gk} = \lim\limits_{(A_1,A_2,\alpha,A_3)\to(0,0,0,-\lambda)} \langle M \rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{63}$$

(3) If we choose, $A_1 = A_2 = 0$ and $A_3 = -D_e$, we obtain improved Kratzer potential, and $\alpha \to 0$, from Eqs. (46) and (54), we deduced eigenvalues correspond to improved Kratzer potential for 3D-ERQM and 3D-ENRQM symmetries as:

$$E_{nc}^{kp}(n, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) = E_{nl}^{r-kp} + \left[\langle M \rangle_{(nlm)}^{kp} \left((\tau \aleph + \chi \Omega) m + \Theta F(j,l,s)\right)\right]^{1/2}, \tag{64}$$

and

$$E_{nc-nl}^{nr-kp}(n, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) = -\frac{2\mu D_e^2 r_e^2}{\left(n + 1/2 + \sqrt{2\mu D_e r_e^2 + (l+1/2)^2}\right)^2} + \langle M \rangle_{(nlm)}^{nr-kp} \left((\tau \aleph + \chi \Omega) m + \Theta F(j,l,s)\right), \tag{65}$$

where the relativistic eigenvalues E_{nl}^{kp} in 3D-RQM symmetries obtained form [1,101]:

$$E_{nl}^{kp2} - M^2 = -\frac{\left(E_{nl}^{kp} + \mu\right) D_e^2 r_e^2}{\left(n + 1/2 + \sqrt{\left(E_{nl}^{kp} + \mu\right) D_e^2 r_e^2 + (l+1/2)^2}\right)^2}, \tag{66}$$

while the new relativistic and non-relativistic expectations values $\langle M\rangle_{(nlm)}^{gk}$ and $\langle M\rangle_{(nlm)}^{nr-gk}$ can be determined from the following limits:

$$\begin{cases} \langle M\rangle_{(nlm)}^{kp} = \lim\limits_{(A_1,A_2,\alpha,A_3)\to(0,0,0,-D_e)} \langle M\rangle_{(nlm)}^{yk} \\ \langle M\rangle_{(nlm)}^{nr-gk} = \lim\limits_{(A_1,A_2,\alpha,A_3)\to(0,0,0,-D_e)} \langle M\rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{67}$$

(4) If we choose, $A_1 = A_3 = 0$ and $A_2 = -A$, we obtain improved modified Kratzer plus screened Coulomb potential (IMKSCP). From Eqs. (46) and (54), we deduced eigenvalues correspond to IMKSCP for 3D-ERQM and 3D-ENRQM symmetries as:

$$E_{nc}^{kc} = E_{nl}^{r-kc} + [\langle M\rangle_{(nlm)}^{kc}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2}, \tag{68}$$

and

$$E_{nc-nl}^{nr-kc} = \frac{\alpha^2 l(l+1)}{2\mu} + D_e + \alpha^2 D_e r_e^2 - 2\alpha D_e r_e - \alpha^2\left[\frac{2\mu\left(\alpha^2 D_e r_e^2 - \alpha\left(A + 2D_e r_e\right)\right) + l(l+1)}{2n+1+2\sqrt{2\mu D_e^2 r_e^2 + (l+1/2)^2}}\right]^2$$
$$+ \langle M\rangle_{(nlm)}^{nr-kc}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right), \tag{69}$$

where the relativistic eigenvalues E_{nl}^{kc} in 3D-RQM symmetries are obtained from Refs. [1,2]:

$$E_{nl}^{kc2} - M^2 = \left(E_{nl}^{kc} + \mu\right)\left(D_e + \alpha^2 D_e r_e^2 - 2\alpha D_e r_e\right) + \alpha^2 l(l+1)$$
$$- \alpha^2\left[\frac{\left(E_{nl}^{kc} + \mu\right)\left(\alpha^2 D_e r_e^2 - \alpha\left(A + D_e r_e\right)\right) + l(l+1)}{2n+1+2\sqrt{\left(E_{nl}^{kc} + \mu\right) D_e^2 r_e^2 + (l+1/2)^2}}\right]^2, \tag{70}$$

while the new relativistic and non-relativistic expectations values $\langle M\rangle_{(nlm)}^{kc}$ and $\langle M\rangle_{(nlm)}^{nr-kc}$ can be determined from the following limits:

$$\begin{cases} \langle M\rangle_{(nlm)}^{kc} = \lim\limits_{(A_1,A_3,A_2)\to(0,0,-A)} \langle M\rangle_{(nlm)}^{yk} \\ \langle M\rangle_{(nlm)}^{nr-kc} = \lim\limits_{(A_1,A_3,A_2)\to(0,0,-A)} \langle M\rangle_{(nlm)}^{nr-yk} \end{cases} \tag{71}$$

(5) If we choose, $A_1 = A_5 = 0$, $A_2 = B$ and $A_4 = C$, we obtain improved Hellmann potential (IHP). From Eqs. (46) and (54), we deduced eigenvalues correspond to IMKSCP for 3D-ERQM and 3D-ENRQM symmetries as [39]:

$$E_{nc}^{hp} = E_{nl}^{r-hp} + [\langle M\rangle_{(nlm)}^{hp}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2}, \tag{72}$$

and

$$E_{nc-nl}^{nr-hp} = \frac{\alpha^2 l(l+1)}{2\mu} - \alpha C - \frac{\alpha^2}{2\mu}\left[\frac{2\mu\,(B-C) + l(l+1)}{2n+2l+2} + \frac{n+l+2}{2}\right]^2$$
$$+ \langle M\rangle_{(nlm)}^{nr-hp}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right), \tag{73}$$

where the relativistic eigenvalues E_{nl}^{hp} in 3D-RQM symmetries obtained from Refs. [1,102]:

$$E_{nl}^{hp2} - M^2 = -\alpha C\left(E_{nl}^{hp} + \mu\right) + \alpha^2 l(l+1)$$
$$- \alpha^2 C^2\left[\frac{\left(E_{nl}^{hp} + \mu\right)(B-C) + l(l+1)}{2n+2l+2} + \frac{n+l+2}{2}\right]^2, \tag{74}$$

while the new relativistic and non-relativistic expectations values $\langle M\rangle_{(nlm)}^{hp}$ and $\langle M\rangle_{(nlm)}^{nr-hp}$ can be determined from the following limits:

$$\begin{cases} \langle M\rangle_{(nlm)}^{hp} = \lim\limits_{(A_1,A_5,A_2,A_4)\to(0,0,B,C)} \langle M\rangle_{(nlm)}^{yk} \\ \langle M\rangle_{(nlm)}^{nr-hp} = \lim\limits_{(A_1,A_5,A_2,A_4)\to(0,0,B,C)} \langle M\rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{75}$$

(6) If we choose, $A_1 = A_3 = D_e = 0$ and $A_2 = -A$, we obtain an improved screened Coulomb potential or improved Yukawa potential (IYP). From Eqs. (46) and (54), we deduced eigenvalues correspond to IMKSCP for 3D-ERQM and 3D-ENRQM symmetries as [38, 88, 103]:

$$E_{nc}^{yp} = E_{nl}^{r-yp} + [\langle M\rangle_{(nlm)}^{yp}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2}, \tag{76}$$

and

$$E_{nc-nl}^{nr-yp} = \frac{\alpha^2 l(l+1)}{2\mu} - \frac{\alpha^2}{2\mu}\left[\frac{l(l+1)-2\mu A\alpha^{-1}}{2n+2l+2} + \frac{n+l+2}{2}\right]^2 + \langle M\rangle_{(nlm)}^{nr-yp}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right). \tag{77}$$

where the relativistic eigenvalues E_{nl}^{yp} in 3D-RQM symmetries obtained from Ref. [104, 105]:

$$E_{nl}^{yp2} - M^2 = \alpha^2 l(l+1) - \alpha^2\left[\frac{-A\alpha^{-1}\left(E_{nl}^{yp}+\mu\right)+l(l+1)}{2n+2l+2} + \frac{n+l+2}{2}\right]^2. \tag{78}$$

while the new relativistic and non-relativistic expectations values $\langle M\rangle_{(nlm)}^{yp}$ and $\langle M\rangle_{(nlm)}^{nr-yp}$ can be determined from the following limits:

$$\begin{cases} \langle M\rangle_{(nlm)}^{yp} = \lim\limits_{(A_1,A_3,D_e,A_2)\to(0,0,0,-A)} \langle M\rangle_{(nlm)}^{yk} \\ \langle M\rangle_{(nlm)}^{nr-yp} = \lim\limits_{(A_1,A_3,D_e,A_2)\to(0,0,0,-A)} \langle M\rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{79}$$

(7) If we choose, $A_2 = A_4 = A_5 = 0$, $A_1 = A$ and $A_2 = D_2$, we obtain improved inversely square Yukawa potential. From Eqs. (46) and (54), we deduced eigenvalues correspond to improved inversely square Yukawa potential for 3D-ERQM and 3D-ENRQM symmetries as [37, 105]:

$$E_{nc}^{iyp} = E_{nl}^{r-iyp} + [\langle M\rangle_{(nlm)}^{iyp}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right)]^{1/2}, \tag{80}$$

and

$$\begin{aligned} E_{nc-nl}^{nr-iyp} = {} & \frac{\alpha^2 l(l+1)}{2\mu} - \frac{\alpha^2}{2\mu} \\ & \times\left[\frac{l(l+1)-2\mu A}{\left(n+1/2+\sqrt{\left(l+\frac{1}{2}\right)^2-2\mu A}\right)} + \frac{\left(n+1/2+\sqrt{\left(l+\frac{1}{2}\right)^2-2\mu A}\right)}{2}\right]^2 \\ & + \langle M\rangle_{(nlm)}^{nr-isy}\left((\tau\aleph + \chi\Omega)\, m + \Theta F\,(j,l,s)\right), \end{aligned} \tag{81}$$

where the relativistic eigenvalues E_{nl}^{yp} in 3D-RQM symmetries obtained from Ref. [5, 103]:

$$\begin{aligned} E_{nl}^{iyp2} - M^2 = {} & \alpha^2 l(l+1) - \alpha^2 \\ & \times\left[\frac{-A\left(E_{nl}^{iyp}+\mu\right)l(l+1)}{2\left(n+1/2+\sqrt{\left(l+\frac{1}{2}\right)^2-A\left(E_{nl}^{iyp}+\mu\right)}\right)^2} + \frac{\left(n+1/2+\sqrt{\left(l+\frac{1}{2}\right)^2-A\left(E_{nl}^{iyp}+\mu\right)}\right)}{2}\right]^2, \end{aligned} \tag{82}$$

while the new relativistic and non-relativistic expectations values $\langle M\rangle_{(nlm)}^{iyp}$ and $\langle M\rangle_{(nlm)}^{nr-iyp}$ can be determined from the following limits:

$$\begin{cases} \langle M\rangle_{(nlm)}^{iyp} = \lim\limits_{(A_2,A_4,A_5,A_1,A_2)\to(0,0,0,A,D_2)} \langle M\rangle_{(nlm)}^{yk} \\ \langle M\rangle_{(nlm)}^{nr-iyp} = \lim\limits_{(A_2,A_4,A_5,A_1,A_2)\to(0,0,0,A,D_2)} \langle M\rangle_{(nlm)}^{nr-yk} \end{cases}. \tag{83}$$

6. Spin-averaged mass spectra of HLM under IMKSCP models in 3D-ENRQM symmetries

The quark-antiquark interaction potentials, are spherically symmetrical and provide a good description of the heavy-light mesons (HLM) such as $c\bar{c}$ and $b\bar{b}$. This would give us a strong incentive to dedicate this section to the purpose to determine the modified spin-averaged mass spectra of the heavy quarkonium system such as $c\bar{c}$ and $b\bar{b}$ under the IMKSCP model interaction, in 3D-ENRQM symmetries, by using the following formula,

$$M_{nc-nl}^{yk-hlm} = 2m_q + \begin{cases} \frac{1}{3}\left(E_{nc-nl}^{nr-yk-u} + E_{nc-nl}^{nr-yk-m} + E_{nc-nl}^{nr-yk-l}\right) \text{ for spin-1} \\ E_{nc-nl}^{nr-yk} \text{ for spin-0} \end{cases}, \tag{84}$$

where $E_{nc-nl}^{nr-yk-u}$, $E_{nc-nl}^{nr-yk-m}$, $E_{nc-nl}^{nr-yk-l}$ and E_{nc-nl}^{nr-yk} are the new energy eigenvalues that correspond ($j = l+1$, $s = 1$), ($j = l$, $s = 1$), ($j = l-1$, $s = 1$) and ($j = l$, $s = 0$) under improved modified Yukawa-Kratzer potential model interactions in 3D-NRNCQM symmetries. We generalized the original formula [106–108]:

$$M_{nl}^{yk-hlm} = 2m_q + E_{nl}^{nr},$$

here m_q is the quark mass which equals the antiquark mass $m_{\overline{q}}$ in the case of charmonium $c\bar{c}$, bottomonium $b\bar{b}$ while E_{nl}^{nr} is the non-relativistic energy under modified Yukawa-Kratzer potential models which is determined by generalizing Eq. (49) in 3-dimensional space ($N = 3$). We need to replace the factor $F\,(j,l,s)$ with new generalized values as follows:

$$F\,(j,l,s) = \begin{cases} l/2 & \text{for} \quad (j = l+1, s = 1) \\ -1 & \text{for} \quad (j = l, s = 1) \\ (-2l-2)/2 & \text{for} \quad (j = l-1, s = 1) \\ 0 & \text{for} \quad (j = l, s = 0) \end{cases}. \tag{85}$$

Permuted us to obtain the new energy eigenvalues (E_{nc-nl}^{yk-u}, E_{nc-nl}^{yk-m} and E_{nc-nl}^{yk-l}) and E_{nc-nl}^{nr-yk} of the heavy quarkonium system such as $c\bar{c}$ and $b\bar{b}$ as:

1_ For the case: $j = l+1$and $s = 1$, E_{nl}^{nc-u} can be expressed by the following formula:

$$E_{nc-nl}^{yk-u} = \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 - \frac{\alpha^2}{2\mu}\left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5+A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2}\right]^2 + \langle M\rangle_{(nlm)}^{nr-yk}\left((\tau\aleph + \chi\Omega)\,m + \Theta\frac{l}{2}\right). \tag{86}$$

2_For the case: $j = l$ and $s = 1$, E_{nc-nl}^{yk-m} can be expressed by the following formula:

$$E_{nc-nl}^{yk-m} = \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 - \frac{\alpha^2}{2\mu}\left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5+A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2}\right]^2 + \langle M\rangle_{(nlm)}^{nr-yk}\,((\tau\aleph + \chi\Omega)\,m - \Theta). \tag{87}$$

3_For the case: $j = l-1$and $s = 1$, can be express on the new energy eigenvalue E_{nl}^{yk-l} by the following formula:

$$E_{nc-nl}^{yk-l} = \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 - \frac{\alpha^2}{2\mu}\left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5+A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2}\right]^2 + \langle M\rangle_{(nlm)}^{nr-yk}\,((\tau\aleph + \chi\Omega)\,m - \Theta\,(l+1)). \tag{88}$$

4_For the case $(j = l,\, s = 0)$, we can be express on the new energy eigenvalue E_{nc-nl}^{nr-yk} by the following formula:

$$E_{nc-nl}^{nr-yk} = \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 - \frac{\alpha^2}{2\mu}\left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5+A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2}\right]^2 + \langle M\rangle_{(nlm)}^{nr-yk}\,(\tau\aleph + \chi\Omega)\,m). \tag{89}$$

By substituting Eqs. (86), (87) and (88) into Eq. (89), the new mass spectrum of the meson systems in 3D-ENRQM symmetries under the IMYKP model for any arbitrary radial and angular momentum quantum numbers becomes:

$$M_{nc-nl}^{yk-hlm} = M_{nl}^{yk-hlm} + \begin{cases} \langle M \rangle_{(nlm)}^{nr-yk} \left((\tau\aleph + \chi\Omega)\, m - \frac{(4+l)}{6}\Theta \right) \text{ for spin-1} \\ \langle M \rangle_{(nlm)}^{nr-yk} \, (\tau\aleph + \chi\Omega)\, m) \text{ for spin-0} \end{cases} \tag{90}$$

Thus the spin-averaged mass spectra M_{nl}^{yk-hlm} of the heavy quarkonium system such as $c\bar{c}$ and $b\bar{b}$ under the modified Yukawa-Kratzer potential model interactions in usual 3D-NRQM symmetries:

$$\begin{aligned} M_{nl}^{yk-hlm} &= 2m_q + \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 \\ &- \frac{\alpha^2}{2\mu} \left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5 + A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2} \right]^2, \end{aligned} \tag{91}$$

is extended to include $\left(\delta M_{nc-nl}^{yk-hlm} = M_{nc-nl}^{yk-hlm} - M_{nl}^{yk-hlm}\right)$ in 3D-ENRQM symmetries:

$$\delta M_{nc-nl}^{yk-hlm} = \begin{cases} \langle M \rangle_{(nlm)}^{nr-yk} \left((\tau\aleph + \chi\Omega)\, m - \frac{(4+l)}{6}\Theta \right) \text{ for } s = 1 \\ \langle M \rangle_{(nlm)}^{nr-yk} \, (\tau\aleph + \chi\Omega)\, m) \text{for } s = 0 \end{cases}. \tag{92}$$

Which is sensitive to the atomic quantum numbers (n, j, l, s, m), potential depths (V_0, α), and the non-commutativity parameters (Θ, τ, χ) under the deformed properties of space-space. This allows us to realize logical physical limits:

$$\underset{(\Theta,\tau,\chi)\to(0,0,0)}{\text{Lim}} M_{nc-nl}^{yk-hlm} = M_{nl}^{yk-hlm}, \tag{93}$$

to be achieved.

6.1. Composite systems

In this section, and in the context of NC algebra, consider composite systems such as molecules comprised of $N = 2$ particles of mass m_α ($\alpha = 1, 2$). In the non-relativistic context, it is important to consider the characteristics of the system descriptions. It was discovered that those composite systems with various mass descriptions need various NC parameters [51, 52, 109]:

$$\left[\widehat{x}_\mu^{(s,h,i)}, \widehat{x}_\nu^{(s,h,i)} \right]_* = i\theta_{m\nu}^c, \tag{94}$$

where the non-commutativity parameter $\theta_{m\nu}^c$ is determined from $\sum_{\alpha=1}^{2} \mu_\alpha^2 \theta_{m\nu}^{(\alpha)}$, with $\mu_1 = m_1/(m_1 + m_2)$ and $\mu_2 = m_2/(m_1 + m_2)$, and $\theta_{m\nu}^{(\alpha)}$ is the new parameter of non-commutativity, corresponding to the mass particle of mass μ_n. Note that in the case of a physical system composed of two identical particles $\mu_1 = \mu_2$ such as the diatomic H_2 molecules under the effect of the improved modified Yukawa-Kratzer potential and a class of Yukawa potential, the parameter $\theta_{\mu\nu}^{(\alpha)} = \theta_{\mu\nu}$. Thus, the three parameters Θ, τ and χ which appear in Eq. (79) are changed to become as follows:

$$\Lambda^{c2} = \left(\sum_{\alpha=1}^{2} \mu_n^2 \Lambda_{12}^{(\alpha)} \right)^2 + \left(\sum_{\alpha=1}^{2} \mu_n^2 \Lambda_{23}^{(\alpha)} \right)^2 + \left(\sum_{\alpha=1}^{2} \mu_n^2 \Lambda_{13}^{(\alpha)} \right)^2, \tag{95}$$

with Λ^c can be present Θ^c, τ^c and χ^c. As mentioned above, in the case of a system of two particles with the same mass $\mu_1 = \mu_2$, we have $\theta_{\mu\nu}^{(n)} = \theta_{\mu\nu}$, $\tau_{\mu\nu}^{(n)} = \tau_{\mu\nu}$ and $\chi_{\mu\nu}^{(n)} = \chi_{\mu\nu}$. Finally, we can generalize our obtained non-relativistic total energy E_{nc-nr}^{sp} (n, V_0, α, D_e, r_e, Θ^c, τ^c, χ^c, j, l, s, m) under the improved modified Yukawa-Kratzer potential considering that composite systems with different masses are described with different new NC parameters in Eq. (94) for the HCl, CH, LiH and ScH diatomic molecules as:

$$\begin{aligned} E_{nc-nl}^{nr-yk}(n, V_0, \alpha, D_e, r_e, \Theta, \tau, \chi, j, l, s, m) &= \frac{\alpha^2 l\,(l+1)}{2\mu} + D_e - A_3 + \alpha^2 A_5 - \alpha A_4 \\ &- \frac{\alpha^2}{2\mu} \left[\frac{l\,(l+1)\,\frac{2\mu}{\alpha^2}\left(\alpha^2\,(A_5 + A_1) + \alpha\,(A_2 - A_4)\right)}{2\,(n + X_{nl}^{nr})} + \frac{n + X_{nl}^{nr}}{2} \right]^2 \\ &+ \langle M \rangle_{(nlm)}^{yk-nr} (n, V_0, \alpha, D_e, r_e) \left((\tau^c\aleph + \chi^c\Omega)\, m + \Theta^c F\,(j, l, s) \right). \end{aligned} \tag{96}$$

It is worth noting that for the three-simultaneous limits $(\Theta, \tau, \chi) \rightarrow (0,0,0)$ and $(\Theta^c, \tau^c, \chi^c) \rightarrow (0,0,0)$, we recover the energy equations for both the KGE and SE with modified Yukawa-Kratzer potential in 3D-RQM and 3D-NRQM symmetries, which are obtained in main Ref. [1].

7. Conclusions

In summary, this paper presents an approximate analytical solution of the 3-dimensional deformed Klein-Gordon and deformed equations in 3D-ERQM symmetries with the improved modified Yukawa-Kratzer potential model models using the parametric Bopp's shift method and standard perturbation theory. Under the deformed features of space-space, we found new bound-state energies that appear sensitive to quantum numbers (n, j, l, s, m), the mixed potential depths (V_0, D_e, r_e), the screening parameter's inverse α and the non-commutativity parameters (Θ,τ,χ). Moreover, the non-relativistic limit of the studied potential in 3D-ENRQM symmetries has been investigated. The modified spin-averaged mass spectra of heavy and heavy-light mesons such as $c\bar{c}$ and $b\bar{b}$ in both 3D-NRQM (commutative space) and 3D-ENRQM symmetries were determined by applying our results of the new non-relativistic energies that represent the binding energy between the quark and anti-quark. In the context of 3D-ERQM and 3D-ENRQM symmetry, we have treated certain significant particular instances that we hope will be valuable to the specialized researcher, such as the improved modified Kratzer potential, improved generalized Kratzer potential, improved Kratzer potential, improved modified Kratzer plus screened Coulomb potential, improved screened Coulomb potential and improved inversely square Yukawa potential. It is shown that the IMYKP model in a 3D-ERQM and 3D-ENRQM symmetry has similar behavior to the dynamics of a bosonic particle and bosonic antiparticle with equal scalar and vector potential for the modified Yukawa-Kratzer potential model models in a 3D-RQM symmetry (commutative space) influenced by the effect of a constant magnetic field, a self-rotational and a perturbed spin-orbit interaction. We recover the energy equations for the KGE and SE with modified Yukawa-Kratzer potential in 3D-RQM and 3D-NRQM symmetries, which were found in the main reference [1], for the three-simultaneous limits $(\Theta, \tau, \chi) \rightarrow (0,0,0)$ and $(\Theta^c, \tau^c, \chi^c) \rightarrow (0,0,0)$.

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