Wheeler-DeWitt canonical quantum gravity in hydrogenoid atoms according to de Broglie-Bohm and the geodesic hypothesis. Einstein's quantum field equation

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We explore the geodesic hypothesis of orbital trajectories of the electrons in hydrogenoid atoms, in the frame of de Broglie-Bohm quantum theory. It is intended that the space-time can be curved, at very short distances, by the effect of the joint action of the energy content of the atomic system and the contribution of the electric and quantum potentials. The geodesic hypothesis would explain the non-lose of energy in the electron orbital trajectories. So we explore a conception where particles and waves interact in a closed system: the waves guide the particles and the particles generate the space-time perturbation that acts as a wave, beyond the pilot-wave theory.
We establish the equivalence, in a local neighborhood, between the electron trajectory of an hydrogenoid atom in the Minkowskian space where the de Broglie-Bohm can be cast with its movement in a Lorentzian manifold, according to the concept of tangential metric. Through the geodesic condition and the invariance of the elemental length, we establish a relationship between some components of the metrics. But as the particles in microphysics do not follow the Einstein's field equation, we consider the 3+1 decomposition according to ADM and the quantization in the Wheeler De Witt theory and with a so-called quantum Einstein field equation, with a decomposition of spacetime into three-dimensional sheets of a spatial character. Then we derive from the moment-energy tensor a further equation between the components of the metrics. It opens the avenue to characterize the metric by an exact solution of the Einstein's field equations.

Keywords: De Broglie-Bohm; curvature of space time; metric tensor; general relativity; hydrogen-like atoms; electron trajectory; quantum potential; wave function; numerical methods; geodesics; Lorenzial geometry.

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1. Introduction

The relationship between the de Broglie-Bohm Quantum Theory and the Canonical Quantum Gravity seems a promising bridge between Quantum Mechanics and General Relativity for particles with low velocity.

The de Broglie-Bohm Quantum theory (dBB), also known as Pilot-Wave theory or Causal Quantum Mechanics Interpretation, appears in the period between 1923 to 1927 by the work of de Broglie [4,5]. In 1951 Bohm developed further the theory [1,2]. (See Ref. [12] for a treatise of the theory).

Our approach is based in the de Broglie-Bohm theory but goes beyond it because, instead of consider that the wave only guides the particle and the particle does not influence the wave, we consider a co-determination between wave and particle, where the deformation of the space-time by the electrical and quantum potentials plays an essential role. And then, the particle's trajectory in stationary systems is considered as a geodesic of this perturbed space-time.

The study of the hydrogenoid atom has the particularity of providing a framework with analytic functions, which facilitates the purpose of delving into the theory dBB. Furthermore, the stationary and symmetric character of its movement eases to go deeper in the essential physical issues of our purposals.

In earlier works [9,8,11] we considered the hydrogenoid atom according to de Broglie-Bohm theory, trying to investigate the conditions for the orbital electronic motion described in it to be interpretable as a geodesic of a curved spacetime, within the framework of a local limit situation that allows us to approach from the properties of a pseudo-Euclidean space to a Lorentzian manifold. Then we obtained a relationship between certain components of the Lorentzian metric, so a family of possible metrics that fullfill our purposes.

But to this geometrical consideration could not be added further physical results in the framework of the Classical General Relativity. In particular, the Einstein's Field Equations are not extensible to microphysics. To delve into this research, we come to quantize the equations of the General Relativity, which addresses quantum gravity.

Among the three branches that divide this theoretical line, started in the 1930s [17], our purpose finds support in the Canonical Quantum Gravity, initiated by Bergmann and Dirac in the 1950s.

It should be mentioned that there is a point of general interest with others lines of investigation in these approaches, especially with regard to cosmological aspects. An important argument consists of basing the quantum conception on a theory that, unlike the conventional or Copenhagen, does not require outside observers or ignore the isolated systems, so that it can spread throughout the universe.

An exponent of the de Broglie-Bohm integration in cosmological research is its use in the frame of Canonical Quantum Cosmology. Its introduction on the study of scalar po-

tentials open the door to it, [10] and this line extended mainly to the relationship with the inflation scenario by means of a scalar field as origin of space-time structure. To model this inflation phenomenon, the de Broglie-Bohm formalism has been used in order to restrict the possibles potentials in quantum cosmology [20-22].

Coming back to our approach, we refer to the 3+1 formalism in the form ADM [16] and its Wheeler-DeWitt quantization [6], which can lead to the so-called *Einstein's Quantum Field Equations.*

According to our knowledge, the first approximation to the definition of an Einstein's Quantum Field Equation is performed by F. Shojai and A. Shojai [19,20], based in the de Broglie-Bohm (dBB) theory and on algebraic considerations that have driven them to a quantum gravity model.

Later, the group in which among others we can cite the recently failed D. Dürr [7], S. Goldstein and N. Zanghì, and followed by other investigators as Tumulka [23] and Struyve [15] has elaborated a consistent theory that allows us to move in that direction.

In this study, we seek to apply the geodesic hypothesis of the hydrogenoid atom to this context of the Canonical Quantum Gravity, especially with relation to the definition of the energy-moment tensor, in order to raise the situation in which Einstein's Quantum Field Equations can be solved within the framework of our hypotheses, thus opening the way to identify the metric of space-time.

The outline of this article is as follows: firstly, we set out the fundamental lines of the approach to the geodesic hypothesis applied to hydrogenoid atoms in the framework of de Broglie-Bohm theory. Then we recall Einstein's Quantum Field Equations following their genesis from the ADM formulation of the General Relativity to their quantization by Wheeler-DeWitt theory and its particle adaptation. We then proceed to adapt our previous results to the theory particularly exposed to the energy-momentum tensor.

Finally, the results obtained are discussed.

2. The de Broglie-Bohm Theory and the hydrogenoid atoms

We start from the de Broglie-Bohm theory of hydrogenoid atoms. The Schrödinger Equation :

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi - V\Psi = 0, \tag{1}$$

when applied to a wave equation $\Psi = Re^{i\frac{S}{\hbar}}$,with R and S real functions of position and time, leads to:

$$\left(i\hbar\left[\partial_t R + R\frac{i}{\hbar}\partial_t S\right] + \frac{\hbar^2}{2m}\left[\nabla^2 R + 2\frac{i}{\hbar}\vec{\nabla}R\cdot\vec{\nabla}S + \frac{i}{\hbar}R\nabla^2 S - \frac{R}{\hbar^2}(\nabla S)^2\right] - VR\right)e^{i\frac{S}{\hbar}} = 0. \tag{2}$$

Both real and imaginary parts must be nul. For the real part we have:

$$-R\,\partial_t S + \frac{\hbar^2}{2m}\left(\nabla^2 R - \frac{R}{\hbar^2}(\nabla S)^2\right) - VR = 0,$$

that can be expressed as:

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0, \tag{3}$$

and with the de Broglie's equation that relate the linear momentum $\vec{p}$ with the gradient of the wave's phase, $\vec{p} = \vec{\nabla}S$ we arrive to the equation:

$$\partial_t S + \frac{p^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0. \tag{4}$$

For the imaginary part of the Eq. (2) we get:

$$i\left[\hbar\partial_t R + \frac{\hbar}{2m}\left(2\vec{\nabla}R\cdot\vec{\nabla}S + R\nabla^2 S\right)\right] = 0.$$

We multiply by $2R$ and get:

$$2R\partial_t R + \frac{1}{m}\left(2R\vec{\nabla}R\cdot\vec{\nabla}S + R^2\nabla^2 S\right) = 0,$$

that can be written as:

$$\partial_t R^2 + \frac{1}{m}\left(\vec{\nabla}R^2\cdot\vec{\nabla}S + R^2\nabla^2 S\right) = 0,$$

and taking into account the divergence of the product of a scalar (R^2) by a vector (∇S) the previous equation reads:

$$\partial_t R^2 + \nabla\cdot\left(\frac{R^2\vec{\nabla}S}{m}\right) = 0.$$

Now we introduce the density of probability $\mathcal{P}$ according with the principle of quantum equilibrium $\mathcal{P} = |\Psi|^2 = R^2$ and the linear moment that allows define the velocity as $\vec{v} = \vec{\nabla}S/m$ and we come to:

$$\partial_t\mathcal{P} + \nabla\cdot(\mathcal{P}\,\vec{v}) = 0.$$

This expression is interpreted as a continuity equation of the density of probability.

The Eq. (4) is interpreted as a Hamilton-Jacobi equation, with an additional potential to the classical one that is called quantum potential:

$$Q(\vec{r}) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}. \tag{5}$$

In the case of stationary states, $\partial_t S$ in Eq. (4) is the quantizied energy of the system corresponding to the level n given by $E_n = -mq_e^4/8\epsilon_0\hbar^2 n^2$. So, replacing values in Eqs. (4), (5) can be written in cylindrical coordinates:

$$Q = \frac{mq_e^4}{8\epsilon_0\hbar^2 n^2} - \frac{u^2\hbar^2}{2m\rho^2} + \frac{q_e^2}{4\pi\epsilon_0\sqrt{\rho^2+z^2}},$$

the force derived from the quantum potential $F_Q = -\nabla Q$ can be decomposed into the sum of two vectors: $-F_E$ which opposes the force derived from the electric potential and F_C, centripetal force which conditions the rotation of the electron in a circular orbit in a plane, which perpendicular through its center crosses the positive nucleus.

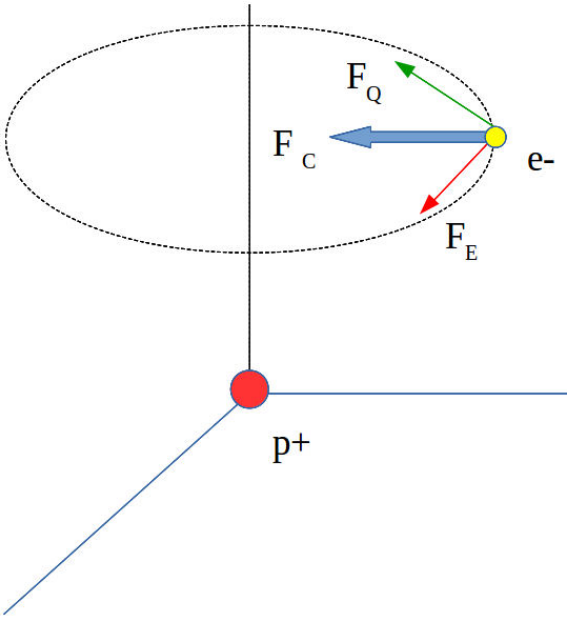

FIGURE 1. Forces acting on the hidrogen electron in the constant phase plane passing through the spin axis, according to dBB theory.

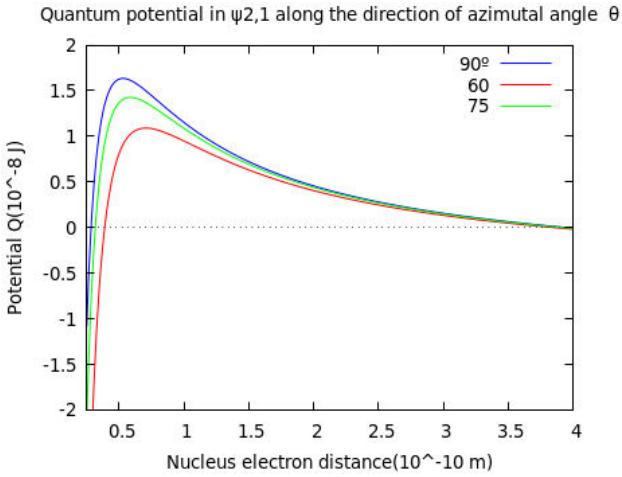

FIGURE 2. Quantum potential versus ρ, distance to the nucleus at constant angle in a hydrogenoid atom.

Figure 1 illustrates, in a hydrogenoid atom, the composition of the electrostatic force $-F_E$ and the quantum force F_Q, derived from the quantum potential, which are in a constant phase plane.

Figure 2 shows how the quantum potential varies in radial directions, depending on the azimuth angle θ. The strongly repulsive character of the quantum force is observed in small values of ρ, evidenced by the growing character of the Q graph, to become attractive for values greater than ρ.

A very convenient approach to describe the quantum system by the dBB theory is to define its Hamiltonian operator (generally starting from the classical equivalent) and to solve the Schrödinger equation. Then we obtain the eigen functions Ψ and from there the R and the S functions. The velocity can then be obtained from the gradient of S and the quantum potential from the quantum Hamilton-Jacobi equation or from its definition and the probability density as R^2.

3. Geodesic hypothesis on a Lorentzian manifold

In dBB theory, the electron's trajectory is described in an orthonormal-based pseudo-Euclidean space as a periodic circular motion governed by a centripetal force.

The geodesic hypothesis is to interpret this trajectory as a geodesic of a Riemannian (Lorentzian) manifold, which the electron would follow in the absence of forces and thus without loss of energy. Of course, we will stay in a simplified model without considering spin or small fluctuations of orbit [24].

3.1. Local correspondence between a Riemannian manifold and an Euclidean space.

In the de Broglie-Bohm theory, the wave guides the particle; so it is very important the structure of the space-time in neighborhood of the particle. Let's consider a hydrogen atom and in it the electron. We will suppose that it moves in a Lorentzial manifold $\mathcal{V}_4$, located at the point M_0 with coordinates y^α and endowed by a metric that will satisfy a condition that the movement of the electron will constitute a geodesic on it.

As we know, a Lorentzial manifold locally appears as a pseudo euclidian space $\mathcal{E}_4$. We will show that this tangent pseudo-euclidean space-time $T_{M_0}\mathcal{V}$ can admit the same metric tensor as the manifold $\mathcal{V}_4$ in this point, that is locally, and admit an image point m_0 in the pseudo-euclidean space-time, endowed with the same metric and signature.

Furthermore, we consider a reference system in the center of mass of the atom with ortogonal unit vectors and the same signature of the tangent pseudo euclidean space previously mentioned. In it, we can describe the dynamics of the electron in the frame of the dBB theory.

Then, we can establish a relationship between the tensors and invariants described in the tangent pseudo-Euclidean system and the reference system by the equations of tensor analysis. By this correspondence between these two pseudo-Euclidean spaces, we can relate *locally* the dynamics of the electron in the dBB theory and in the manifold.

The legitimation of a correspondence between a pseudo Euclidean space and a Riemann (Lorentzian) differential manifold, *at local scope, so in the neighborhood of the point*, rests on the concept of *first-order representation* corresponding to the *tangent Euclidean metric* [14].

Although the theory is general for any Riemann manifold and any dimension, we will refer here for the sake of brevity only to a Lorentzian manifold with dimension 4 and signature $(-+++)$.

Let $\mathcal{V}_4$ be a Lorentzian manifold with a metric tensor $g_{\mu\nu}$ in its point M_0 with coordinates $(y_0^\mu)^i$. Let also $\mathcal{E}_n$ be a

pseudo-Euclidean space of the same dimension and signature as the manifold, so the tangent space in M_0 endowed with a reference system $(m_0, \vec{e}_\mu)$ with the condition:

$$\vec{e}_\mu.\vec{e}_\nu = (g_{\mu\nu})_0. \tag{6}$$

We can put into correspondence the point M_0 of the manifold and a point m_0 of the tangent pseudo-Euclidean space, therefore with the same signature of the Lorentzial manifold. This fact makes not necessary the use of complex transformations like the Wick rotation, to relate both the manifold and the pseudo-euclidean space. Every point M of the neighborhood of M_0 can be mapped to a point m of the neighborhood of m_0, using second-degree functions $\Lambda_{(2)}$ of the difference of coordinates $y^\nu - y_0^\mu$ at the point of the manifold,

$$\overrightarrow{m_0 m} = [(y^\mu - y_0^\mu) + \Lambda^\mu_{(2)}(y^\nu - y_0^\nu)]\vec{e}_\mu. \tag{7}$$

From (7), passing to the limit, it follows that:

$$\left(\frac{\partial \vec{m}}{\partial y^\mu}\right)_0 = \vec{e}_\mu.$$

Thus, the point m in this pseudo-Euclidean space is defined by the coordinates of the manifold y^μ; the values $y^\mu - y_0^\mu$ act as curvilinear coordinates of the point m in the tangent space. It must be highlighted that the signatures of both the manifold and the Euclidean tangent space are the same.

Let us now consider $\bar{g}_{\mu\nu}$ the metric of the tangent space, defined by:

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dy^\mu dy^\nu.$$

Then for $y^\mu = y_0^\mu$ the pseudo-Euclidean and Lorentzian metrics have the same values, and both metrics are said to be *tangent* at this point.

$$g_{\mu\nu} = \bar{g}_{\mu\nu}.$$

The pseudo-Euclidean point m is called the image of the point M of the manifold and the relation between they is called *representation of first order*.

Now we must relate this metric and coordinates with those of the reference system at the atom's mass center. For our purpose, we need only the relations at local level, so between variables' derivates and differentials.

The tensor η has, in cilindrical coordinates, the following value:

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Concerning the tensor $g_{\mu\nu}$, since general cylindrical and axially space-time allows the following metric's structure [13]:

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & g_{20} \\ 0 & 0 & g_{33} & 0 \\ 0 & g_{02} & 0 & g_{00} \end{pmatrix}. \tag{8}$$

Furthermore, the elemental distance between two points of the neighborhood considered in the Lorentzial manifold and the pseudo- Euclidean space is the same:

$$\begin{aligned} \overline{m_0 m}^2 &= (g^-_{\mu\nu})_0 \, dy^\mu dy^\nu \\ &= (g_{\mu\nu})_0 \, dy^\mu dy^\nu = \overline{M_0 M}^2. \end{aligned} \tag{9}$$

Therefore, the elemental distance ds^2 is conserved between the Lorentzian manifold and the pseudo-Euclidean space. So, this correspondence of first order allows to identify the manifold metrics and the metrics of the pseudo-Euclidean tangent space and conserve the elemental distance ds^2 .

Now, we can use the coordinates $x^\alpha = y^\alpha - y_0^\alpha$ for the sake of brevity. The coordinates $x^{'\beta}$ will be used in the reference system. Both tensors and coordinates are related by the equation:

$$g_{\mu\nu} = \frac{\partial x^{'\alpha}}{\partial x^\mu} \frac{\partial x^{'\beta}}{\partial x^\nu} \eta_{\alpha\beta}.$$

The ds^2 as invariant will have the same value in both reference systems. And being $\eta_{\alpha\beta}$ the orthogonal pseudo-Euclidean metric of the cylindrical reference system and $x^{'\alpha}$ its coordinates, we can write this equivalence as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\alpha\beta} dx^{'\alpha} dx^{'\beta}.$$

For the left hand side of this equation, according with the orthogonal system of reference with the cylindrical metric we can write:

$$ds^2 = -d^2 x^{'0} + d^2 x^{'1} + (x^{'1})^2 d^2 x^{'2} + d^2 x^{'3},$$

and for the central member:

$$\begin{aligned} ds^2 = &-g_{00} d^2 x^0 + 2 g_{02} dx^0 dx^2 \\ &+ g_{11} d^2 x^1 + g_{22} d^2 x^2 + g_{33} d^2 x^3. \end{aligned} \tag{10}$$

Now, we can establish a relation between both tangent and reference systems. We can make the approximation of identify the differentials at both tangent and reference system, as we will detail later on. By equalling the elemental interval in the tangent space and the space in the reference system, runs:

$$2 g_{02} dx^0 dx^2 + g_{22} d^2 x^2 = (x^{'1})^2 d^2 x^{'2},$$

so

$$g_{22} = (x^{'1})^2 - \frac{2 g_{02}}{\dfrac{dx^2}{dx^0}}. \tag{11}$$

Let be the reference system where we represent the atomic system in the dBB theory, endowed with cylindrical

coordinates with origin in the mass center of the atomic system. Consider the motion of the electron around the proton in the pseudo-Euclidean space and absolute time as a Minkowskian spacetime with time equations:

$$x^{'1} = \rho_0, \qquad x^{'2} = \phi = \frac{u\hbar t}{m\rho_0^2},$$
$$x^{'3} = z, \qquad x^{'0} = ct. \tag{12}$$

Indeed, in the frame of the above-mentioned hypothesis we can establish that approximation:

- $dx^0 = dx^{'0} = cdt$, because we work with velocities little respect c.
- $dx^1 = dx^{'1} = 0$ because the electron has a circular trajectory in the dBB theory, so with constant radius.
- $dx^2 = dx^{'2} = \phi$ because symmetry considerations.
- $dx^3 = dx^{'3} = z$ because the trajectory is plane in $\mathcal{E}_3$.

So we can write dx^α instead of $dx^{'\alpha}$. Then we can establish the **quadri-velocity** as:

$$\frac{dx^1}{dt} = \dot{\rho} = 0, \qquad \frac{dx^2}{dt} = \dot{\phi} = \omega = \frac{u\hbar}{m\rho_0^2},$$
$$\frac{dx^3}{dt} = \dot{z} = 0, \qquad \frac{dx^0}{dt} = c, \tag{13}$$

Then the Eq. (11) simply runs:

$$g_{22} = \rho_0^2 - \frac{2cg_{02}}{\omega}. \tag{14}$$

We must now introduce a physical condition to go deeper in the previous equation: the quantum condition that all electrons that may belong to the same quantum state corresponding to the same atomic orbital must possess the same angular or kinetic moment. That is, it must be accomplished:

$$m\omega\rho_0^2 = u\hbar,$$

where $\hbar$ is the reduced Planck constant, u is the magnetic quantum number and ρ_0 *is the radius of the orbit, while* ρ *is the generic coordinate of spacetime*, which matches ρ_0 in the electron's trajectory. So ρ_0 is a *particle* coordinate and ρ a *field* coordinate in the particle's neighborhood. The following relation is thus established:

$$\omega = \frac{u\hbar}{m\rho_0^2},$$

and we introduce a constant to simplify notation: f, the reduced Compton length:

$$f = \frac{\hbar}{mc}, \tag{15}$$

with which we can express:

$$\omega = \frac{ufc}{\rho_0^2}. \tag{16}$$

By substituting ω in Eq. (14) we get:

$$g_{22} = \rho_0^2 \left(1 - \frac{2}{uf} g_{02}\right). \tag{17}$$

This relation is very important in our purpose; we will see that is a particular geodesic of the Lorentzian manifold. Obviously, we must have $g_{02} < (u\, f/2)$.

So, we have defined a coherent first-order correspondence that justifies our transit from the Lorentzial manifold to the pseudo- Euclidean reference space-time at the (local) differential level.

3.2. Geodesics in the Lorentzian manifold

We will impose now to the metric the condition that the electron trajectory be a geodesic of the manifold.

The geodesics of spacetime represented by the manifold will be given by taking as parameter the proper time that, since its velocity is of the order of $10^{-2}c$, we assimilate to the time of the inertial observer:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0, \tag{18}$$

where the $\Gamma^\mu_{\nu\lambda}$ are the Levi-Civita affine connectors. The above equation has the physical meaning of imposing a zero acceleration on the particle, calculated as a covariant derivative of the quadrivelocity. Quadrivector velocity experiences a *parallel transport* along the trajectory.

The "force" in the direction μ depends on the velocities in each direction ν and λ. The coefficients $\Gamma^\mu_{\nu\lambda}$ thus represent the influence of combinations of velocities on the force.

We have previously evaluated the dBB quadrivelocity in (13). So, replacing the velocities (13) to (18) we get:

$$\omega^2\Gamma^\mu_{22} + 2\omega c\Gamma^\mu_{02} + c^2\Gamma^\mu_{00} = 0. \tag{19}$$

Connectors can be expressed with respect to the metric tensor as follows:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\delta}(\partial_\nu g_{\lambda\delta} + \partial_\lambda g_{\nu\delta} - \partial_\delta g_{\nu\lambda}).$$

We now hypothesize that the components of the metric do not depend on the angle, $\phi = x^2$, the $z = x^3$ or the time $ct = x^0$. In the components of the metrics, we will only consider variation with respect to the $\rho = x^1$ coordinate as possible.

Replacing in the equation that expresses the connector depending on the metric g_{ij} we can write the non-zero connectors:

$$\Gamma^{\mu}_{22} = \frac{1}{2}g^{\mu\delta}(\partial_2 g_{2\delta} + \partial_2 g_{\delta 2} - \partial_\delta g_{22})$$
$$= -\frac{1}{2}g^{\mu\delta}\partial_\delta g_{22} = -\frac{1}{2}g^{\mu 1}\partial_1 g_{22}, \quad (20)$$

$$\Gamma^{\mu}_{02} = \frac{1}{2}g^{\mu\delta}(\partial_2 g_{0\delta} + \partial_0 g_{\delta 2} - \partial_\delta g_{02})$$
$$= -\frac{1}{2}g^{\mu\delta}\partial_\delta g_{02} = -\frac{1}{2}g^{\mu 1}\partial_1 g_{02}, \quad (21)$$

$$\Gamma^{\mu}_{00} = \frac{1}{2}g^{\mu\delta}(\partial_0 g_{0\delta} + \partial_0 g_{\delta 0} - \partial_\delta g_{00})$$
$$= -\frac{1}{2}g^{\mu\delta}\partial_\delta g_{00} = -\frac{1}{2}g^{\mu 1}\partial_1 g_{00}. \quad (22)$$

We now replace the calculated connectors at the four trajectory/geodesic Eqs. (19):

$$\omega^2 g^{\mu 1}\partial_1 g_{22} + 2\omega c g^{\mu 1}\partial_1 g_{02} + c^2 g^{\mu 1}\partial_1 g_{00} = 0, \quad (23)$$

contracted form of 4 equations with $\mu = 1, 2, 3, 0$.

To simplify these equations by dividing by $g^{\mu 1}$ we make sure that the values of this equation which interest us are not null. That is, we need to evaluate the contravariant metric tensor $g^{\mu\nu}$. We know that if $\alpha^{\mu\nu}$ is the adjoint of $g^{\mu\nu}$, we have:

$$g^{\mu\nu} = \frac{\alpha^{\mu\nu}}{g},$$

where g is the determinant of its matrix, $g_{\mu\nu}$.

As we have adopted the metric structure (8), we evaluate its determinant g, which will be $\neq 0$ (Lorentzian metric):

$$g = det(g_{\mu\nu}) = g_{11}g_{33}(g_{22}g_{00} - g_{02}^2) \neq 0,$$

and consequently tensor $g^{\mu\nu}$ has the form:

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{g_{11}} & 0 & 0 & 0 \\ 0 & \frac{g_{00}}{g_{22}g_{00}-g_{02}^2} & 0 & \frac{g_{02}}{g_{22}g_{00}-g_{02}^2)} \\ 0 & 0 & \frac{1}{g_{33}(g_{22}g_{00}-g_{02}^2)} & 0 \\ 0 & \frac{g_{02}}{g_{33}(g_{22}g_{00}-g_{02}^2)} & 0 & \frac{g_{02}}{g_{33}(g_{22}g_{00}-g_{02}^2)} \end{pmatrix}$$

We replace g^{11} in Eq. (23) and we are interested in $\mu = 1$ as the four equations are reduced to one:

$$\frac{\omega^2}{g_{11}}\partial_1 g_{22} + \frac{2c\omega}{g_{11}}\partial_1 g_{02} + \frac{c^2}{g_{11}}\partial_1 g_{00} = 0. \quad (24)$$

Simplify with $g_{11} \neq 0$, and replace partial derivatives by totals with respect to ρ, that we will indicate by $'$. We have:

$$\omega^2 g'_{22} + 2c\omega g'_{02} + c^2 g'_{00} = 0. \quad (25)$$

This equation is the condition of the geodesic equation if the metric only depends on ρ, but does not yet assume the condition of constant kinetic moment of quantum states. It is therefore a necessary condition, but not sufficient to implement the above hypothesis when considering the quantum condition that all electrons that may belong to the same quantum state corresponding to the atomic orbital: all of them must possess the same angular or kinetic moment. That is, it must be accomplished:

$$m\omega\rho_0^2 = u\hbar,$$

where ρ_0 is the radius of the orbit, while ρ is the generic coordinate of spacetime, which matches ρ_0 in the electron's trajectory. So ρ_0 is a *particle* coordinate and ρ a *field* coordinate in the proximity of the particle. As was previously established:

$$\omega = \frac{u\hbar}{m\rho_0^2},$$

and with the reduced Compton length f:

$$\omega = \frac{ufc}{\rho_0^2}. \quad (26)$$

Let us now enter the *kinetic moment constancy condition in the congruence of trajectories* represented by the equation obtained in the previous paragraph (25), obtaining:

$$u^2 f^2 g'_{22} + 2uf\rho_0^2 g'_{02} + \rho_0^4 g'_{00} = 0. \quad (27)$$

In our stationary and cylindrical symmetric case $g'_{00} = 0$ and we can write:

$$ufg'_{22} + 2\rho_0^2 g'_{02} = 0, \quad (28)$$

which constitutes a corollary of the geodesic condition.

3.3. dBB Geodesic condition

The general conditions for the metric of the previous paragraph allow us to state the following condition on the metric:

A Lorentzian metric g_{ij} compatible with de Broglie-Bohm theory for hydrogenoid atoms, of the form (8) and with the dependence of the components of the metric only on the cylindrical coordinate ρ, fulfills the following equation on the geodesic, in cylindrical coordinates:

$$u^2 f^2 g'_{22} + 2uf\rho_0^2 g'_{02} + \rho_0^4 g'_{00} = 0. \tag{29}$$

Since $g'_{00} = 0$ we can finally write:

$$g'_{02} = -\frac{uf}{2\rho_0^2} g'_{22}. \tag{30}$$

The metric of the manifold we are looking for must meet the above equation for each value of ρ_0 as well as the derivatives of the metric must be evaluated at this value.

We must remark again that the coordinate ρ pays there a twofold role: in the metrics component is a field variable, while with 0 subindex is a particle trajectory variable. But at local level, as ρ and ρ_0 have very close values, we can represent it by the same symbol.

This approximation is *locally* valid, as we want, because our study only has local scope. But it shall be taken into account in operations as derive or integrate against ρ, as ρ_0 must be considered then as a constant. Furthermore, it defines a geodesic condition for all trajectories of the electron for any value of ρ_0, *i.e.* for all corresponding orbital, defined by Ψ_{ijk}. We indeed achieve this through the equation:

$$g'_{02} = -\frac{uf}{2\rho^2} g'_{22}. \tag{31}$$

Therefore the integral of (30) is:

$$g_{02} = -\frac{uf}{2\rho_0^2} g_{22} + K. \tag{32}$$

The above equation is a condition on the metrics that meet the relativistic geodesic hypothesis with de Broglie-Bohm theory. Coming back to the condition (17) derived of the invariance of the elemental distance between the tangent space at the manifold and the reference space-time, expressed as:

$$g_{02} = \frac{uf}{2}\left(1 - \frac{g_{22}}{\rho_0^2}\right), \tag{33}$$

that we see that is a particular case of (32) for $K = uf/2$. So the Eq. (33) is, between the bundle of solutions given by (32), the particular that fulfill the both conditions of geodesic and invariance of the elemental distance.

4. Quantum gravity and quantum field equation of Einstein

The three-plus-one (ADM) formulation of relativity, separating time from spatial coordinates is cast, from its parametric form to the Hamiltonian to allow its quantization and thus to define the equivalent of the Einstein field equations in the quantum scope. Quantization takes place according to the Wheeler-DeWitt formulation.

In this section and the following we will adopt the metric (+ - - -) and units according to $\hbar = c = 1$, as the reference authors do [3,7,15]. Greek indices vary from 0 to 3 and Latins from 1 to 3.

4.1. 3+1 formalism, ADM

3+1 formalism was initiated by Darmois in the 1920s, followed by Lichnerowitz in the 1930s and by Mesmer and other authors in the 50s (ADM formalism). It involves decomposing the Lorentzian variety of spacetime into a family of 3-dimensional hypersurfaces each according to a time value. So, hypersurfaces have a spatial character: all points in each hypersurface have the same time coordinate value.

Let a coordinate point $(t, x^i(t))$ be in the Σ_t hypersurface that has $h_{ij}(t)$ three-dimensional metric and the coordinate point $(t + dt, x^i)$ in the Σ_{t+dt} hypersurface that has $h_{ij}(t + dt)$ metric. The *proper time* between the two events will not usually be dt, but Ndt, where N will be a function that is called *lapse function*. The space coordinates of the point of Σ_{t+dt} will not be just either x^i, but $x^i + N^i dt$ with N^i as functions that determine the displacement vector.

So we can put the differential line element, according to the Fig. 3:

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

or explicitly, downloading index with h_{ij} in the hypersurface:

$$ds^2 = (N^2 - N_i N^i)\, dt^2 - 2N_i dx^i dt - h_{ij} dx^i d^j. \tag{34}$$

In matrix form, the covariant and contravariant metric tensors can be expressed as follows:

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_i N^i & -N_i \\ -N_i & -h_{ij} \end{pmatrix},$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^i}{N^2} & \frac{N^i N^j}{N^2} - h^{ij} \end{pmatrix}.$$

Our purpose is to express the dynamic equations and the moment energy tensor based on canonical variables, so that

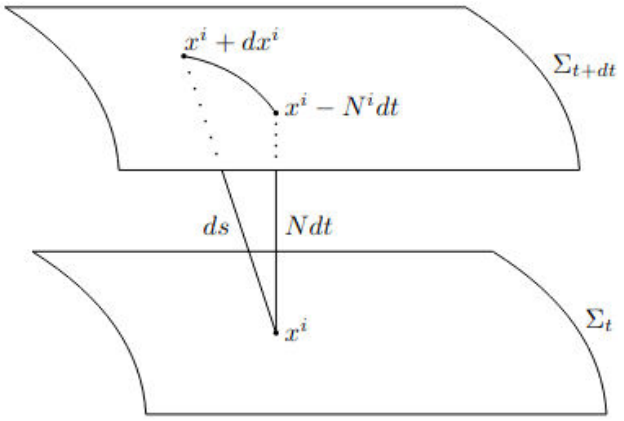

FIGURE 3. Dynamics of a point between two hypersurfaces separated by a time differential (S. Carlip, 2019).

we can quantize later. The Hamiltonian formulation can be derived from the action S, which is an additive magnitude; in our case:

$$S = S_{cl} + S_Q,$$

where S_{cl} is the classical action and S_Q the quantum action, which also allows additive solutions, particularly the energy-momentum tensor.

We consider non-relativistic particles [15] instead of scalar fields as is commonly done.

It must be noted that the phase variables are not only the coordinates of a particle but the functions that make up the metric matrix in the hypersurface, h_{ij}. To formulate the equations in Hamiltonian form, one must determine their conjugate moments from the Lagrangian.

4.2. Classical relativistic approach in the ADM theory

In a classical relativistic approach, the action of a particle S_{cl} (extension to more particles is immediate) can be considered as an addition of the Einstein-Hilbert action S_G and that of a non-relativistic particle in a gravitational field (relativistic case limit) S_M:

$$S_{cl} = S_G + S_M = -\frac{1}{\kappa}\int R\sqrt{-g}d^4x - m\int N dt + \frac{m}{2}\int \frac{h_{ij}}{N}(\dot{X}^i + N^i)(\dot{X}^j + N^j)dt,$$

where R is the scalar curvature of the manifold; N and N^i are functions of t; $\kappa = 16\pi G$.

The above expression allows the identification of total classical Lagrangians. Because:

$$S_{cl} = \int L_{cl}dt = \int \mathcal{L}_{cl}d^4x,$$

then identifying the integrand of the above equation:

$$L_{cl} = -\frac{1}{\kappa}\int R\sqrt{-g}d^3x - mN + \frac{m}{2}\frac{h_{ij}}{N}(\dot{X}^i + N^i)(\dot{X}^j + N^j).$$

Let be D_i the covariant derivative operator with respect to the h_{ij} metric of the hypersurface. This allows us to define its extrinsic curvature:

$$K_{ij} = \frac{1}{2N}\left(D_iN_j + D_jN_i - \dot{h_{ij}}\right),$$
$$K = K_{ij}h^{ij}.$$

Using the above equations, we can define the conjugate canonical moments:

$$P_i = \partial_{\dot{X}^i}L_{cl} = \frac{m}{N(X)}\left(\dot{X}_i + N_i(X)\right),$$
$$\pi_{ij} = \partial_{\dot{h}_{ij}}\mathcal{L}_{cl} = -\frac{1}{\kappa}\sqrt{h}(K^{ij} - Kh^{ij}).$$

so we can express the action based on conjugate variables:

$$S_{cl} = \int dt\int_\Sigma d^3x\dot{h}_{ij}\pi^{ij} + \int dt\dot{X}^iP^i - \int dtH.$$

To calculate the Hamiltonian we need to define the DeWitt metric:

$$G_{ijkl} = \frac{h_{ij}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}}{2\sqrt{h}}, \tag{35}$$

the potential density $\mathcal{V}$:

$$\mathcal{V} = \frac{\sqrt{h}(2\Lambda - R^{(3)})}{\kappa} \approx -\frac{\sqrt{h}R^{(3)}}{\kappa},$$

and the total potential as a function of time:

$$V(t) = \int_\Sigma d^3xN(t,\vec{x})\mathcal{V}(h(t,\vec{x})).$$

Then we can describe the equations of motion according to ADM theory:

$$\begin{aligned}
\dot{X}_i &= \frac{N(X)}{m}P^i - N^i(X),\\
\dot{P}_i &= -\partial_iN\left(m + \frac{1}{2m}P^kP_k\right)\\
&\quad - \frac{N}{2m}\partial_ih^{kl}P_kP_l + \partial_iN^kP_k,\\
\dot{h}_{ij} &= 2\kappa NG_{ijkl}\pi^{kl} + D_iN_j + D_jN_i,\\
\dot{\pi}^{ij} &= -N\kappa\,\partial_{h_{ij}}G_{mnkl}\pi^{mn}\pi^{kl}\\
&\quad - \delta_{h_{ij}}V + 2\delta_{h_{ij}}\int_\Sigma d^3yN_kD_l\pi^{kl}\\
&\quad + \delta(x-X)\frac{N(X)}{2m}P^iP^j,\\
\mathcal{H} &= 0,\\
\mathcal{H}_i &= 0.
\end{aligned}$$

From the Einstein's field equations so defined or directly from S_M

$$T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}.$$

can directly derive the equation of the energy-momentum tensor, the energy component of which is:

$$T^{00}_{cl} = \frac{m + \frac{1}{2m}P_k(t)P^k(t)}{N^2(t,x)\sqrt{h(t,x)}}\delta(\mathbf{x} - \mathbf{X(t)}).$$

The energy density describing T^{00}_{cl} becomes total energy if we integrate into a volume. Specifically, integrating over hypersurface Σ gets:

$$\int_\Sigma d^3xT^{00}_{cl} = \frac{m + \frac{1}{2m}P_k(t,X)P^k(t,X)}{N^2(t,X)\sqrt{h(t,X)}}.$$

The remaining components of the tensor $T^{\mu\nu}_{cl}$ are:

$$T^{0i}(t,x) = \frac{P^i(t)}{N(t,x)\sqrt{h(t,x)}} \times \delta(\mathbf{x} - \mathbf{X(t)}) - N^i(t,\mathbf{x})\, T^{00},$$

$$T^{ij}(t,x) = \frac{1}{\sqrt{h(t,\mathbf{x})}}\left[\frac{P^i(t)P^j(t)}{m} - \frac{P^i(t)N^j(t,\mathbf{x})) + P^j(t)N^i(t,\mathbf{x}))}{N(t,\mathbf{x}}\right] \times \delta(\mathbf{x} - \mathbf{X(t)}) + N^i(t,\mathbf{x})N^j(t,\mathbf{x})\, T^{00}(t,\mathbf{x}).$$

Until here, we have considered the classical relativistic approach. Additional physical concept of the quantum condition must be introduced, which we shall do following Wheeler-DeWitt theory.

4.3. Quantum approach:Wheeler-DeWitt Quantization

The quantization of the ADM theory is performed by providing Hamiltonian prime derivatives with respect to canonical variables and their moments, which can be replaced by the corresponding operators.

In Wheeler-DeWitt theory the "state" of the system is a functional of particle coordinates and 3-metric, $\Psi(X, h_{ij})$; the Wheeler-Dewitt equation holds:

$$\hat{\mathcal{H}}\Psi = \hat{\mathcal{H}}_G\Psi + \hat{\mathcal{H}}_M\Psi,$$

where

$$\hat{\mathcal{H}}_G = -\kappa G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \mathcal{V}(h,x),$$

$$\hat{\mathcal{H}}_M = \delta(x-X)(m - \frac{\nabla^2}{2m}),$$

with $\mathcal{V}(h,x)$ as effective potential density and G_{ijkl} the DeWitt metrics (35). Also, note the diffeomorphic constraint:

$$\hat{\mathcal{H}}_i\Psi = \hat{\mathcal{H}}_{Gi}\Psi + \hat{\mathcal{H}}_{Mi}\Psi,$$

with the operators (D_i covariant derivation):

$$\hat{\mathcal{H}}_{Gi} = 2h_{ik}D_j\frac{\delta}{\delta_{h_{jk}}},$$

$$\hat{\mathcal{H}}_{Mi} = \delta(x-X)\nabla_i.$$

Writing the wave equation in polar form, $\psi = |\psi|e^{iS}$ and entering the conditions:

$$P_i = \nabla_i S,$$

$$\pi^{ij} = \frac{\delta S}{\delta h_{ij}}.$$

We get the particle guide equations like:

$$\dot{X}^i = \frac{N(X)}{m}\nabla^i S - N^i(X),$$

$$\dot{h}_{ij} = 2\kappa N G_{ijkl}\frac{\delta S}{\delta h_{ij}} + D_iN_j + D_jN_i,$$

that could be used to modify the dBB guide equation. But we will not follow this approach here.

4.4. Einstein's quantum field equations. Energy-momentum tensor

The *additivity of the action* in the hamiltonian formulation allows us to consider the total action as the sum of the classical and quantum action:

$$S = S_{cl} + S_Q,$$

where

$$S_Q = -\int dtQ = -\int dt\int_\Sigma d^3x N\mathcal{Q}.$$

From the action, we can derive the Einstein's *Quantum* Field Equations, which will be applied to classical and quantum action. Varying action to the metric $g_{\mu\nu}$ we get, with the cosmological constant $\Lambda = 0$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = 8\pi G(T_{C\mu\nu} + T_{Q\mu\nu}).$$

The first member is covariantly conserved; so must happen in the second member. The energy moment tensor appears with two summands, the classic, $8\pi\, G\, T_{C\mu\nu}$ and the quantum $8\pi\, G\, T_{Q\mu\nu}$ in correspondence with both parts of Hamiltonian. We will focus first on the energy density component of the tensor, T^{00}, as the other components derive from it.

The quantum contribution to the momentum tensor is inferred from:

$$T^{\mu\nu}_Q = -\frac{2}{\sqrt{-g(x)}}\frac{\delta S_Q}{\delta g_{\mu\nu}(x)},$$

providing the following expressions. For the energy density of quantum origin:

$$T^{00}_Q = \frac{\mathcal{Q}(t,x)}{N^2(t,x)\sqrt{h(t,x)}}.$$

For temporary space components:

$$T^{0i}_Q = -N^i(tx)\, T^{00}_Q,$$

and for the fully spacial ones:

$$T^{ij}_Q = N^i(tx)\, N^j(tx)\, T^{00}_Q - \frac{2}{N(x,t)\sqrt{h(x,t)}} \times \frac{\delta}{\delta h_{ij}}\int_\Sigma d^3y N(y,t)\mathcal{Q}(y,t).$$

Thus, in both the temporal and fully space components of $T^{\mu\nu}$ is the expression of T^{00}_Q. We intend to calculate this component to refer to our geodesic hypothesis.

The quantum potential density $\mathcal{Q}(t,\mathbf{x})$ has a gravitational component and another massive one:

$$\mathcal{Q} = \mathcal{Q}_G + \mathcal{Q}_M,$$

the gravitational component is given by:

$$\mathcal{Q}_G = -\kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}.$$

For the mass quantum potential density we have:

$$\mathcal{Q}_M = -\delta(\mathbf{x}-\mathbf{X})\frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}.$$

That is:

$$T^{00}_Q = \frac{1}{N^2(t,x)\sqrt{h(t,x)}}\left(-\kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}} - \delta(\mathbf{x}-\mathbf{X})\frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}\right).$$

The general expression for a particle is:

$$T^{00} = \frac{1}{N^2(t,x)\sqrt{h(t,x)}}\left(m + \frac{1}{2m}P_k(t)P^k(t) - \frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}\right)\delta(\mathbf{x}-\mathbf{X(t)}) - \kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}. \tag{36}$$

The energy density allows the calculation of the total energy by integrating it into a hypersurface, which we assume is closed:

$$E_t = \int_\Sigma d^3x\, T^{00} = \frac{1}{N^2(t,x)\sqrt{h(t,x)}}\left(m + \frac{1}{2m}P_k(t)P^k(t) - \frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}\right) - \kappa\int_\Sigma d^3x G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}$$

For various particles, the formulation is additive. In particular, for the hydrogenoid atoms the mass of the first term of the parentheses should include the sum of the proton and electron.

5. Applying the quantum field equation to hydrogenoid atoms according to geodesic hypothesis

We are now going to apply the above theory to the geodesic hypothesis, which we have previously proposed [8,9,11] and in previous points of this work.

In two-particle systems such as the hydrogen atom, in which one particle is much heavier than the other one, the Schrödinger wave function coincides with that of Wheeler-DeWitt, [7], which allows us to use atomic orbitals as such.

In a stationary atomic system such as a hydrogenoid atom, the formulation of the above paragraph is additive as stated and the covariance of the energy-impulse tensor is ensured. Additionally, the spatial metric h_{ij} remains constant in time: $\dot{h_{ij}} = 0$.

The geodesic hypothesis links the derivatives of the components of the metric g'_{22} and g'_{24}. This relationship is a restriction to be taken into account.

5.1. Lapse and vector shift function

Consider a reference system with origin in the centre of mass of the hydrogenoid atom:

$$x^0 = t, \qquad x^1 = \rho, \qquad x^2 = \phi, \qquad x^3 = z.$$

We are looking to determine a metric tensor of the type:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & -h_{02} & 0 \\ 0 & -h_{11} & 0 & 0 \\ -h_{02} & 0 & -h_{22} & 0 \\ 0 & 0 & 0 & -h_{33} \end{pmatrix}. \tag{37}$$

As we have advanced (34), the metric $g_{\mu\nu}$ allows the following expression:

$$g_{\mu\nu}dx^\mu dx^\nu = (N^2 - N_kN^k)dt^2 - 2N_kdx^kdt - h_{ij}dx^idx^j.$$

Expressing the differential element ds^2 based on (37) and matching the above equation yields:

$$g_{00}dt^2 = (N^2 - N_kN^k)dt^2 - 2h_{02}dx^2dx^0 = -2N_kdx^kdt,$$

where the displacement vector has components:

$$N_1 = 0 \qquad N_2 = h_{02} \qquad N_3 = 0.$$

For calculating the N lapse function, we do:

$$g_{00}dt^2 = (N^2 - N_k N^k)dt^2$$
$$g_{00} = N^2 - N_k N^k$$

and considering our $g_{00} = 1$ hypothesis, we can write:

$$N^2 - N_2 N^2 = 1.$$

Note that to download N^i index we must use the three-dimensional metric matrix g^{ij} which is simply ($g_{ij}g^{ij} = \delta_{ij}$):

$$g^{ij} = \begin{pmatrix} -\frac{1}{h_{11}} & 0 & 0 \\ 0 & -\frac{1}{h_{22}} & 0 \\ 0 & 0 & -\frac{1}{h_{33}} \end{pmatrix},$$

where $h_{02}h^{02} = -(h_{02}^2/h_{22})1$ and we can replace in the above equation the value of N depending on the metric tensor:

$$N = \sqrt{1 - \frac{h_{02}^2}{h_{22}}}.$$

6. The Energy-moment tensor

6.1. Non-energy components of the energy impulsion tensor

The components of the energy-momentum tensor are given by a classical and a quantum part. Components with a time index will be:

$$T_{cl}^{0i} = \frac{P^i}{N\sqrt{h}} - N^i T_{cl}^{00}, \qquad T_Q^{0i} = -N^i T_Q^{00}.$$

Then for $i \neq 0$:

$$T^{0i} = \frac{P^i}{N\sqrt{h}} - N^i T^{00},$$

which gives the values:

$$T^{01} = 0,$$
$$T^{02} = \frac{1}{h_{11}h_{22}^{\frac{3}{2}}}\left(\frac{u}{\rho^2} + mh_{02}\right) - h_{02}T^{00},$$
$$T^{03} = 0.$$

Components with both spatial indexes:

$$T_{cl}^{ij} = \frac{1}{h}\left[\frac{P^iP^j}{m} - \frac{1}{N}\left(P^iN^j + P^jN^i\right)\right],$$
$$T_Q^{ij} = -N^iN^jT_Q^{00} - \frac{2}{N\sqrt{h}}\frac{\delta}{\delta h_{ij}}\int_\Sigma d^3yN\mathcal{Q}.$$

Then for $i \neq 0$ y $j \neq 0$:

$$T^{ij} = \frac{1}{h}\left[\frac{P^iP^j}{m} - \frac{1}{N}\left(P^iN^j + P^jN^i\right)\right] - N^iN^jT_Q^{00} - \frac{2}{N\sqrt{h}}\frac{\delta}{\delta h_{ij}}\int_\Sigma d^3yN.$$

The component with both temporal indexes is associated with the energy density of the system and is considerated separately.

6.2. The density of energy component of the energy-moment tensor

In the hydrogenoid atom we have two particles and motion is constant in time. For two particles the whole process is additive and one must count in the expression of the energy of the mass continent, the two masses $M = m_p + m$, being m_p the mass of the proton and m the mas of the electron. On the other hand, we can replace the value of $(N)^2$ and the function h is the value of the determinant of h_{ij}, product of the three diagonal components of the spatial metric; assuming in our case $g_{33} = g_{11}$ in a tetradimensional matrix, we can write:

$$\frac{1}{\sqrt{h}} = \frac{1}{h_{11}\sqrt{h_{22}}}.$$

The Eq. (36) becomes:

$$T^{00} = \frac{\sqrt{h_{22}}}{(h_{22} - h_{02}^2)h_{11}}\left[\left(M + \frac{1}{2m}P_kP^k - \frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}\right) \times \delta(\mathbf{x} - \mathbf{X(t)}) - \kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}\right].$$

The interior of the parentheses contains four terms. The first three have energy character multiplied by factor $\delta(\mathbf{x} - \mathbf{X(t)})$, which has the value 1 when the general coordinate x coincides with a value of the particle's trajectory and in the nucleous position.

The total energy value E_T can then be calculated as a hypersurface Σ containing both the electron and the nucleus. Then, we get:

$$E_T = \int d^3xT^{00} = \frac{\sqrt{h_{22}}}{(h_{22} - h_{02}^2)h_{11}} \times \left[\left(M + \frac{1}{2m}P_kP^k - \frac{\hbar^2\nabla^2|\Psi|}{2m|\Psi|}\right) - \kappa\int_\Sigma d^3xG_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}\right].$$

6.2.1. *Kinetical term*

The term $(1/2m)P_kP^k$ is much lower in value than the mass term M; it contains only the mass of the electron and the velocity is of the order of $0,1 \cdot c$; that is about 10^{-5} times less than the mass term. On the other hand, linear moments are time-independent. Explicitly putting the coordinate dependency on the trajectory remains:

$$\frac{1}{2m}P_k(\rho)P^k(\rho).$$

Notice that the moment P_k is given by:

$$P_k(\rho) = \partial_{\dot{X}^i} L_{cl} = \frac{m}{N}\left(\dot{X}_k + N_k\right),$$

and $P^k = P_k/g_{kk}$. On the other hand, $\dot{X}_2 = \omega = u\rho^2$. Therefore, this term keeps:

$$\frac{1}{2m}P_k(\rho)P^k(\rho) = \frac{1}{2m}\left(-\frac{1}{h_{22}}\right)\frac{m^2}{N^2}P_2(\rho)P_2(\rho) = \frac{m}{2(h_{02}^2 - h_{22})}\left(\frac{u}{m\rho^2} + h_{02}\right)^2.$$

6.2.2. *dBB quantum potential term*

The $\hbar^2\nabla^2|\Psi|/2m|\Psi|$ term is the quantum potential of dBB theory, which we have evaluated for the orbital Ψ_{211}, of hydrogenoid atoms [11].

In cylindrical coordinates and in the above units it has the following expression: [11]

$$Q = E_n - \frac{u^2}{2m\rho^2} + \frac{q_e^2}{4\pi\epsilon_0\sqrt{\rho^2+z^2}}.$$

In atomic systems, the quantum potential is closely linked to the energy deficit of the atom, caused by its formation from the component particles. In our case, the energy level, corresponding to ionisation energy.

6.2.3. *Geodesic condition*

To the aforementioned constraints must be added the geodesic condition (17), that in the formulation of this section it should be expressed:

$$h_{02} = \frac{uf}{2}\left(1 - \frac{h_{22}}{\rho^2}\right). \tag{38}$$

It should be remarked that $h_{ij} = g_{ij}$.

6.2.4. *Gravitational quantum potential term*

Computing the term

$$-\kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}}, \tag{39}$$

and taking into account the diagonal character of h_{ij} and the definition of G_{ijkl} (35), then (39) is expressed as follows:

$$-\kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}} = -\frac{\kappa}{2\sqrt{h}|\Psi|} \times \left[h_{ii}^2\frac{\delta^2|\Psi|}{\delta h_{ii}\delta h_{ii}} - 2h_{ii}h_{jj}\frac{\delta^2|\Psi|}{\delta h_{ii}\delta h_{jj}}\right],$$

with $ii \neq jj$ in the second term. Considering that the functional derivation is expressible as the partial and substituing the general coordinates of the space for those of the trajectory and the assumed equality of h_{11} and h_{33}, we can write the term like this:

$$-\kappa G_{ijkl}\frac{1}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}} = \frac{\kappa\sqrt{h_{22}}}{|\Psi|} \times \left[2\partial^2_{h_{11}h_{22}}|\Psi| - \frac{h_{22}}{2h_{11}}\partial^2_{h_{22}h_{22}}|\Psi|\right].$$

This term is affected by the coefficient $\kappa \approx 10^{-30}$. We can assume that it has a negligible value with respect to the rest.

6.2.5. *Final expression*

Putting all terms together, without the gravitational term, T^{00} remains:

$$T^{00} = \frac{\sqrt{h_{22}}}{(h_{22}-h_{02}^2)h_{11}}\left(M + \frac{m\left(\frac{u}{m\rho^2}+h_{02}\right)^2}{2(h_{02}^2-h_{22})} - E_n + \frac{u^2}{2m\rho^2} - \frac{q_e^2}{4\pi\epsilon_0\sqrt{\rho^2+z^2}}\right)\delta(\mathbf{x}-\mathbf{X}). \tag{40}$$

7. New conditions on the metric tensor

We can add a new condition for the metric tensor, concerning also to the component g_{11}. In our treatment of the hydrogenoid atom we can come back to the expression of T^{00} (40), that allows the calculation of the total integral energy on a hypersurface. The $\delta(x-X)$ in the mentioned reference indicates that the integral can be done in a very reduced neighborhood of ρ as the energy of the system is reduced to the nucleus and the electron. This energy must be, for low particle momentum, simply $M - En$. Therefore, we get:

$$E_T = M - E_n = \int_\Sigma dx^3 T^{00} = \frac{\sqrt{g_{22}}}{(g_{22}-g_{02}^2)g_{11}}\left[M + \frac{m\left(\frac{u}{m\rho^2}+g_{02}\right)^2}{2(g_{02}^2-g_{22})} - E_n + \frac{u^2}{2m\rho^2} - \frac{q_e^2}{4\pi\epsilon_0\sqrt{\rho^2+z^2}}\right],$$

allowing g_{11} to be defined as a function of g_{22} and g_{02}, previously defined.

$$g_{11} = \frac{\sqrt{g_{22}}}{(g_{22} - g_{02}^2)(M - E_n)} \left[M + \frac{m\left(\frac{u}{m\rho^2} + g_{02}\right)^2}{2(g_{02}^2 - g_{22})} - E_n + \frac{u^2}{2m\rho^2} - \frac{q_e^2}{4\pi\epsilon_0\sqrt{\rho^2 + z^2}} \right].$$

Finally, we would indicate that a direct consequence of the quantum Einstein equation is that the total energy - momentum tensor is covariantly conserved. Therefore, its divergence must be null. That will supply a further relation within the metric components. These items and their implications will be treated in future works.

8. Summary, discussion and future projection

The geodesic hypothesis of dBB theory for hydrogenoid atoms, advanced by us in earlier works [8,9,11], is to suppose that the trajectory of the electron described by dBB theory in an pseudo-Euclidean space can be interpreted as a geodesic on a Lorentzian manifold, *i.e.* a curved spacetime, therefore without energy emission. The curvature of the space-time should be produced by the *joint action of the quantum and the electromagnetical fields*; the field contribution from the mass atraction could be ignored.

In this work, we established the relations between the pseudo-Euclidean de Broglie-Bohm representation of the electron trajectory and this trajectory in a Lorentzian manifold. The mathematical foundation of these equivalences has been based on the concept of first-order representation and tangent metric at *local* neighborhood. We defined the condition for geodesics that requires the de Broglie-Bohm theory and the conservation of the elemental distance in the manifold, its tangent space and the space-time of reference. We find out that in the bundle of geodesics that accomplishes the geodesic condition, one fulfills the condition of elemental distance conservation, so the founded condition between two elements of the metrics has a firm soil. The conclusion of it is that we are able to find geometries of the Lorentzial manifold where our hypothesys are coherent.

To progress in the physical significance, we go to the Wheeler-de Witt formulation of the Einstein's quantum field equations, applied for particles. We conserve in it the values of two components of the metrics, in order to prevail the geodesic character of the trajectory and the invariance of the elemental distance, and we focus on the energy conservation. With that, we arrive to a further relation between the metric's components that finally relate all the independent components. Furthermore, an additional relationship can be derived from the null divergence of the energy-moment tensor.

It all opens the way to look for an appropriate metric that, being a solution of the Einstein's quantum field equations, could be adapted to the relations that we expressed, concerning the geodesic trajectory and the conservation of the elemental distance. The results obtained make it possible to see their continuation in subsequent research.

We must once again remark that our conception goes beyond the de Broglie Bohm or pilot wave. In dBB theory, the wave guides the particle, but the particle does not play any role of determination in the wave. Its ontology have two differentiated elements: wave, as a real field, and particle, that is guided by the field. In our conception, still we have two elements, particle and fields, represented by a perturbation of the space time that surrounds the particle; but this perturbation is created by the particles that form the quantum system. We have, then, an active and a passive part if we can speak so, that, together, form a closed system and that are in mutual interaction.

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