An exact treatment of localization of electromagnetic plus separable potential

A. K. Behera
Department of Physics, National Institute of Science and Technology, Berhampur-761008, Odisha, India.

B. Swain and U. Laha
Department of Physics, National Institute of Technology, Jamshedpur-831014, Jharkhand, India.

J. Bhoi
Department of Physics, Veer Surendra Sai University of Technology, Burla-768018, Odisha, India.

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Equivalent local potential with energy-momentum dependence is developed for the combined interaction of Hulthén modified Yamaguchi potential by developing its exact Jost solution. The generated local potentials are applied to compute scattering phase parameters for (p-p) and (p-d) systems through the phase function method (PFM). The same are also calculated for the nonlocal potential from the expression of the Fredholm determinant. Our obtained data for both the energy-momentum dependent local and the pure nonlocal interactions are in reasonable agreement with the standard data. Reasonable correspondence between the results for the phase equivalent and the nonlocal potentials indicate that our equivalent local analysis is in proper order.

Keywords: Hulthén plus Yamaguchi potential; phase equivalent potential; phase parameters; (p-p) and (p-d) systems.

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1. Introduction

In optical model calculation the pure nonlocal interactions are replaced by its equivalent energy-dependent local potentials. These make a comparison between the features of nonlocal potentials and the well-known phenomenology of the local interactions. Such studies are concerned with the comparison of the phase parameters or equivalently the transition matrix elements of both the potentials. Also, many groups [1–8] have studied the supersymmetric aspects of the N-N scattering in the context of the one pion exchange approximation and the transformation of the deep potential into a shallow potential with a repulsive core. Nucleus-nucleus interactions are frequently discussed in nuclear physics using both shallow and deep potentials. Baye [3] has exploited Supersymmetric Quantum Mechanics (SQM) to derive shallow nucleus-nucleus potentials that are phase equivalent to deep ones. Amado [4] applied Baye's method to the partial wave Coulomb problem and obtained a new potential that is phase identical to the Coulomb potential. Amado's analytical approach to the problem, in contrast to Baye's treatment, sheds light on the function of SQM in obtaining a family of phase equivalent potentials. Khare and Sukhatme [5] and Talukdar *et al*. [6] have also attempted to extend Amado's work in order to obtain a parametric relationship between several potentials. Using supersymmetric algebra, Majumder *et al*. [8] recently produced phase-equivalent potentials to Manning-Rosen ones in which the created interactions have a weak energy dependence and nearly the same shape. The single channel (elastic) scattering inside the core potential mode is the subject of this text.

In past, several groups [9–16] constructed phase equivalent local potentials to a nonlocal one of nuclear origin to understand the nature of interactions. Coz *et al*. [9] proposed a generalization of the method to have a momentum-dependent local potential by exploiting two linearly independent solutions of the Schrödinger equation with nonlocal interaction. Afterwards, Arnold and MacKellar [10] judiciously applied the method of Ref. [1] with nonlocal separable two-nucleon interactions to fit $^1S_0$ phase shifts. Following the approach of Ref. [10], in the recent past, we have constructed a local potential equivalent to a local plus a nonlocal interaction and examine the merit of our method through some model calculations namely the nucleon-nucleon and alpha-nucleon scattering [17]. In Ref. [17] we have added the electromagnetic part externally to the energy-dependent nuclear potential to compute charged hadron scattering phase parameters. However, from the perspective of additive interaction like p-p, α-p, p-d, α-^{3}H, α-^{3}He etc., the scenario of the scattering theory is different from the pure nuclear interaction because of the combined effect of nuclear as well as electromagnetic potential. For electromagnetic plus a nonlocal separable nuclear potential the situation can be treated exactly, while for a local nuclear potential the problem is handled through some approximation techniques [18,19]. For charged hadron systems the nonlocal Schrödinger equation admits exact analytical solution. As the electromagnetic interaction is very significant in the low energy range, it is quite obvious to include this interaction explicitly in such problems. The present article is an attempt in this direction.

Here we propose to construct energy dependent local potential for the Hulthén [20] modified Yamaguchi separable [21] interaction using the prescription of Ref. [10]. For the purpose of establishing an energy-momentum dependent local potential, that is phase equivalent to a local plus a nonlocal interaction, the problem is studied in the coordinate representation. Only the s-wave case is considered here. Separable nonlocal interactions have immense importance in various branches of physics. In nuclear scattering the effect of recoil of the target due to particle exchange should be taken care of properly and as a consequence the interaction is represented by a nonlocal potential $V(r,s)$. In addition to the phenomenological separable nuclear interaction, the short ranged electromagnetic interaction is represented by a screened Coulomb, the Hulthén potential [20]. The purpose of the present text is to study hadron-hadron systems with an energy-dependent local potential with rigorous inclusion of the electromagnetic effect. As a primary requisite for the localisation process, we need any convenient pair of linearly independent solutions to the effective nonlocal-like interaction for the development of equivalent local potential. The next section is directed towards the derivation of the irregular solutions for the Hulthén modified Yamaguchi potential and method of localisation. The constructed local potentials are applied to compute scattering phase shifts of (p-p) and (p-d) systems through the Phase Function Method (PFM) in Sec. 3. Finally, we conclude in Sec. 4.

2. Construction of an equivalent local potential

2.1. Irregular solution

The wave equation with irregular boundary condition for Hulthén [20] plus Yamaguchi potential [21] is written as

$$\left\{\frac{d^2}{dr^2}+k^2-V_0\frac{e^{-r/a}}{1-\,e^{-r/a}}\right\}f(k,r) = \lambda e^{-\alpha\, r}d^{(I)}(k)f(k,r), \tag{1}$$

where

$$d^{(I)}(k)=\int_0^\infty ds\, e^{-\alpha\, s}f(k,s). \tag{2}$$

Here V_0 and a are termed as the strength and the inverse range of the atomic Hulthén potential [20] whereas the quantities λ and α are the parameters of the Yamaguchi potential [21]. Applying Green's function technique, the irregular or the Jost solution $f(k,\,r)$ for the concerned combined potential reads as

$$f(k,r)=f_H(k,r)+\lambda d^{(I)}(k)\int_r^\infty e^{-\alpha s}G_H^{(I)}(r,s)ds, \tag{3}$$

with the irregular Hulthén Green's function [22, 23]

$$G_H^{(I)}(r,s)=\frac{1}{f_H\,(k)}[\phi_H(k,s)f_H(k,r) - \phi_H(k,r)f_H(k,s)]. \tag{4}$$

Here $\varphi_H(k,r)$, $f_H(k,r)$ and $f_H(k)$ are the s-wave regular, irregular/Jost solution and Jost function of the same potential under consideration, respectively. These are given by [24]

$$\varphi_H(k,r)=ae^{ikr}\left(1-e^{-r/a}\right) \times {}_2F_1(1+A,1+B;2;1-e^{-r/a}), \tag{5}$$

$$f_H(k,r)=e^{ikr}\,{}_2F_1\,(A\,,\,B;\,C\,;\,e^{-r/a}), \tag{6}$$

and

$$f_H(k)=\frac{\Gamma(C)}{\Gamma(1+A)\,\Gamma(1+B)}, \tag{7}$$

where $A=-iak+ia\sqrt{(k^2+V_0^2)}$, $B=-iak-ia\sqrt{(k^2+V_0^2)}$ and $C=1-2iak$.

From Eq. (3) one can easily evaluate the factor $d^{(I)}(k)$ as

$$d^{(I)}(k)=\frac{1}{D^{(I)}(k)}\int_0^\infty e^{-\alpha\, r}f_H(k,r)\,dr, \tag{8}$$

with $D^{(I)}(k)$, the Fredholm determinant reads as

$$D^{(I)}(k)=1-\lambda\int_0^\infty\int_r^\infty e^{-\alpha r}\,e^{-\alpha s}G_H^{(I)}(r,s)\,dr\,ds. \tag{9}$$

To deduce the expression for required irregular solution from Eq. (3), one has to first solve the indefinite integration $G_H^{(I)}(r,\alpha)=\int_r^\infty G_H^{(I)}(r,s)\,e^{-\alpha s}ds$. In such case we can write the quantity $G_H^{(I)}(r,\alpha)$ as

$$G_H^{(I)}(r,\alpha)=\int_0^\infty G_H^{(I)}(r,s)\,e^{-\alpha s}ds - \int_0^r G_H^{(I)}(r,s)\,e^{-\alpha s}ds. \tag{10}$$

Substituting Eqs. (5)-(7) in Eq. (4) along with the following analytic continuation for Gaussian hypergeometric function [25–27]

$${}_2F_1(A\,,B;C\,;\,Z)=\frac{\Gamma\,(C)\,\Gamma\,(C-A-B)}{\Gamma\,(C-A)\,\Gamma\,(C-B)}{}_2F_1(A\,,B;A+B-C+1\,;\,1-Z) + (1-Z)^{C-A-B}\,\frac{\Gamma\,(C)\,\Gamma\,(A+B-C)}{\Gamma\,(A)\,\Gamma\,(B)}{}_2F_1(C-A,C-B;C-A-B+1;1-Z), \tag{11}$$

the second term in Eq. (10) becomes

$$\int_0^r G_H^{(I)}(r,s)e^{-\alpha s}ds = -\lim_{\varepsilon\to 0} a\, e^{ikr}\Bigg[\left(1-e^{-r/a}\right){}_2F_1(1+A,1+B;2;1-e^{-r/a})\int_0^r e^{-(\alpha-ik)s} \times {}_2F_1(A,B;\varepsilon;1-e^{-s/a})ds - {}_2F_1(A,B;\varepsilon;1-e^{-r/a}) \times \int_0^r e^{-(\alpha-ik)s}\left(1-e^{-s/a}\right){}_2F_1(1+A,1+B;2;1-e^{-s/a})ds\Bigg]. \tag{12}$$

Now by applying the integral relations [28, 29]

$$f_\sigma(a,b;c;z) = \frac{1}{c-1}\Bigg\{{}_2F_1(a,b\,;c\,;z)\int_0^z s^{\sigma-1}\ (1-s)_2^{a+b-c}\,F_1(a-c+1\,,b-c+1\,;\,2-c\,;\,s)\,ds - z^{1-c}\,{}_2F_1(a-c+1\,,b-c+1\,;\,2-c\,;\,z)\int_0^z s^{\sigma+c-2}\ (1-s)^{a+b-c}\,{}_2F_1(a\,,b\,;\,c\,;\,s)\,ds\Bigg\}, \tag{13}$$

Eq. (12) simplifies to

$$\int_0^r G_H^{(I)}(r,s)\ e^{-\alpha s}ds = -a^2 e^{ikr}\left(1-e^{-r/a}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+1-(\alpha+ik)\,a\right)}{\Gamma\left(1-(\alpha+ik)\,a\right)n!}f_{n+1}(1+A,1+B;2;1-e^{-r/a}). \tag{14}$$

However, the first term in Eq. (10) involving definite integration can be easily solved through the application of standard integral relation for hypergeometric function [25–27]

$$\int_0^1 x^{\rho\,-\,1}\,(1-x)^{\sigma\,-\,1}\,{}_2F_1\,(a,\,b;\,c;x)\,dx = \frac{\Gamma\,(\rho)\ \Gamma\,(\sigma)}{\Gamma\,(\rho+\sigma)}{}_3F_2\,(a,b,\rho;c,\rho+\sigma;\,1)$$
$$Re\rho>0;\quad Re\sigma>0\,;\quad Re(c+\sigma-a-b)>0. \tag{15}$$

Combining Eqs. (4)-(7) together with Eq. (15) simplifies the first term of Eq. (10) as

$$\int_0^\infty G_H^{(I)}(r,s)\ e^{-\alpha s}ds = -a\ e^{ikr}\Bigg[\frac{\left(1-e^{-r/a}\right)}{(\alpha-ik)\,f_H(k)} \times {}_3F_2(A,B,(\alpha-ik)\,a;C,1+(\alpha-ik)\,a;1){}_2F_1\,(1\,+\,A\,,\,1\,+\,B;\,2\,;\,1-\,e^{-r/a}) - \frac{a\,\Gamma\left((\alpha+ik)\,a\right)\Gamma\left((\alpha-ik)\,a\right)}{f_H(k)\Gamma\left(1+(\alpha-ik)\,a-A\right)\Gamma\left(1+(\alpha-ik)\,a-B\right)}{}_2F_1\,(\,A\,,\,B;\,C\,;\,e^{-r/a})\Bigg]. \tag{16}$$

Now, by utilising Eqs. (10), (14) and (16) in Eq. (9) jointly with the application of Eq. (15) and the following formula and relations for non-homogenous Gaussian hypergeometric functions [25–28]

$$\int_0^1 z^{c-1}\,(1-z)^{\upsilon-1}\ f_\sigma(a\,,b\,;c\,;pz)\ dz = \frac{\Gamma\left(\sigma+c-1\right)\Gamma\left(\upsilon\right)}{\Gamma\left(\sigma+c+\upsilon-1\right)}f_\sigma(a\,,b\,;c+\upsilon\,;p), \tag{17}$$

$$f_\sigma(a\,,b\,;c\,;z) = \frac{z^\sigma}{\sigma\left(\sigma+c-1\right)}{}_3F_2(1\,,\sigma+a,\sigma+b\,;\,\sigma+1\,,\,\sigma+c\,;\,z), \tag{18}$$

we get

$$D^{(I)}(k) = 1-\lambda\,a^3\sum_{n=0}^{\infty}\frac{\Gamma\left(n+1-(\alpha+ik)\,a\right)}{\Gamma\left(1-(\alpha+ik)\,a\right)}\frac{\Gamma\left((\alpha-ik)\,a\right)}{\Gamma\left(n+3+(\alpha-ik)\,a\right)} \times {}_3F_2(1,n+2+A,n+2+B;n+2,n+3+(\alpha-ik)\,a;1). \tag{19}$$

Equation (6) in Eq. (8) together with the use of Eq. (15), one has

$$d^{(I)}(k) = \frac{1}{(\alpha-ik)D^{(I)}(k)}{}_3F_2(A,B,(\alpha-ik)a;C,1+(\alpha-ik)a;\,1)\,. \tag{20}$$

Substituting Eqs. (10), (14), (16) and (20) in Eq. (3), the desired irregular solution for effective nonlocal interaction under consideration is expressed as

$$f(k,r) = f_H(k,r) + \lambda\, d(k)\, a\, e^{ikr}\Bigg[a\left(1-e^{-r/a}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+1-(\alpha+ik)\,a\right)}{\Gamma\left(1-(\alpha+ik)\,a\right)n!}f_{n+1}(1+A,1+B;2;1-e^{-r/a})$$
$$-\frac{1}{(\alpha-ik)\,f_H(k)}{}_3F_2(A,B,(\alpha-ik)\,a;C,1+(\alpha-ik)\,a;1)\left(1-e^{-r/a}\right)\ {}_2F_1\left(1+A,1+B;2;1-e^{-r/a}\right)$$
$$+\,a\frac{\Gamma\left((\alpha+ik)\,a\right)\Gamma\left((\alpha-ik)\,a\right)}{f_H(k)\Gamma\left(1+(\alpha-ik)\,a-A\right)\Gamma\left(1+(\alpha-ik)\,a-B\right)}{}_2F_1\left(A,B;C;e^{-r/a}\right)\Bigg]. \tag{21}$$

2.2. Equivalent local potential

In this text we use two linearly independent irregular solutions $f_{\pm}(k,r)$ where $f_{-}(k,r) = f_{+}^{*}(k,r)$ satisfying the boundary conditions $\lim_{r\to\infty} e^{\mp ikr}f_{\pm}(k,r) = 1$. With knowledge of two independent solutions the desired equivalent local potential can be constructed from the relation [9,10]

$$V_{EQ}(k,r) = -\frac{1}{2}\frac{J''(k,r)}{J(k,r)} + \frac{3}{4}\left(\frac{J'(k,r)}{J(k,r)}\right)^2 - \frac{\lambda}{J(k,r)}e^{-\alpha r}\int_0^{\infty} e^{-\alpha s}\,Q'(k,r,s)\,ds, \tag{22}$$

where the Wronskian $J(k,r)$ of the pair of irregular solution is written as

$$J(k,r) = \frac{1}{k}\,\text{Img}\left[f_{-}(k,r)f_{+}'(k,r)\right], \tag{23}$$

and the quantity $Q_{\ell}(k,r,s)$ is defined as [9,10]

$$Q(k,r,s) = \frac{1}{k}\,\text{Img}\left[f_{\ell-}(k,r)f_{\ell+}(k,s)\right]. \tag{24}$$

From Eqs. (23) and (24), one can write $J'(k,r)$, $J''(k,r)$ and $Q'(k,r,s)$ as

$$J'(k,r) = \frac{1}{k}\,\text{Img}\left[f_{-}(k,r)f_{+}''(k,r) + f_{-}'(k,r)f_{+}'(k,r)\right], \tag{25}$$

$$J''(k,r) = \frac{1}{k}\,\text{Img}\left[f_{-}(k,r)f_{+}'''(k,r) + 2\times f_{-}'(k,r)f_{+}''(k,r) + f_{-}''(k,r)f_{+}'(k,r)\right], \tag{26}$$

and

$$Q'(k,r,s) = \frac{1}{k}\,\text{Img}\left[f_{\ell-}'(k,r)f_{\ell+}(k,s)\right]. \tag{27}$$

Here the higher order partial differentiation of $f_{\pm}(k,r)$ have to be carried out with respect to r using the following formulae [25,28]

$$\frac{d^n}{dz^n}{}_2F_1\ (a,b,c;z) = \frac{(a)_n(b)_n}{(c)_n}\ {}_2F_1\ (a+n,b+n,c+n;z), \tag{28}$$

and

$$\frac{d}{dz}f_{\sigma}(a,b,c;z) = (\sigma-1)\quad f_{\sigma-1}(a+1,b+1,c+1;z). \tag{29}$$

Now from the Eq. (27), the quantity $\int_0^{\infty} e^{-\alpha s}\,Q'(k,r,s)\,ds$ can be expressed as

$$\int_0^{\infty} e^{-\alpha s}\,Q'(k,r,s)\,ds = \frac{1}{k}Img\Bigg[f_{\ell-}'(k,r)\Bigg\{\frac{1}{(\alpha-ik)}{}_3F_2(A,B,(\alpha-ik)\,a;C,1+(\alpha-ik)\,a;1)$$
$$\times\Bigg\{\lambda d(k)a^2\Bigg\{a\sum_{n=0}^{\infty}(n+1)\,\frac{\Gamma\left(n+1-(\alpha+ik)\,a\right)}{\Gamma\left(1-(\alpha+ik)\,a\right)}\Bigg\}\frac{\Gamma\left((\alpha-ik)\,a\right)}{\Gamma\left(n+2+(\alpha-ik)\,a\right)}$$
$$\times f_{n+1}(1+A,1+B;2+(\alpha-ik)\,a;1)\Bigg\}\Bigg\}\Bigg]. \tag{30}$$

The above Eq. (30) is deduced by using integral formulae mentioned in Eqs. (15) and (17).

Thus, having the knowledge of Eqs. (23), (25), (26) and (30) in conjunction with Eq. (22), one can develop the required expression for energy dependent equivalent local potential for the Hulthén modified Yamaguchi potential.

3. Results and discussion

The energy dependent local potential, developed here, for Hulthén plus Yamaguchi separable interaction is applied to compute scattering phase shifts of (p-p) and (p-d) systems through PFM. In past, PFM was successfully used by us for various phase equivalent potentials [14–17] in order to analyse nuclear scattering phase shifts. The phase equation for a local potential [30] is given by

$$\delta_\ell'(k,r) = -k^{-1}V_\ell(k,r)\left[\hat{\mathrm{j}}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)\right]^2, \tag{31}$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati–Bessel functions. For the partial wave $\ell = 0$ one gets

$$\delta_0{}'(k,r) = -k^{-1}V_0(r)\ \left[\sin\left(\delta_0(k,r)+kr\right)\right]^2. \tag{32}$$

Here $V_0(r)$ is $V_{EQ}(k,r)$, the energy-dependent equivalent local potential for the effective interaction of Hulthén modified Yamaguchi potential, developed from Eq. (22).

The merit of the construction of energy dependent potential can be judged when it will effectively reproduce same observables of its parent nonlocal potential. Therefore, phase shifts of p-p and p-d systems are also computed from the Fredholm determinant $D^{(+)}(k)$ [31] associated with physical boundary condition for Hulthén distorted Yamaguchi separable interaction. From Ref. [31] $D^{(+)}(k)$ for s-wave is expressed as

$$D^{(+)}(k) = 1 + \lambda a^3 \frac{\Gamma\left(2\alpha a - 1\right)}{(A+1)\ \ (B+1)\ \ \Gamma\left(2\alpha a+1\right)} \times {}_4F_3(1,2,1-(\alpha+ik)\,a, 1-(\alpha+ik)\,a; A+2, B+2, 2-2\alpha a; 1). \tag{33}$$

For numerical computation, we have taken $\hbar^2/m_p = 41.47$ MeV fm^2 and $V_0 a = 0.0347$ fm^{-1} and 0.04629 fm^{-1} for (p-p) and (p-d) systems, respectively. Here the screening radius is considered to be $a = 10$ fm for (p-p) and (p-d) systems. We have portrayed our results for the generated equivalent local potential, labelled as "Eq. Local-Exact" phase shift values, along with results of its nonlocal partner, in Figs. 1-3. For better understanding, we have also presented phase shifts "Eq. Local-Appx" for the energy dependent equivalent local potential calculated from Ref. [17].

In Fig. 1, phase data $\delta_{^1s_0}$-pp (Eq. Local-Exact) computed from the exact energy dependent local potential constructed for Hulthén plus Yamaguchi separable interaction are in better agreements with standard results [32] than those of phase data $\delta_{^1s_0}$-pp (Eq. Local-Appx.) computed from the approximate equivalent local one [17]. It is also observed that phase shift values $\delta_{^1s_0}$-pp (Eq. Local-Exact) are in more consistent with nonlocal results $\delta_{^1s_0}$-pp (Nonlocal) than $\delta_{^1s_0}$-pp (Eq. Local-Appx.) data.

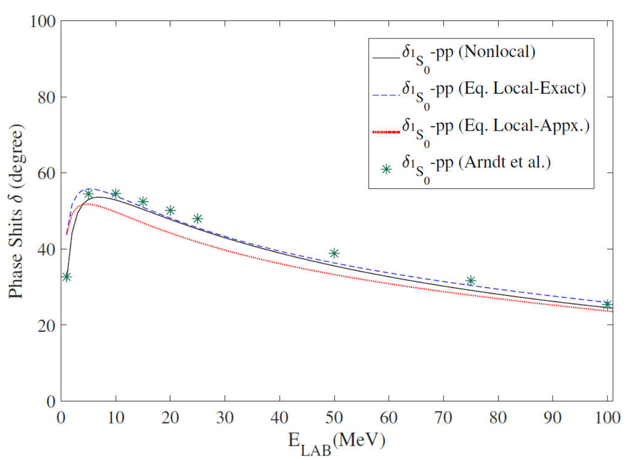

FIGURE 1. ^{1}S$_0$ (p-p) phase shifts as a function of E_{LAB}.

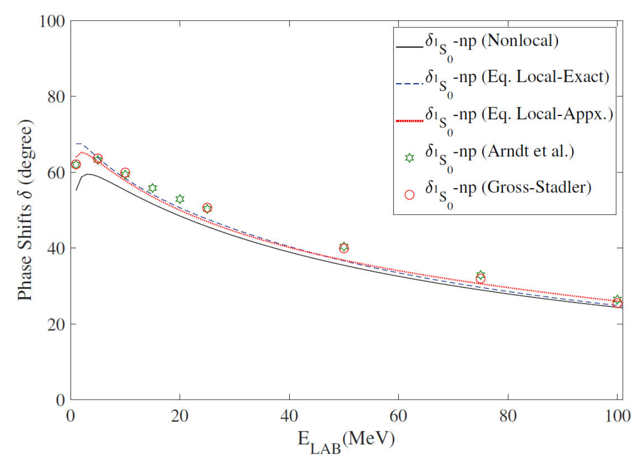

FIGURE 2. ^{1}S$_0$ (n-p) phase shifts as a function of E_{LAB}.

The phase parameters for the (n-p) system are calculated by turning off the Hulthén potential in the numerical programme of the (p-p) system and are plotted in Fig. 2 along with the standard data of Arndt *et al*. [32] and Gross-Stadler [33]. The ^{1}S$_0$ states of (p-p) and (n-p) systems are unbound and phase shift values are obtained by using $\lambda = -2.405$ MeVfm^{-1} and $\alpha = 1.1$ fm^{-1} [34]. In the absence of electromagnetic interaction both exact and approximate [17] energy dependent local potentials analytically represent the same

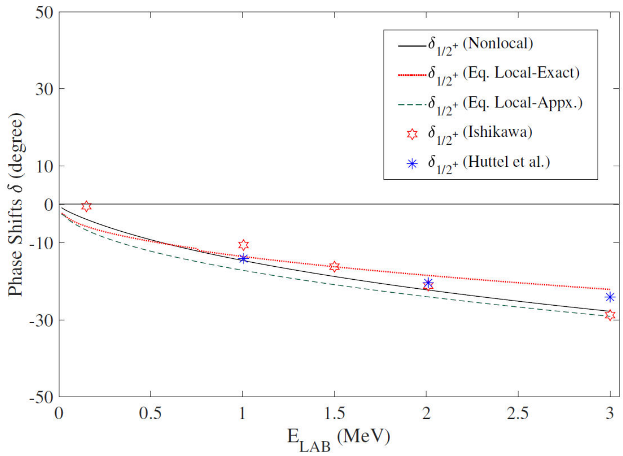

FIGURE 3. 1/2$^+$ (p-d) phase shifts as a function of $E_{\rm LAB}$.

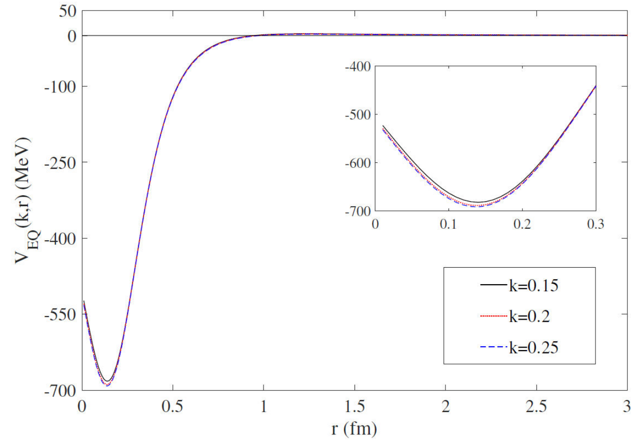

FIGURE 5. S-wave (p-d) potential at various energies as a function of r.

localisation to pure Yamaguchi interactions, thereby showing numerically equivalent results except the peak value. In Fig. 2, we find the peak point in phase shifts only for the local potential obtained in Ref. [17]. However, in s-wave p-p scattering, our phase shift estimation for exact local potential exhibits nearly exact peak value at $E_{\rm LAB} = 5$ MeV.

Parameters of Yamaguchi potential for the 1/2$^+$ state of the p-d system with binding energy $E_B = -7.761$ MeV are found to be $\lambda = -162.7959$ MeVfm^{-1} and $\alpha = 3.97$ fm^{-1}. In case of bound state, zeros of the $D^{(+)}(k)$ for $k = ik_B$ with $k_B = \sqrt{2\mu E_B/\hbar^2}$ reproduce the bound state energy, thus providing a relation between λ, α and k_B. Hence, either by adjusting λ or α, one can produce proper phase shift values satisfying the binding energy of the system under consideration. In Fig. 3, we have presented phase shifts for 1/2$^+$ state of the p-d system along with standard data [35, 36]. Phase shifts for exact local potentials $\delta_{1/2^+}$ (Eq. Local-Exact) match quite well with Huttel *et al*. [35] at laboratory energies 1−3 MeV. Though our phase data shows light difference with those of Ishikawa [36] beyond 2 MeV, the overall quality of matching is noteworthy. On the other hand, nonlocal phase shifts $\delta_{1/2^+}$ (Nonlocal) are in reasonable order with results of Ref. [36] while they discern from those of Ref. [35] in the energy range $2-3$ MeV. Despite large dissimilarity in phase shifts with approximate energy dependent potential they follow correct trend of the phase parameters. At very low laboratory energies (less than 0.5 MeV) the quantitative disagreement of both local (Exact) and nonlocal phase values may be due to improper accountability of the electromagnetic interaction which dominates over the nuclear force in this region.

For (p-p) and (p-d) systems, natures of the equivalent local potentials are investigated and displayed in Figs. 4 and 5. Here we have plotted potentials for a few k-values considered within the corresponding energy range of phase shifts evaluated. Unlike ^{1}S$_0$ state of the (p-p) system, potentials for 1/2$^+$ state of the (p-d) system die out quickly beyond 1 fm. In Fig. 5, depth of potentials slowly increases as the energy increases. The similar tendency is also observed in case of the ^{1}S$_0$ p-p potential within $k < 0.2$ fm^{-1} ($E_{\rm LAB} < 3.3176$ MeV). For instance, in ^{1}S$_0$ state of p-p system, maximum depth of -57.03 MeV is noted in potential for $k = 0.15$ fm^{-1} whereas depth of -57.24 MeV is observed for $k = 0.2$ fm^{-1}. However, we have checked that this tendency holds for very low energies. For $k > 1$ fm^{-1} ($E_{\rm LAB} > 46.6537$ MeV), the depth of potentials gradually decreases, similar to ^{1}S$_0$ p-p potential in Fig. 4. In Ref. [17], it is shown that the ^{1}S$_0$ state (p-p system) local potentials possess repulsive cores. The same trend is also noticed in the present case below laboratory energy 3.3176 MeV ($k = 0.08$ fm^{-1}) due to repulsive electromagnetic potential which are not shown in the figure.

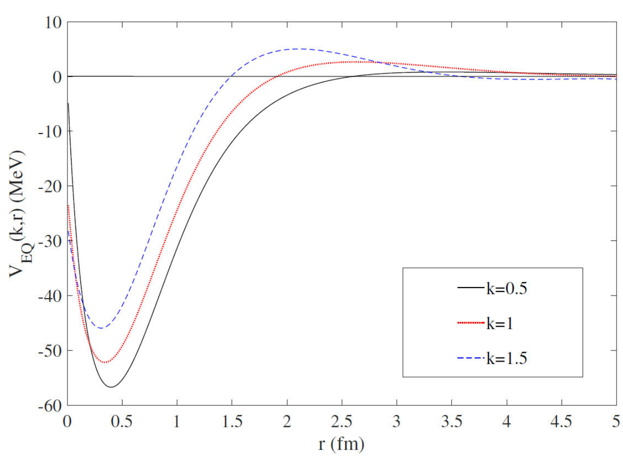

FIGURE 4. ^{1}S$_0$ (p-p) potential at various energies as a function of r.

4. Conclusions

It is well known that separable two body interactions, because of their simplicity, are frequently applied to various physical situations which are extremely difficult to solve with local two body interactions. In this context, the equivalent local analysis may lead physicists to understanding the properties of these phenomenological nonlocal potentials in terms of most familiar concepts of a local potential. Phase equivalent potentials have also been constructed by several researchers by exploiting the methodology of Supersymmetric Quantum Mechanics and variable phase approach. Supersymmetric formalism involves the construction of phase equivalent potential to a parent deep or shallow local potential. In this text we have adapted a different point of view which is entirely different from the algebra of Supersymmetric Quantum Mechanics. Here we have developed equivalent local potential for the effective nonlocal interaction of Hulthén plus Yamaguchi separable potential following the approach of Ref. [10]. The constructed local potentials applied to study asymptotic phase shifts for elastic scattering of (p-p) and (p-d) systems through PFM and achieve good agreement with standard data [32–35]. We have also found reasonable correspondence between the results produced by the nonlocal and the generated local potentials. In our previous work [17], Hulthén potential was added to the generated equivalent local one to represent the net equivalent local interaction of the charged particle scattering. The (p-p) and the (p-d) phase shifts calculated for the present local potential are superior than those computed from the previous one [17]. This can be attributed to the fact that the energy and the angular momentum dependency of our equivalent local potentials are described properly in terms of the solutions to the nonlocal equation and the potential. However, the construction of this type of equivalent local potential to nonlocal interaction does not depend explicitly on the boundary conditions imposed on the solutions [10]. For higher partial wave generalization, one has to consider separable potential in partial waves [14] along with the addition of a centrifugal barrier to the wave equation. Therefore, within the framework of partial wave analysis, the generalization of the present approach connected with introducing equivalent local interaction for a combined electromagnetic plus separable nuclear interaction is still remain to be investigated. Such an approach involves lots of mathematical complications. However, it is in our active contemplation and will be tackled in future.

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