

Comparison of energy-dependent and independent interactions-A case study

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In the present text we construct velocity-dependent or equivalently energy-dependent potential (EDP) to an energy-independent nonlocal potential (EIP) of rank-1 by Taylor series method. The phase shifts for the nucleon-nucleon and alpha-nucleon systems are computed for these two potentials by exploiting the variable phase method and the Fredholm determinant, respectively. The velocity-dependent potential is found to be superior to central nonlocal interaction in generating the scattering phase shifts up to high energy region.

Keywords: Graz separable potential; equivalent energy-dependent interaction; phase function method; scattering phase shifts.

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1. Introduction

The study of the two-nucleon interaction is of essential importance for the understanding of nuclear properties on a microscopic level. Although in a system of many nucleons there may subsist three and many-body forces, the role of the two-nucleon force plays a predominant character. Generally, in nuclear physics nucleons are considered as non-relativistic particles interacting through the N-N interaction which is represented by a potential either central or non-central. Most of the information about the N-N interaction is obtained from the nucleon-nucleon scattering experiments which is supplemented by the properties of the deuteron- the only bound state of the two nucleon system. Analyses of these data yield phase shifts for the different partial waves regarding to the nucleon-nucleon scattering. The short range part of the nuclear interaction mainly arises due to multiplication exchange processes when the effect of recoil of the nucleons is indispensable. In such situation the nucleon-nucleon interaction cannot be represented by one radial variable r , *i.e.*, the relative separation of two interacting nucleons. Thus the nucleon-nucleon interaction should be represented by a nonlocal potential which depends on two variables r and r' . The wave function $\psi(r)$ at any point r will now depend on its value at other neighbouring points r' through the interaction $V(r, r')$ [1-7]. The momentum-dependent potentials are generally applied in the nucleon-nucleon scattering to restore the hard core in nuclear interactions. The intention of constructing a phase equivalent local potential is to understand the properties of the parent nonlocal interaction in terms of those models which are familiar to physicists. As the equivalent local potential is derived from the knowledge of the solutions to the nonlocal one, it becomes energy-momentum dependent in a partial wave analysis. The main motivation of such method consists in studying to what degree the angular momentum barriers remain effective. Therefore, several attempts have been made

towards the development of equivalent local potential to a nonlocal one which provides a greater understanding of nuclear interaction [8–16]. Various approaches to study energy-momentum dependent interactions in the context of atomic and nuclear physics have also been advocated by a number of researchers [17–34]. The present text addresses itself to the phase shift analysis of the nucleon-nucleon and nucleon-nucleus elastic scattering in the low to high energy region. In Sec. 2 we briefly outline our methodology. Section 3 is devoted to results and discussion part. We put our concluding remarks in Sec. 4.

2. Methodology

The Schrödinger equation for a non-local potential $V_l(r, r')$ in all partial waves is given by [3–6]

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right] \phi_l(k, r) = \int_0^\infty V_l(r, r') \phi_l(k, r') dr'. \quad (1)$$

The Taylor series expansion of the radial wave function $\phi_l(k, r')$ yields

$$\phi_l(k, r') = \sum_{m=0}^{\infty} \frac{(r' - r)^\lambda}{m!} \frac{d^m}{dr^m} \phi_l(k, r). \quad (2)$$

From Eqs. (1) and (2) one gets

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right] \phi_l(k, r) = \sum_{m=0}^{\infty} \phi_l(k, r) \times \int_0^\infty \frac{(r' - r)^\lambda}{m!} V_l(r, r') dr'. \quad (3)$$

Considering the Taylor series expansion up to $m = 2$ in Eq. (3) one gets an energy-dependent local potential of the form

$$V_L(k, r) = V_L^{(1)}(k, r) + V_L^{(2)}(r) \frac{d}{dr}, \quad (4)$$

where the quantities $V_L^{(1)}(k, r)$ and $V_L^{(2)}(r)$ are given by

$$V_L^{(1)}(k, r) = \frac{I_l^{(0)}(s) - k^2 I_l^{(2)}(s) + \frac{l(l+1)}{r^2} I_l^2(r)}{1 - I_l^{(2)}(r)}, \quad (5)$$

and

$$V_L^{(2)}(r) = \frac{I_l^{(1)}(r)}{1 - I_l^{(2)}(r)}. \quad (6)$$

Rank-one separable potentials with its simplest mathematical form are used in different areas of physics. The rank one separable potential considered here reads as [3–7]

$$V_l(r, r') = \lambda_l g_l(r) g_l(r'), \quad (7)$$

where

$$g_l(r) = 2^{-l} (l!)^{-1} r^l e^{-\beta_l r}. \quad (8)$$

We consider the partial waves $l = 0, 1$ for which we obtain

$$I_0^{(0)}(r) = \frac{\lambda_0}{\beta_0} e^{-\beta_0 r}, \quad (9)$$

$$I_1^{(0)}(r) = \frac{\lambda_1 r}{4\beta_1^2} e^{-\beta_1 r}, \quad (10)$$

$$I_0^{(1)}(r) = \frac{\lambda_0(1 - r\beta_0)}{\beta_0^2} e^{-\beta_0 r}, \quad (11)$$

$$I_1^{(1)}(r) = \frac{\lambda_1 r(2 - r\beta_1)}{4\beta_1^3} e^{-\beta_1 r}, \quad (12)$$

$$I_0^{(2)}(r) = \frac{\lambda_0 e^{-\beta_1 r}}{2\beta_0^3} (2 + r^2\beta_0^2 - 2r\beta_0), \quad (13)$$

and

$$I_1^{(2)}(r) = \frac{\lambda_1 r e^{-\beta_1 r}}{8\beta_1^4} (6 + r^2\beta_1^2 - 4r\beta_1). \quad (14)$$

For velocity-dependent potential one has to use the phase equation of McKellar and May [35] to calculate the scattering phase shifts. According to McKellar and May [35] and Behera *et al.* [14] the phase equation corresponding to Eqs. (4) is given by

$$\begin{aligned} \delta_l'(k, r) &= k^{-1} [V_L^{(1)}(k, r) (\hat{j}_l'(kr) \cos \delta_l(k, r) \\ &\quad - \hat{\eta}_l'(kr) \sin \delta_l(kr)) - V_L^{(2)}(r) \\ &\quad \times [\hat{\eta}_l(kr) \sin \delta_l(k, r) - \hat{j}_l(kr) \cos \delta_l(k, r)]. \end{aligned} \quad (15)$$

The scattering phase shift $\delta(k)$ is accomplished by cracking the above equation from origin to asymptotic region with the

initial condition $\delta(k, 0) = 0$. Finally, one gets the phase shift $\delta(k)$ when $r \rightarrow \infty$. For pure nonlocal plus the Coulomb potential under consideration the Fredholm determinant for physical boundary condition reads as [3–7]

$$\begin{aligned} D_l^+(k) &= 1 + \lambda_l \left[\frac{\Gamma(2l+2)(\beta_l - ik)^{-2}}{2^{2l}(l!)(2\beta_l)^{2l+1}(l+1)^2} \right. \\ &\quad \left. \times {}_2F_1 \left(1, -l; l+2 \left(\frac{\beta_l + ik}{\beta_l - ik} \right)^2 \right) \right]. \end{aligned} \quad (16)$$

The Jost function is related to $D_l^{(+)}(k)$ and $D_l(k)$ by $f_l(k) = D_l^{(+)}(k)/D_l(k)$, where the quantity $D_l(k)$ is the Fredholm determinant associated with the regular and irregular boundary conditions and is always a real quantity while $D_l^{(+)}(k)$, the Fredholm determinant associated with the physical boundary condition, is complex. The Jost function has two distinct properties: its zeros in the complex momentum plane's top half repeat the energies of bound and resonance states, and its phase is the opposite of the scattering phase shift. Therefore, by exploiting the Jost function one can calculate the scattering phase shifts. For computing scattering phase shifts we follow two different approaches. For the velocity-dependent potential in Eq. (4) we consider the phase function method [14, 35] and solve the problem numerically by using Chi square fitting to get appropriate potential parameters. Utilizing these parameters Eq. (16) is solved numerically to have the same phase parameters. Our aim is to compare the phase parameters computed for velocity-dependent and parent central nonlocal interactions and to examine the effectiveness of these two potentials.

3. Result and discussion

For numerical processes we have chosen to work with $\hbar^2/2m = 41.47 \text{ MeV fm}^2$ and 25.92 MeV fm^2 and $2k\eta = 0.03472$ and 0.11152 fm^{-1} for nucleon-nucleon and alpha-nucleon systems, respectively. The best fitted potential parameters for nucleon-nucleon, alpha-neutron and alpha-proton systems are presented in Table I.

TABLE I. Parameters for the systems under consideration.

System	State	$\lambda(\text{fm}^{-2l-3})$	$\beta(\text{fm}^{-1})$
Nucleon-nucleon	${}^3s_1(n-p)$	-17.201	1.909
	${}^1s_0(n-p/p-p)$	-5.237	1.4054
$\alpha - n$	$1/2^+$	-35.55	1.53
	$1/2^-$	-42.28	1.53
	$3/2^-$	-56.50	1.53
	$3/2^+$	-62.3	1.56
	$5/2^+$	-65.21	1.56
$\alpha - p$	$1/2^+$	-45.55	1.94
	$1/2^-$	-49.28	1.95
	$3/2^-$	-76.20	1.95
	$3/2^+$	-65.45	2.02
	$5/2^+$	-72.54	2.02

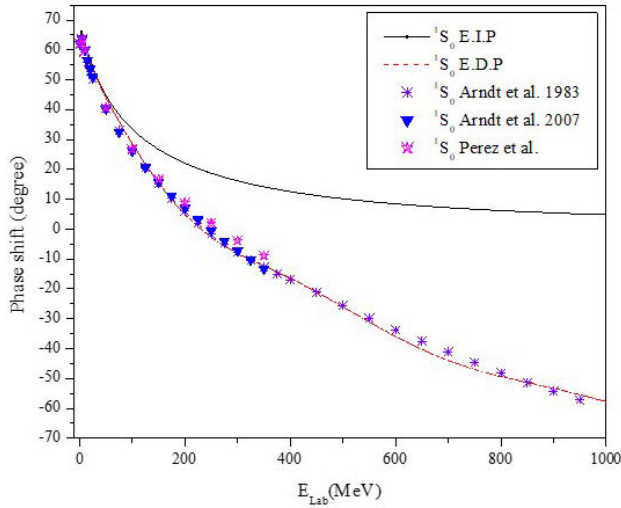


FIGURE 1. (n-p) 1S_0 phase shifts as a function of E_{Lab} . The standard data are from Refs. [36–38].

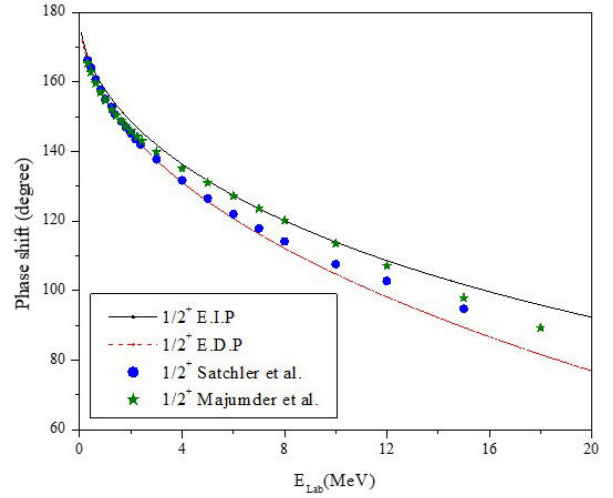


FIGURE 4. $(\alpha - n)$ $1/2^+$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39, 40].

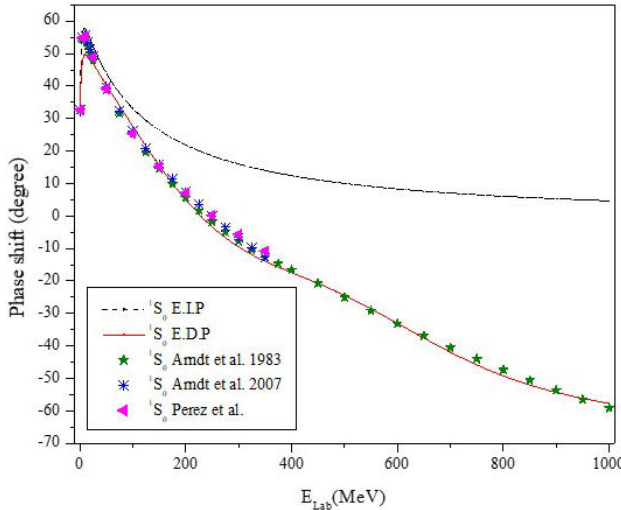


FIGURE 2. (p-p) 1S_0 phase shifts as a function of E_{Lab} . The standard data are from Refs. [36–38].

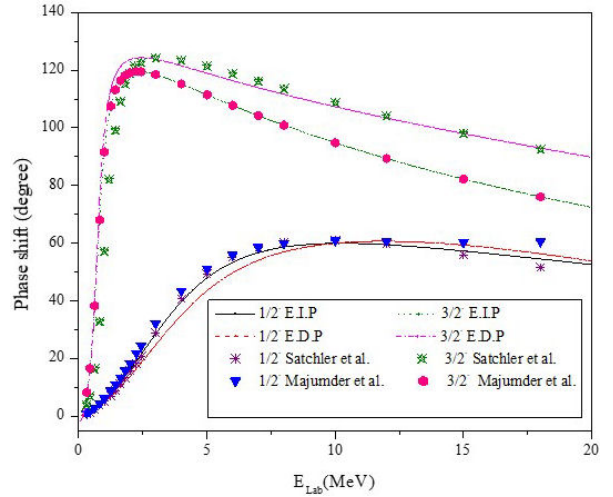


FIGURE 5. $(\alpha - n)$ $1/2^-$ and $3/2^-$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39, 40].

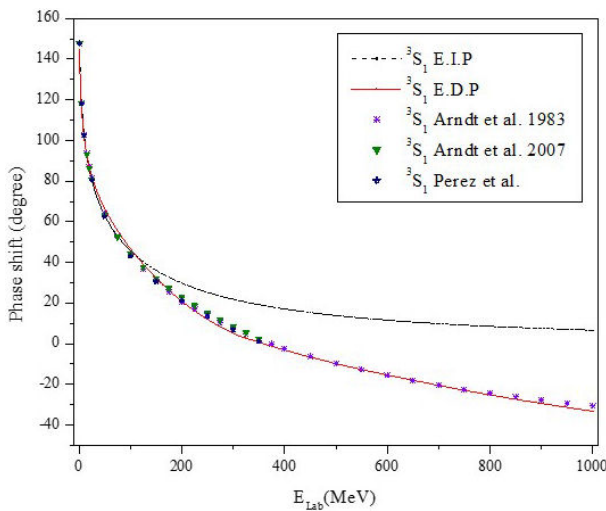


FIGURE 3. (n-p) 3S_1 phase shifts as a function of E_{Lab} . The standard data are from Refs. [36–38].

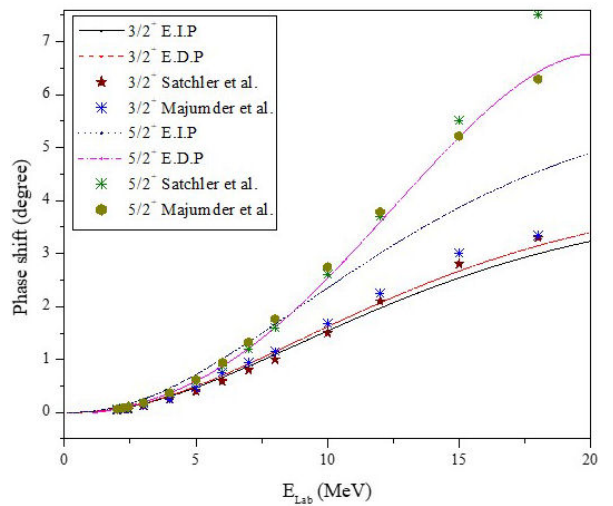


FIGURE 6. $(\alpha - n)$ $3/2^+$ and $5/2^+$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39, 40].

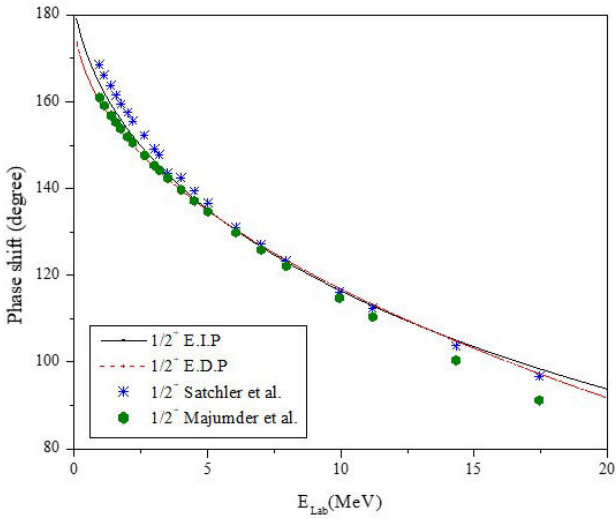


FIGURE 7. $(\alpha - p)$ $1/2^+$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39,40].

The computed phase parameters are presented in Figs. 1-9 along with those of standard data [36–40].

Looking closely at Figs. 1-3 it is noticed that our velocity-dependent potential reproduces phase shifts in conformity with those of Arndt *et al.* [37] and Perez *et al.* [38] up to 350 MeV and over the entire energy range under consideration with Arndt *et al.* [36] for both n-p and p-p systems while those with central nonlocal potential match quite well with Ref. [36–38], up to 50 MeV and beyond that they discern from Refs. [36–38] quite significantly. From the foregoing observation it is quite clear that the energy-dependent potentials are superior to their energy-independent counterpart. This may be attributed to the fact that the energy-dependent interactions have the ability to reproduce the effect of non-central nature of the N-N force to generate correct phase parameters up to high energy region.

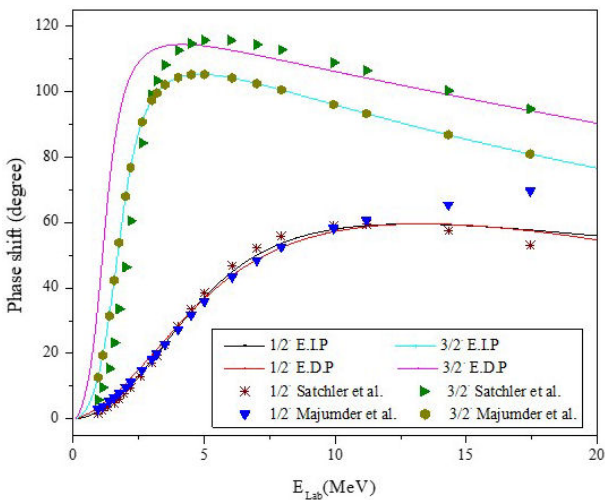


FIGURE 8. $(\alpha - p)$ $1/2^-$ and $3/2^-$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39,40].

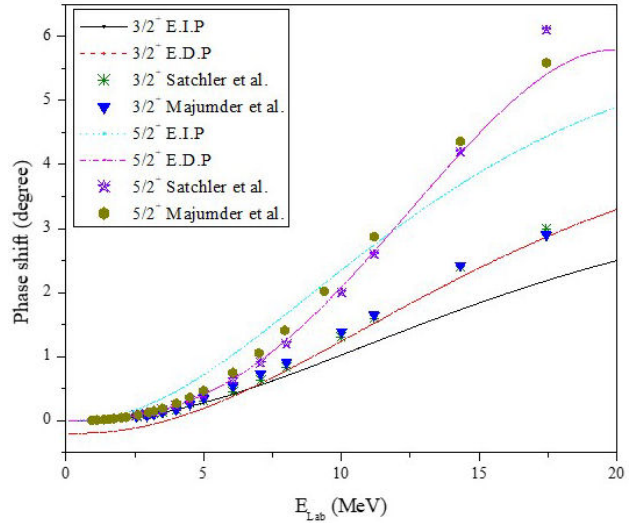


FIGURE 9. $(\alpha - p)$ $3/2^+$ and $5/2^+$ phase shifts as a function of E_{Lab} . The standard data are from Refs. [39,40].

From Figs. 4-9 it is noticed that our energy-dependent phase shifts for different states of $(\alpha-n)$ and $(\alpha-p)$ systems reproduce better agreement with those of Satchler *et al.* [39]. However, the energy-independent phase parameters match better with Majumder *et al.* [40]. It is quite obvious because Majumder *et al.* [40] analyzed the phase shifts by using a central potential model. The phase parameters with the velocity-dependent interaction are in close conformity with those of Satchler *et al.* [39], Kumar *et al.* [41] and Laha-Bhoi [25]. The phase parameters of Refs. [25,41] are in close conformity with those of Satchler *et al.* [39] and are not portrayed in Figs.4-9 for clarity of presentation. However, the data from Ref. [40] are plotted for comparison with our result. Small discrepancies which are observed in our phase parameters with those of standard data are within the permissible error. They may appear due to improper electromagnetic effect which does not account perfectly as observed in experiment in s-wave cases and for higher partial waves may be due to strong centrifugal barrier effects. By combining the resonating-group method with a microscopic description of the nucleon clusters with more realistic N3LO and CD-Bonn NN potentials, Quaglioni and Navrátil [42] calculated $(n-\alpha)$ and $(p-\alpha)$ phase shifts which are in good agreement with Ref. [39] for s-wave only but discern from p-wave data. Lee and Robson [43] also generated a nucleon-nucleus optical potential in the folding model approach, including spin-orbit and tensor forces. When it comes to the p-wave, they were able to describe the alpha-nucleon phase shifts with appropriate splitting and magnitude in comparison to the experimental results [39]. The phase shifts do not change sign in their values as the central nonlocal potential is rank-one attractive interaction. The velocity-dependent potential has two components which take care of the sign change in phase shift as energy increases. The same phenomenon is also observed for $\alpha-n$ and $\alpha-p$ scattering with velocity-dependent and independent interactions. The energy-dependent potential takes

care of the sign change in nucleon-nucleon phase shifts and reproduces reasonable agreement in numerical values up to 1 GeV. This may be attributed to the fact that the energy-dependent potential includes implicitly the effects of spin-orbit and tensor parts of the interaction. Thus, the energy-momentum dependent potential reflect to be superior to its nonlocal counterpart.

4. Conclusions

For a number of different nuclear systems, elastic scattering phase shifts have been satisfactorily fitted using separable nonlocal potentials [6, 7, 44, 45]. For systems where the phase shifts change sign with energy, higher-rank separable potentials are typically used. The rank one separable potential is purely appealing and is unable to explain how the phase shifts' sign changed. But the phase changes for nucleon-nucleon and alpha-nucleon systems reproduced correctly by our built-in equivalent energy-dependent potential

up to very high energy. From a practical point of view, comparison of the potential models is often made through phase-shift analyses of experimental data. In order to verify various components of the potential, it is also important to compare to n-p scattering observables, especially the n-p analyzing power. Our results for the elastic scattering phase shifts for the nucleon-nucleon and alpha-nucleon systems are in conformity with the standard data. The calculations of cross sections and analyzing powers can be performed using the basic ingredients like phase parameters. A synchronized description of all obtainable theoretical and experimental data over a large energy range would expose more information on the nucleon-nucleon interaction and possibly on the significance of three-body forces for the nucleon-nucleon and alpha-nucleon systems. Therefore, computing phase shifts and other physical observables for complex nuclear systems may start with this straightforward model calculation. Our simple minded approach to the problem may turn out to be interesting to physicists.

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