# PDM-Coulombic effects of non-inertial cosmic strings on a Klein-Gordon oscillator

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This paper addresses the problem of a three-dimensional Klein-Gordon oscillator with position-dependent mass in a non-inertial cosmic string background. We provide solutions to this problem and analyze the eigensolutions, considering the influence of non-inertial effects and the presence of position-dependent mass (PDM) on the eigenvalues. Expressions are obtained for the bound state energies and wave functions.

*Keywords:* Non-inertial cosmic string; Klein-Gordon oscillator.

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## 1. Introduction

Domain walls, cosmic strings, and monopoles are topological defects formed during the vacuum phase transition in the early universe [1–3]. Among them, cosmic strings have garnered significant attention in particle physics, particularly in cosmology and astrophysics, where gravitational effects play a crucial role [4–16]. Cosmic strings do not induce local gravitational interactions; however, they alter the spacetime geometry, resulting in planar and solid angle deficits, respectively [17].

The term Dirac oscillator (DO) was coined by Moshinsky and Szczepaniak [18, 19] in their study of a harmonic oscillator, which introduces strong spin-orbit coupling through the substitution $\vec{p} \to \vec{p} - im\omega\beta\vec{r}$. Its physical applications have been extensively explored by various researchers [20–22], making it the most renowned interaction due to its myriad physical applications and its role in exact solutions of Dirac's equation examples (see Ref. [21] and references therein).

Relativistic wave equations for the (DO) interaction in a cosmic string background constitute a significant field in current research. Their solutions are employed to determine the curvature's influence on various physical properties and to derive the quantum states of these systems [22–38, 40].

The interaction induced by position-dependent mass (PDM) has been a focal point in recent years [41–50]. These quantum systems with dependent effective mass have been studied extensively in both theoretical and applied physics. The primary objective of these studies is to derive eigenfunctions and energy levels using the Schrödinger equation for a system with mass varying with position and subject to a specific potential. Among the applied aspects of PDM are semiconductor heterostructures [45], Helium clusters [51], neutrino mass oscillations [52], quantum wells, and quantum dots [53–58].

Non-inertial effects resulting from a rotating frame in a cosmic string background have been investigated extensively. Recently, several intriguing papers have been published on this subject: Zare *et al*. [27] examined the relativistic generalized DKP oscillator for a spin-zero field in a cosmic string background spacetime characterized by a stationary cylindrical symmetric metric. Santos *et al*. [28] studied the non-inertial effects on a non-relativistic quantum harmonic oscillator in the presence of a screw dislocation. The non-inertial effects of a rotating frame on a spin-zero system with non-commutative geometry in momentum space have been addressed in Ref. [29]. Ahmed [30] investigated the effects of rotation on the KG oscillator with a scalar potential in magnetic cosmic string spacetime using Kaluza–Klein theory. The effects induced by a rotating frame on the Klein–Gordon equation in spacetime with a screw dislocation have been discussed in Ref. [31].

The primary objective of this paper is to analyze the effects of non-inertial forces on the dynamics of a KG oscillator in cosmic string spacetime (with and without position-dependent mass (PDM) settings) characterized by a stationary cylindrical symmetric metric. Our contribution thus introduces a novel aspect to the work by Zare *et al*. [27], which considered the same system but for the DKP oscillator.

The outline of our letter is as follows: In Sec. 2, we present the solution to the KG equation and KG oscillator in a non-inertial cosmic string background. In Sec. 3, we investigate PDM KG-particles in non-inertial cosmic string spacetime. Finally, in Sec. 4, we present our conclusions.

## 2. An overview of the Klein-Gordon equation in curved spacetime

In this section, we delve into the relativistic quantum description of a spin-0 particle propagating in Minkowski spacetime, characterized by the metric tensor $\eta_{\mu\nu} = \text{diag}\,(1,-1,-1,-1)$. For a scalar massive field $\Phi$ with mass $m>0$, the standard covariant Klein-Gordon (KG) equation is given by

$$\left(\eta^{\mu\nu}D_\mu D_\nu + m^2\right)\Phi(x,t) = 0, \tag{1}$$

where $D_\mu = i\left(p_\mu - eA_\mu\right)$ denotes the minimally-coupled covariant derivative.

The equation of motion for a scalar particle in a Riemannian spacetime, characterized by the metric tensor $g_{\mu\nu}$, can be obtained by reformulating the KG equation as [32, 33]

$$\left(\Box + m^2 - \xi R\right)\Phi(x,t) = 0, \tag{2}$$

where

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right), \tag{3}$$

represents the Laplace-Beltrami operator, $\xi$ is a real dimensionless coupling constant, $R$ is the Ricci scalar curvature defined by $R = g^{\mu\nu}R_{\mu\nu}$ where $R_{\mu\nu}$ is the Ricci curvature tensor, $g^{\mu\nu}$ is the inverse metric tensor, and $g = \det\left(g^{\mu\nu}\right)$.

Now, we aim to investigate the quantum dynamics of spin-0 particles in the spacetime induced by non-inertial effects on a cosmic string's Klein-Gordon oscillators.

### 2.1. Free Klein-Gordon equation in non-inertial cosmic string space-time

Let us first derive the KG wave equation for the free relativistic scalar particle propagating in the cosmic string space-time that is assumed to be static and cylindrically symmetric.

The general expression for a (3+1)-dimensional cosmic string metric is defined by the line element [31]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - \alpha^2 r^2 d\varphi^2 - dz^2, \tag{4}$$

in cylindrical coordinates[i]. Here $-\infty \leq t \leq +\infty$, $r \geq 0$, $0 \leq \varphi \leq 2\pi$, $-\infty \leq z \leq +\infty$, and $\alpha \in [0,1[$ is the angular parameter which determines the angular deficit $\delta\varphi = 2\pi(1-\alpha)$, and it is related to the linear mass density $\mu$ of the string by $\alpha = 1-4\mu$.

Consider a string with a linear mass density $\mu$ along the $z$-axis, with the Lorentz metric $ds^2 = dt^2 - dx'^2 - dy'^2 - dz'^2$, and the coordinate changes

$$x' = R\,\cos(\alpha\boldsymbol{\Phi}), y = R\,\sin(\alpha\boldsymbol{\Phi}), z = \boldsymbol{Z}, \text{and}\; t = \boldsymbol{T},$$

which leads to the line element of a cosmic string space-time with the cylindrical coordinates [34–38]

$$ds^2 = d\boldsymbol{T}^2 - d\boldsymbol{R}^2 - (\alpha\boldsymbol{R})^2\,d\boldsymbol{\Phi}^2 - d\boldsymbol{Z}^2. \tag{5}$$

In addition to the presence of a dislocation, we will analyze a frame that rotates evenly with a constant angular velocity $\Omega$. To include this rotation into our line element, we use the coordinate transformation used in Refs. [35–38] to obtain

$$ds^2 = \left(1-\alpha^2\Omega^2 r^2\right)dt^2 - 2\Omega\alpha^2 r^2 d\varphi dt - dr^2 - \alpha^2 r^2 d\varphi^2 - dz^2. \tag{6}$$

Since we do not want the term $g^{00}$ to become positive, we impose that the corresponding radial wave function must vanish at some

$$r \to r_0 = \frac{1}{\alpha\Omega}. \tag{7}$$

When the metric and inverse metric tensor components are, respectively,

$$g_{\mu\nu} = \begin{pmatrix} \left(1-(\alpha\Omega r)^2\right) & 0 & -\Omega\,(\alpha r)^2 & 0 \\ 0 & -1 & 0 & 0 \\ -\Omega\,(\alpha r)^2 & 0 & -(\alpha r)^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & -\Omega & 0 \\ 0 & -1 & 0 & 0 \\ -\Omega & 0 & \left(\Omega^2 - \frac{1}{(\alpha r)^2}\right) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{8}$$

we arrive at the following second order differential equation for the radial function $\psi(r)$ , after some simple algebraic manipulations.

$$\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left(\sqrt{-g}\frac{\partial}{\partial r}\right) - \frac{(j)^2}{\alpha^2 r^2} + (E+\mathit{\Omega} j)^2 - m^2 - K^2\right]\psi\,(r) = 0, \tag{9}$$

and consequently

$$\gamma^2 = (E+\mathit{\Omega} j)^2 - m^2 - K^2, \tag{10}$$

and the rest of the results should be corrected accordingly. Setting

$$\varsigma^2 = \frac{j^2}{\alpha^2}, \tag{11}$$

yields

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\varsigma^2}{r^2} + \gamma^2\right]\varphi_s\,(r) = 0. \tag{12}$$

Equation (12) represents a Bessel differential equation, and its solutions can be expressed in terms of a first-order Bessel function as:

$$\psi\,(r) = A'\,J_{\frac{|j|}{\alpha}}\,(\gamma r)\,, \tag{13}$$

where $A'$ denotes the normalization factor. We may now seek finiteness of such a radial part at $r = r_0 = 1/\Omega\alpha$ so that $J_{|j|/\alpha}(\gamma r_0) = 0$ for $\gamma r_0 \gg 1$. In this case,

$$J_{\frac{|j|}{\alpha}}(\gamma r_0) \rightarrow \sqrt{\frac{2}{\pi\gamma r_0}} \cos\left(\gamma r_0 - \frac{\pi|\varsigma|}{2} - \frac{\pi}{4}\right) = 0, \quad (14)$$

to imply the energies in the form of

$$E_n^{\pm} = -\Omega j \pm \sqrt{m^2 + K^2 + \alpha^2\pi^2\Omega^2\left(n + \frac{|j|}{2\alpha} + 3/4\right)^2}. \quad (15)$$

### 2.2. Klein-Gordon oscillator in non-inertial Cosmic String Space-time

We start by considering a scalar quantum particle embedded in the background of gravitational field space-time described by metric (5). To retrieve the Klein-Gordon oscillators we replace the momentum operator $p_k = -i\partial_k$ by its non-minimal coupling form $p_k = -i\partial_k - i\mathcal{F}_k$, where $\mathcal{F}_k = (\mathcal{F}_r, 0, 0); \mathcal{F}_r = ar$ for KG-oscillators[iii] and $\mathcal{F}_r = \partial_r - (\partial_r f(r)/4f(r))$, where $p_k$ is the position-dependent mass momentum operator.

In this case, one would rewrite the KG-equation as

$$\left[\frac{1}{\sqrt{-g}}(\partial_\mu + \mathcal{F}_\mu)\sqrt{-g}\,(\partial_\nu - \mathcal{F}_\nu) + m^2\right] \times \Psi(t, r, \phi, z) = 0, \quad (16)$$

to obtain

$$\left[\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial r} + \mathcal{F}_r\right)\sqrt{-g}\left(\frac{\partial}{\partial r} - \mathcal{F}_r\right) - \frac{(j)^2}{\alpha^2 r^2} + (E + \Omega j)^2 - m^2 - K^2\right]\psi(r) = 0. \quad (17)$$

Following the same procedure as described in section II, we get

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \mathcal{M}(r) - \frac{\tilde{j}^2}{r^2} + \delta\right]\psi(r) = 0, \quad (18)$$

where $\tilde{j} = j/\alpha$, and

$$\delta = (E + \Omega j)^2 - m^2 - K^2,$$
$$\mathcal{M}(r) = \mathcal{F}_r/r + \mathcal{F}_r' + \mathcal{F}_r^2. \quad (19)$$

Let us now take $\mathcal{F}_r = ar$, to obtain KG-oscillators equation, and use $\psi(r) = R(r)/\sqrt{r}$ to yield

$$\left[\frac{\partial^2}{\partial r^2} - \frac{(\tilde{j}^2 - 1/4)}{r^2} - a^2 r^2 + \tilde{\delta}\right]R(r) = 0, \quad (20)$$

where

$$\tilde{\delta} = \delta - 2\,a. \quad (21)$$

This equation represents KG-oscillators as it resembles the two-dimensional Schrödinger radial oscillators (hence the notion KG-oscillators) and admits exact solution.

Manipulating exactly the same steps before, we obtain the following radial equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - a^2 r^2 - \frac{\vartheta^2}{r^2} + \tilde{\delta}\right]\psi(r) = 0, \quad (22)$$

where we have defined

$$\vartheta^2 = \frac{j^2}{\alpha^2}, \qquad \tilde{\delta} = (E + \Omega j)^2 - m^2 - K^2 - 2a. \quad (23)$$

To solve Eq. (22), we introduce a new dimensionless variable $\mathcal{U} = ar^2$, and by substitution into Eq. (22),the resulting equation reads

$$\left[\frac{d^2}{d\mathcal{U}^2} + \frac{1}{\mathcal{U}}\frac{d}{d\mathcal{U}} - \frac{\vartheta^2}{4\mathcal{U}^2} - \frac{1}{4} + \frac{\tilde{\delta}}{4a\mathcal{U}}\right]\psi(\mathcal{U}) = 0. \quad (24)$$

Consider the following change of variable

$$\psi(\mathcal{U}) = \mathcal{U}^{-\frac{1}{2}}\mathcal{J}(\mathcal{U}), \quad (25)$$

then, Eq. (24) becomes

$$\frac{d^2\mathcal{J}(\mathcal{U})}{d\mathcal{U}^2} + \left[-\frac{1}{4} + \frac{\tilde{\delta}}{4a\mathcal{U}} + \frac{\frac{1}{4} - \left(\frac{\vartheta}{2}\right)^2}{\mathcal{U}^2}\right]\mathcal{J}(\mathcal{U}) = 0, \quad (26)$$

which has the form of the Whittaker differential equation [59, 60]. The general solution of this equation, which is regular at the origin, is given by

$$\mathcal{J}(\mathcal{U}) = |C|\mathcal{U}^{-\frac{1}{2}} M\left(\frac{\tilde{\delta}}{4a}, \frac{|\vartheta|}{2}, \mathcal{U}\right), \quad (27)$$

where $|C|$ is an arbitrary constant and $M\left(\tilde{\delta}/4a, |\vartheta|/2, \mathcal{R}\right)$ is the Whittaker M-function defined via the confluent hypergeometric functions as [34]

$$M_{\delta,\vartheta}\left(\frac{\tilde{\tilde{\delta}}}{4a}, \frac{|\vartheta|}{2}, \mathcal{U}\right) = e^{-\frac{\mathcal{U}}{2}}\mathcal{U}^{\frac{|\vartheta|+1}{2}} \times {}_1F_1\left(\frac{|\vartheta|}{2} - \frac{\tilde{\delta}}{4a} + \frac{1}{2}, |\vartheta| + 1, \mathcal{U}\right). \quad (28)$$

The other solution is the Whittaker W-function given by

$$W_{\tilde{\delta},\vartheta}\left(\frac{\tilde{\delta}}{4a}, \frac{|\vartheta|}{2}, \mathcal{U}\right) = e^{-\frac{\mathcal{U}}{2}}\varrho^{\frac{|\vartheta|+1}{2}} \times U\left(\frac{|\vartheta|}{2} - \frac{\tilde{\delta}}{4a} + \frac{1}{2}, |\vartheta| + 1, \mathcal{U}\right), \quad (29)$$

which $U\left([|\vartheta|/2]-[\tilde{\delta}/4a]+[1/2],|\vartheta|+1,\varrho\right)$ is the Tricomi confluent hypergeometric function (or Kummer's function of the second kind) [59, 60]. In general, $U(a,b,z)$ has a branch point at $z=0$, *i.e*, it has a singularity at zero. Thus we keep only the solution described by (28).

Using the definition (28), the final expression of the wavefunction of the spinless KGO propagating in the non-inertial effects on a cosmic strings can be represented as

$$\psi(\mathbf{r})=|\mathcal{C}_2|\left(ar^2\right)^{\frac{|\vartheta|}{2}}e^{-\frac{a}{2}r^2}e^{-i(Et-j\varphi-iKz)}\times{}_1F_1\left(\frac{|\vartheta|}{2}-\frac{\tilde{\delta}}{4a}+\frac{1}{2},|\vartheta|+1,ar^2\right),\qquad(30)$$

where the parameters $\vartheta$ and $\delta$ are defined in Eq.(23).

Again, The asymptotic behavior of the confluent hypergeometric function implies that

$$\frac{|\vartheta|}{2}-\frac{\tilde{\delta}}{4a}+\frac{1}{2}=-n,\qquad(31)$$

hence, after inserting $\vartheta$ and $\tilde{\delta}$, and by solving Eq. (31) for $E$, we obtain the energy levels for our scalar particle

$$E_n^{\pm}=-\Omega j\pm\sqrt{m^2+K^2+2a\left(2n+\frac{|j|}{\alpha}+2\right)}.\qquad(32)$$

The relativistic energy levels (32) represent the energy spectrum of the Klein-Gordon oscillator within a non-inertial cosmic string spacetime background. The inclusion of distinct geometrical parameters $\alpha$ and $\Omega$ modifies the degenerate spectrum of the particle. Unlike the scenario with the Dirac oscillator [14, 16] and in contrast to flat space, the presence of these defects disrupts the degeneracy of energy levels. As $\alpha$ tends toward 1 and $\Omega$ tends toward 0, and with $N=2n+j$ denoting the principal quantum number, we retrieve the energy spectrum of the Klein-Gordon oscillator in flat space [14, 16].

In the classical limit, utilizing the relation $E'=E+\Omega j=\varepsilon+m$, and considering the non-relativistic condition $\varepsilon\ll m$, Eq. (32) simplifies to

$$\frac{E'^2-m^2}{2m}\approx\epsilon=2a\left(2n+\frac{|j|}{\alpha}+2\right).\qquad(33)$$

We observe that only the presence of $\alpha$ in equation ((33)) breaks the degeneracy of the spectrum. The other parameter $\Omega$ has no influence in the classical limit.

To examine the impact of both $\alpha$ and $\Omega$ parameters on the energy spectrum, we graphed the energy of the Klein-Gordon oscillator in non-inertial cosmic string spacetime across different values of $\alpha$ and $\Omega$ (where $j=a=1$).

Figure 1 shows that when the value of $\alpha$ is bigger, the energy spectrum becomes cramped. Also, we plotted the energy density of Eq. (30)

By analyzing these results, we can estimate the density of the probability of KGO in the non-inertial cosmic string space-time. The purpose of this study is to determine how $\alpha$ and $\Omega$ affect this density. The density of the positive energy spectrum of KGO is provided by the following equation

$$\rho_{\mathrm{KG}}(r)=2E\psi^*\psi.\qquad(34)$$

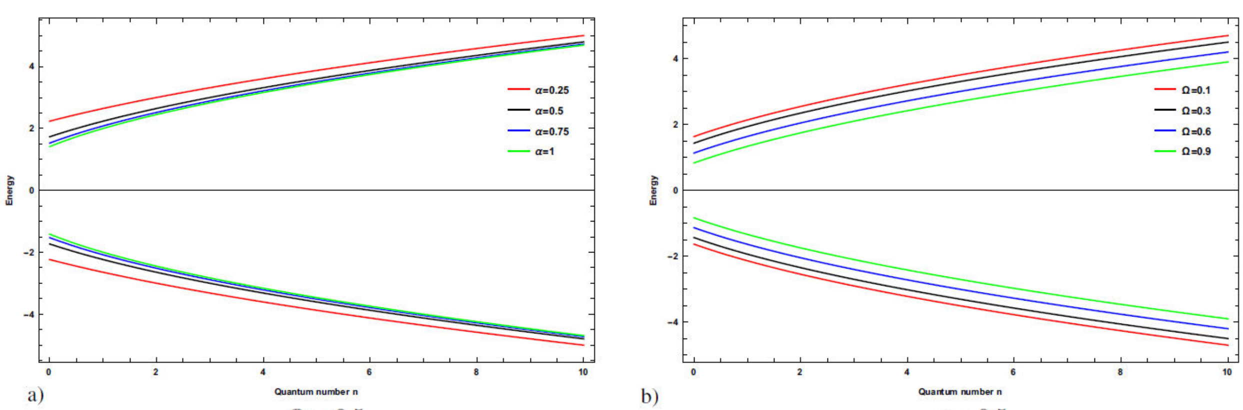
Figures 2 and 3 illustrate the probability density of KGO in the non-inertial cosmic string space-time with respect to the radial distance $r$ for four levels $n=0,1,2,3$ for different values of $\alpha$ and $\Omega$.

Based on the depicted figure, several observations emerge:

- The probability density of Klein-Gordon oscillators (KGO) in non-inertial cosmic string spacetime is notably affected by the choice of the quantum number $n$, alongside the parameters $\alpha$ and $\Omega$.

- With a constant value of $\alpha$ (refer to Fig. 2) and varying $\Omega$, the following trends are evident:

  - The magnitudes of each density peak exhibit considerable variations.




a) $\Omega=0.5$ b) $\alpha=0.5$

FIGURE 1. Energy Spectrum of KG oscillator in Non-inertial cosmic string for different values of $n$.

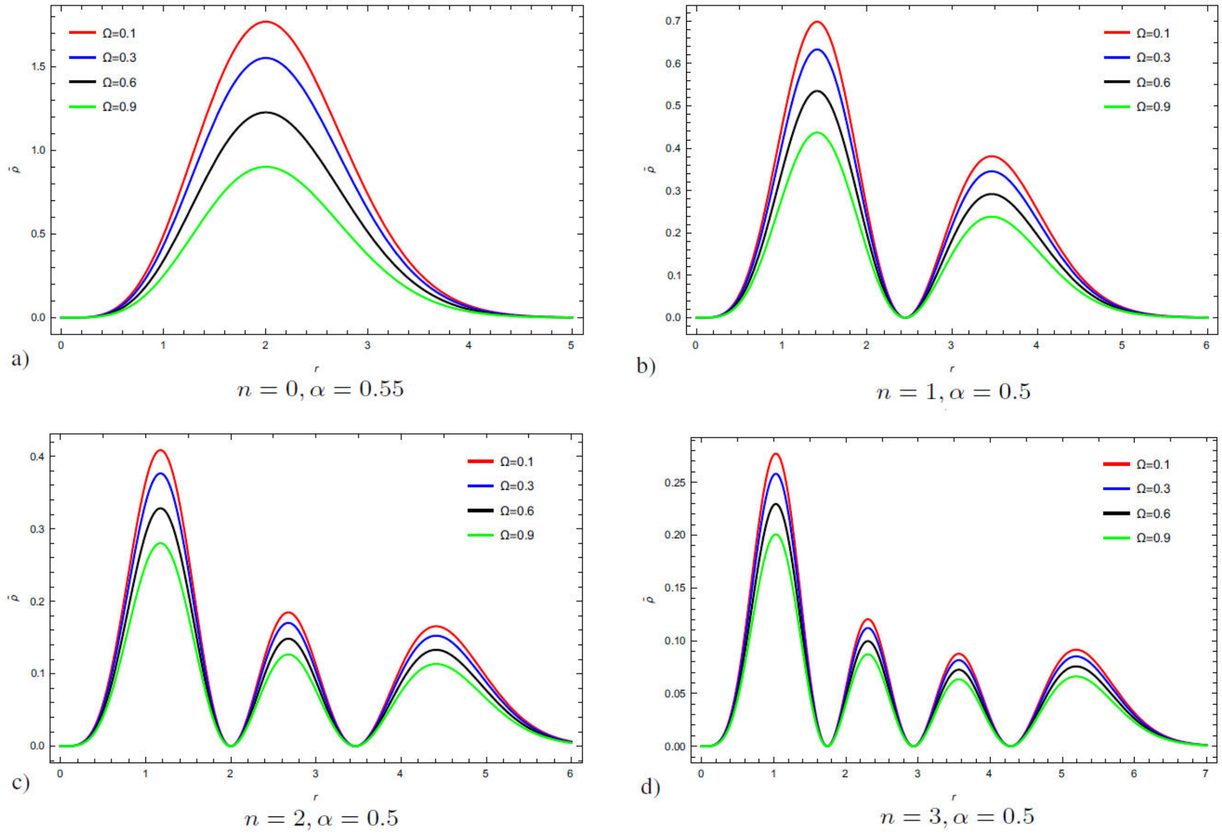



FIGURE 2. Plots of the positive density of KG oscillator in Non-inertial cosmic string for different values of $\Omega$.

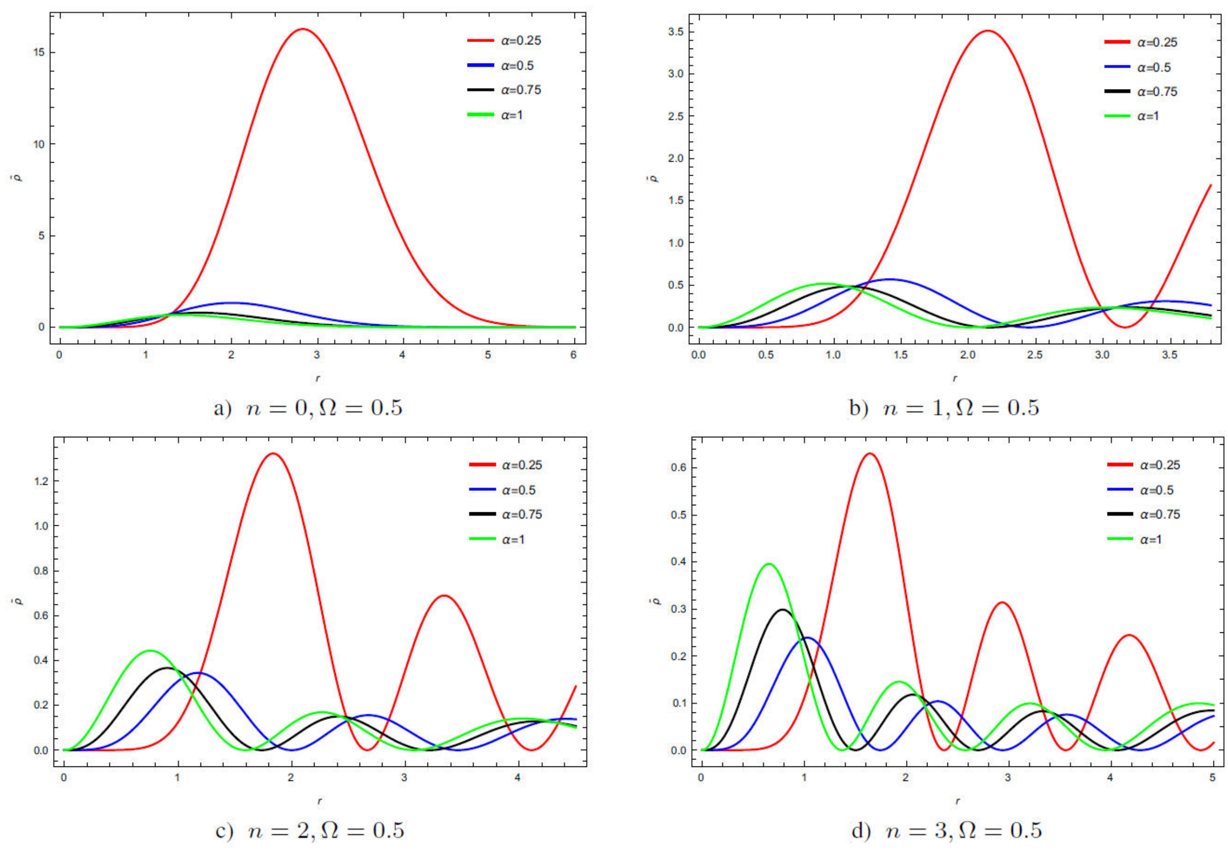



FIGURE 3. Plots of the positive density of KG oscillator in Non-inertial cosmic string for different values of $\alpha$.

  - The quantity of these peaks increases as $n$ rises. Additionally, they display symmetry at a fixed point $r$, with the width of each peak diminishing as the quantum number $n$ escalates.

- Conversely, for a constant $\Omega$ (as shown in Fig. 3) and diverse $\alpha$ values, the scenario differs markedly from the previous case:
  - At a fixed rotational value $\Omega$, the intensity of each density peak declines as the angular $\alpha$ deficit increases. Moreover, the peaks lack symmetry.
  - Similar to the prior case, the number of peaks multiplies with increasing $n$.
  - As with the previous observation, the width of each peak diminishes with the increasing quantum number $n$.

We are now ready to discuss the PDM KG-particles in non-inertial cosmic string space-time.

## 3. PDM KG-particles in non-inertial Cosmic String Space-time

In this section, we wish to discuss PDM KG-particles in non-inertial Cosmic String Space-time using the substitution of

$$\mathcal{F}_r = ar + \frac{f'(r)}{4f(r)}, \tag{35}$$

in Eq. (17) and consider two cases of fundamental interest, PDM KG-oscillators and KG-Coulombic like [41–50] particles.

### 3.1. PDM KG-oscillators

We consider the case $a = 0$ and $f(r) = \exp(2\eta r^2)$ to obtain $\mathcal{M}(r) = 2\eta + \eta^2 r^2$ (19). Using such settings in Eq. (18) we obtain

$$\left[\frac{\partial^2}{\partial r^2} - \frac{(\tilde{j}^2 - 1/4)}{r^2} - \eta^2 r^2 + \tilde{\delta}\right] R(r) = 0, \tag{36}$$

where

$$\tilde{\delta} = \delta - 2\eta. \tag{37}$$

This equation represents KG-oscillators for it resembles the two-dimensional Schrödinger radial oscillators and admits exact solution, with $\psi(r) = R(r)/\sqrt{r}$,

$$\psi(r) = C \exp\left(-\frac{\eta}{2} r^2\right) r^{|\tilde{j}|} \times {}_1F_1\left(\frac{1}{2} + \frac{|\tilde{j}|}{2} - \frac{\tilde{\delta}}{4\eta}, 1 + |\tilde{j}|, \eta r^2\right), \tag{38}$$

and the condition of finite polynomial solution requires that

$$\frac{1}{2} + \frac{|\tilde{j}|}{2} - \frac{\tilde{\delta}}{4\eta} = -n_r, \tag{39}$$

where $n_r = 0, 1, 2, \cdots$ is the radial quantum number.

Consequently,

$$\tilde{\delta} = 2\eta\left(2n_r + |\tilde{j}| + 1\right), \tag{40}$$

or

$$\delta = 2\eta\left(2n_r + \left|\frac{j}{\alpha}\right| + 2\right), \tag{41}$$

that would, using (19), yield

$$E_n^{\pm} = -\Omega j \pm \sqrt{m^2 + K^2 + 2\eta\left(2n'_r + \frac{|j|}{\alpha} + 2\right)}. \tag{42}$$

with $n'_r = n_r + 1$. Equation (42) represents the energy spectrum of the Position-Dependent Mass (PDM) Klein-Gordon oscillators. This expression bears resemblance to (32). Here $\eta$ has a role of frequency compared with (32). It depends on various geometrical parameters $\alpha$ and $\Omega$, along with the parameter $\eta$ denoting the position-dependent masses (PDM). Additionally, the geometrical parameter also breaks the degeneracy of the energy spectrum. In the classical limit, employing $E' = E + \Omega j = \varepsilon + m$, and under the non-relativistic condition $\varepsilon << m$, Eq. (32) simplifies to

$$\frac{E'^2 - m^2}{2m} \approx \epsilon = 2\eta\left(2n'_r + \frac{|j|}{\alpha} + 2\right). \tag{43}$$

Similar to the previous scenario, we observe that only the presence of $\alpha$ in Eq. (43) disrupts the degeneracy of the spectrum. The parameter $\Omega$ exerts no influence in the classical limit.

### 3.2. PDM KG-Coulombic like particles

We now consider $f(r) = J_0(2\sqrt{Ar})^4$ to imply that $\mathcal{M}(r) = -(A/r)$ (19). This would allow us to write (18), with $\psi(r) = R(r)/\sqrt{r}$, as

$$\left[\frac{\partial^2}{\partial r^2} - \frac{(\tilde{j}^2 - 1/4)}{r^2} + \frac{A}{r} + \delta\right] \psi(r) = 0. \tag{44}$$

This is represents the KG-Coulombic particles for it resembles the two-dimensional Schrödinger radial Coulombic particles and admits exact solution in the form of

$$R(r) = B\, e^{-\tilde{A}r/2}\, r^{|\tilde{j}|+1/2} \times {}_1F_1\left(\frac{1}{2} + |\tilde{j}| + \frac{iA}{2\sqrt{\delta}}, 1 + 2|\tilde{j}|, 2i\sqrt{\delta} r\right). \tag{45}$$

where$B$ is a normalization factor. We now need to satisfy the condition

$$\frac{1}{2} + |\tilde{j}| + \frac{iA}{2\sqrt{\delta}} = -n_r$$

so that the confluent hypergeometric series is truncated into a polynomial of of order $n_r$. This would yield to

$$\sqrt{\delta} = -i\,\tilde{A} \Rightarrow \delta = -\tilde{A}^2, \quad \tilde{A} = \frac{A}{2(n_r + \frac{|j|}{\alpha} + \frac{1}{2})}, \tag{46}$$

and with $\psi(r) = R(r)/\sqrt{r}$ we obtain

$$\psi(r) = N_{\text{norm}}\, e^{-\tilde{A}r/2}\, r^{\frac{|j|}{\alpha}} \times {}_1F_1\left(-n_r, 1 + 2\frac{|j|}{\alpha}, 2\tilde{A}r\right), \tag{47}$$

where $N_{\text{norm}}$ denotes the normalization factor.

With $\delta$ in Eq. (19) we obtain

$$(E + \Omega j)^2 - m^2 - K^2 = -\tilde{A}^2, \tag{48}$$

and

$$E_n^{\pm} = -\Omega j \pm \sqrt{m^2 + K^2 - \frac{A^2}{4(n_r + \frac{|j|}{\alpha} + \frac{1}{2})^2}}. \tag{49}$$

This form can be rewritten in the two dimensions as

$$E_n^{\pm} = -\Omega j \pm m\sqrt{1 - \frac{A^2}{4m^2(n_r + \frac{|j|}{\alpha} + \frac{1}{2})^2}}. \tag{50}$$

Equation (50) depicts the energy spectrum form within the PDM KG-Coulombic scenario. It is evident that various parameters such as $A$ from the Coulombic potential, along with $\alpha$ and $\Omega$, the geometric parameters of the curved spacetime, significantly influence this spectrum. In the non-relativistic approximation, the behavior of the spectrum of energy, for very small values of the constant $A$, can be expanding in a power series in $A$ as follows

$$E_n^{\pm} = m - \Omega j - \frac{A^2}{8mN^2} - \frac{A^4}{64mN^4}, \tag{51}$$

where

$$N = \left[n_r + \frac{|j|}{\alpha} + \frac{1}{2}\right], \tag{52}$$

is the principal quantum number, and $[N]$ means the biggest integer inferior to $N$. The components of Eq. (51) can be understood as follows: the initial term represents the particle's rest energy adjusted by the factor $j\Omega$. The second segment mirrors the energy of a particle with mass $m$ in a Coulombic field under non-relativistic conditions. This portion is contingent upon the spatial geometric parameters denoted by $\alpha$. The third segment elucidates the relativistic adjustment to the energy. Notably, this correction relies on the geometric parameter of spacetime, $\alpha$.

## 4. Conclusion

In this study, we investigate the fascinating interaction between the Klein-Gordon oscillator and a non-inertial cosmic string defect. Our primary objective is to ascertain the system's energy and elucidate the repercussions of this interaction. Notably, we observe that the energy density remains entirely positive for positive energy states $E^+$, whereas it turns negative for negative energy states $E^-$. To tackle this issue, we advocate for the utilization of the Feshbach-Villars Approximation method, previously examined in [34], which has consistently produced entirely positive outcomes. Additionally, we have derived the eigensolutions of the problem at hand and scrutinized the influence of non-inertial effects and the presence of the PDM on the eigenvalues.

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$i$. Note that this metric is an exact solution to Einstein's field equations for $0 \leq \mu < 1/4$, and by setting $\varphi' = \alpha\varphi$, then it represents a flat conical exterior space with angle deficit $\delta\phi = 8\pi\mu$.

$ii$. Not $\mathcal{F}_r = m\omega r$ that yields dimensional inconsistency, see our equation below, where $m^2 = m^2c^4$ multiplied by $\omega^2 r^2$ to give an overall dimensionality of energy to power 3 instead of power 2.

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