ATTEMPTS AT QUANTIZATION OF THE GRAVITATIONAL FIELD * +

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The discovery of many new types of strange particles during recent years has drawn new attention to the fact that we really don't understand *why* those particles exist with the properties we observe. Why is a proton 1836 times heavier than an electron? Why is there no neutral μ meson of mass 200? Why is $\hbar c/e^2$ equal to 137? An ultimate theory of matter should explain such things.

Heisenberg[1] thinks that such an ultimate theory will describe all particles and all of their interactions by the behavior of one single field. Such a field theory would necessarily be *non-linear*; otherwise, it could not account for the interactions. Describing everything, it must also describe *gravity*. In order to get used to these two features of the ultimate theory of matter, we do well if we prepare for the future by studying now the non-linear theory of gravitation which we do already possess.

What keeps elementary particles together? Pais[2] once suggested a glue made out of hypothetical "f-particles". Others[3-5] have suggested that perhaps

* Supported by National Science Foundation.

+ Invited paper read before the joint meeting of the Mexican and American Physical Societies in Mexico, August 31, 1955.

it is the crookedness of spacetime itself which under interaction with the field cannot untangle itself. This makes it interesting to consider gravity in the domain of the quantum. The ultimate problem is complicated by the fact that perturbation methods probably cannot handle it.

A complete and rigorous solution of the problem of interaction of all quantum fields known, including gravity, is beyond our present capacity. All that the scientist can do is nibble at the circumference of our present knowledge. Such nibbling sets definite goals; insufficient goals in themselves; but, anyhow, we hope that the small results which we *can* attain will help us at least *towards* the ideal we have in mind. The question I have been asked to discuss with you to-day, then, is how far we have already nibbled, and, perhaps, at what we may be nibbling next.

For saving time, let me be schematic. The attempts at quantizing the gravitational field I want to divide into three groups, as follows:

(1) Linear theories of gravitation in flat space, which seek to be complete in themselves.

(2) Theories, usually in flat space, which seek to be approximations to Einstein's theory, or a perturbation-theoretical treatment of Einstein's theory.

(3) Einstein's non-linear but generally-covariant theory itself.

Our ultimate goal may be this third theory, but it also is the hardest of the three. For an easy start, one would prefer a theory of type (1). This is just the problem at which we worked at Purdue until last year. Some of it has been published in a report based on the doctor's thesis of Swihart[8].

Such a linear theory of gravitation obviously cannot be generally covariant, so we postulate Lorentz-covariance only. We start by postulating the most general Lorentz-covariant Lagrangian for the gravitational potential tensor, that is compatible with a linear theory and with the other usual requirements of simple field theories. Because the heavy mass of matter is to be equal to the inert mass, the gravity field should interact with the energy density tensor of other fields; possibly also with its trace as calculated neglecting gravity. This gives two interaction constants for the theory; and the free-gravity Lagrangian contains four un-known coefficients.

ıese six constants, or, rather, their five ratios, then are to be adjusted later so as to fit the experimental data.

Our theory therefore is a generalization of Moshinsky's[7] theory of 1950, whose Lagrangian had specific values for our constants. For the gravity field created by given matter, Moshinsky's Lagrangian led to Birkhoff's equations[8]. For the force by which gravity acts back on matter, Moshinsky's Lagrangian leads to a result different from that postulated by Birkhoff. Birkhoff's postulate, in fact, probably violates the principle of action and reaction. Moshinsky's interaction of gravity with matter would lead to an advance of the perihelion of planets which is six times too small.

We use the following principles for determining the values of the five ratios of the six constants in our Lagrangian. Obvious requirements are:

1) In first approximation, the theory must predict the Kepler orbits for planets;
2) In second approximation, the perihelions of such orbits have to advance in the observed way[9].
3) The bending of light rays passing the sun at some distance should take the experimental value[10-11], which is equal to or larger than Einstein's theoretical value.
4) & 5) Merely for convenience, we have arbitrarily postulated that the static gravitational potential tensor at a point P, near the sun stationary at O, shall be independent of the direction of OP. This considerably simplifies the theory, and allows us to put one of our constants equal to zero, if at the same time we use our fifth freedom of choice by putting another constant equal to zero, Thus, in the free-gravity Lagrangian only those terms are left, in which the two gradient operators are contracted with each other.

We cannot freely choose the value to be predicted by our theory for the gravitational red shift. Our theoretical value for this red shift is already uniquely determined by our first requirement of obtaining approximately Kepler orbits for the planets. We thus find for the red shift the same value as predicted by Einstein's theory. If Freundlich's experimental criticism of this value would be correct[11-12], this would defeat not only Einstein's theory, but it would kill our linear theory as well.

We thus obtain a Lagrangian different from a mere linear approximation to Einstein's theory, and also different from Moshinsky's Lagrangian. Moshinsky's Lagrangian should lead to only half the observed bending of light rays. The only reason why he got in his paper the full observed amount, and why from his spinor matter interaction he would have found a perihelion motion only three instead of six times too small if he had calculated it, is because he used Birkhoff's assumption of a pressure inside the perfectly fluid sun probably some million times too large[7-8]. The effect of the difference between his and our interactions between light and gravity is negligible.

Once the Lagrangian has been determined, one can quantize. We found a canonically conjugate momentum for every gravitational potential component. As I will mention later, this is never so in a generally covariant theory[13-14], and thus it seems that our linear theory cannot possibly be an approximation to Einstein's theory, whatever otherwise its merits may be.

We have completed the entire quantization program for our theory. We have given a proof of the existence of an interaction representation, and thus we have demonstrated the Lorentz-covariance of the commutation relations. This proof turned out to be extremely complicated, and took us many months. Just one source of complication was in the fact that in Heisenberg representation one cannot simply use the Dirac wave function ψ as a probability amplitude to be quantized in the usual way. First a new wave function has to be introduced. The work of the DeWitt's[15] has shown that a similar difficulty arises in the generally-covariant theory of interaction between the electron spinor field and Einstein's gravitational field.

When we expressed our free-graviton energy in terms of occupation number operators, it became clear that for avoiding negative-energy gravitons one should use a Gupta indefinite metric[16-18], and then impose an auxiliary condition on the state vector. Unfortunately, this auxiliary condition is not rigorously conserved. This is due to the fact that the ***source*** of the gravitational field in a linear theory cannot be conserved. For, on the one hand, one would destroy the linearity of the theory, if one would *include* among the sources of gravity the gravitational energy itself; while, on the other hand, a falling stone transforming gravitational energy into matter energy shows that, *without* such inclusion of gravitational energy, the matter energy alone is not conserved.

The best we would do in our linear theory was postulating that the auxiliary condition was satisfied when God created the universe. Since then, its validity slowly may have been worsening. This turns out to be equivalent with postulating that, since the creation of the universe, matter may have been radiating retarded gravitational fields[19]. Therefore one may ask how much energy has been lost by accelerated matter such as a rotating double star in, say 10^{9}, or 10^{10} years of existence of the universe. Luckily I found this amount to be so small that there is no reason for worry. More important is that matter in absence of incident external gravitational fields can be shown to emit always a bit more positive gravitational energy than negative gravitational energy. Therefore, there is no danger either of an explosion of the universe in which matter suddenly goes wild under emission of negative-energy gravitons.

Thus our theory saves face; but I agree that it is not a nice solution. Therefore, let us now consider theories of type(2); those that are meant as approximations to Einstein's rigorous theory.

The earliest attempts simply took the first linear approximation in an expansion of Einstein's theory in powers of the difference between the actual metric $g_{\mu\nu}$, and its Lorentz-invariant flat-space appoximation $\gamma_{\mu\nu}$. This leads to an advance of the perihelion of planetary orbits, but not quite enough of it. Therefore one does not feel happy unless one knows at least how to deal with the second app roximation.

Progress was made by Papapetrou[20] in 1948. He used a mathematical trick first proposed and later mis-used by Rosen[21]. The trick consists in first choosing some preferred frame of reference, and then in that frame introducing the Lorentz flat-space metric $\gamma_{\mu\nu}$ besides the actual metric $g_{\mu\nu}$. Under *general* transformations then, $\gamma_{\mu\nu}$ transforms as a tensor, and will lose its simple form, except in a Lorentz manifold of frames of reference of which the original frame was one. If somebody else uses a different Lorentz manifold as his starting point, this would amount to a kind of gauge transformation of $\gamma_{\mu\nu}$.[5]

Next, using this γ-metric, Papapetrou defined a new energy-momentum density tensor of matter and gravity together, which had the following two interesting properties: First, was rigorously conserved on account of the Einsteinian gravitational field equations. And, second, it was a symetric tensor.

Essentially, this energy density tensor was little else than the sum of the old-fashioned ***symmetric matter*** tensor and ***non***-symmetric ***gravity*** energy tensor plus what one might call the "***gravity spin*** energy density tensor". Thus, one obtains a mixed tensor, if the first index distinguishing energy from momentum is pulled down by $g_{\mu\nu}$ as a covariant index, and the other index referring to density or flux is kept in a contravariant top position. Then, if in the result the first covariant index is pushed up again, but now by the γ-metric, one obtains the symmetric tensor just mentioned.

Papapetrou then re-wrote Einstein's gravitational field equations using the contravariant density tensor german- $\mathfrak{g}^{\mu\nu}$ - superscripts instead of the variables ordinary- $g_{\mu\nu}$-subscripts. He found this to take the form of a second-order differential equation much alike that for the potentials in Lorentz-covariant Maxwell theory. His new symmetric energy tensor figured in it as source of the field. In the Maxwell case, the differential operator simplifies to the Dalembertian, if one imposes the Lorentz condition on the potentials. Similarly, the differential operator in Papapetrou's equation simplifies to the Dalembertian, if one imposes De Donder's coordinate condition $\partial \mathfrak{g}^{\mu\nu}/\partial x^{\nu} = 0$.

These facts served as the basis of Gupta's work on gravity[17-22]. He starts by suggesting that Papapetrou's equation may be used for solving for the gravitational field in successive approximations, using the lower approximations to the gravity field in the Papapetrou energy tensor, for calculating then the next-higher approximation as retarded solution from the simplified field equation.

Unfortunately, such a procedure breaks down, because the intermediate approximations to the Papapetrou stress tensor do not satisfy a conservation law in this way, and then are meaningless as source for a gravitational field.

As to Gupta's quantization of the field, let me first remark that Gupta never proved the Lorentz-covariance of his commutation relations. This Lorentz-covariance is not entirely obvious, since Gupta, after expanding his Lagrangian, destroyed its general covariance by using the auxiliary condition for simplifying it, much like Fermi's simplification of the Lorentz-covariant Lagrangian of Maxwell's theory.

There is also an arbitrariness in the extent to which this simplification is to be used in the higher-order terms.

Finally, Gupta's proof of conservation of the De Donder auxiliary condition is incorrect, as it ignores the fact that the change of the Lagrangian leads to a change in the conserved symmetric Papapetrou energy tensor as well. One can give a more complicated proof of the conservation of the De Donder condition; but with Gupta's quantum-mechanical interpretation of the auxiliary condition that proof breaks down, too. A way out of this difficulty is by first replacing all ordinary derivatives in the term added to the Lagrangian by Rosen's γ-covariant derivatives[21], and then quantizing the $\gamma_{\mu\nu}$-field as well as the $g^{\mu\nu}$-field. But the result is a generally covariant theory even more complicated than Einstein's theory itself.

Therefore, let us rather proceed to the real problem: Quantization of Einstein's theory itself. The work done by Dirac[23], by Schild, Pirani, and Skinner[24], by the DeWitt's[15,25] but before and after all by Bergmann and his school[14,26-34], is of utmost interest, even if it has not been able yet to give the theory a practicable form. What these workers have shown, though, is that in principle it is ***possible*** to establish a consistent and covariant set of ***commutation relations*** for the generally covariant theory of gravity.

This result is obtained mainly by a careful study of the *non*-quantized theory. The fundamental difficulty, which ***quantization*** meets, was first discovered by Rosenfeld[13] in a much neglected paper of 1930, and was later rediscovered by Bergmann[14] in 1949. This difficulty consists in the fact that, in any theory invariant under a group of transformations characterized by arbitrary ***functions*** $\xi(xyzt)$ -and not just by constants $\underset{\sim}{v}$ -, there necessarily must exist identities which hold between the ***canonical variables*** of such a theory. In electrodynamics you all know such a relation: due to its gauge invariance, the momentum P_4 canonically conjugate to the electric potential Φ vanishes. Similarly, due to the *general* covariance of Einstein's theory of gravitation, there hold four so-called "coordinate constraints" between the field variables $g_{\mu\nu}$ and their conjugate momenta These constraints, also called "ϕ-equations" [23], can be given explicitly [15,24]. The reason why they did *not* appear in my work with Swihart [6], or in Gupta's work [17,22], is that we never had general covariance, and that Gupta destroyed his.

Dirac[23] has shown that these so-called "primary" constraints should appear added to the Hamiltonian with arbitrary coefficients. The Hamiltonian equation of motion then express these arbitrary coefficients in terms of time derivatives of the field variables, the so-called "velocities". An important feature of Dirac's theory is that, from the start, it distinguishes so-called "strong" equations, which shall remain valid after quantization, from "weak" equations, which become auxiliary conditions. The quantum-mechanical interpretation of the latter, and Dirac's rule that the product of two weak equations shall be strong, are points which need some qualification, and have been explained by Bergmann[31,34] on the basis of ideas earlier used by myself in quantum electrodynamics[35,36], and which essentially mean using only a subspace of the conventional Hilbert space.

Typical for Dirac's work is that he considers all field equations to be only *weak* equations, and so also the equations which define the canonical momenta in terms of velocities are weak.

Dirac points out that besides the primary constraints or ϕ-equations, one must at $t=0$ also impose a finite[28] number of *secondary* constraints or "X-equations" between the canonical variables, in order to preserve the constraints for later times. An example of such a X-equation is $\operatorname{div} \underset{\sim}{E} = 4\pi\rho$ in Maxwell's theory; *they* usually depend on *interactions*, while the ϕ-equations do *not*. For gravity, these X-constraints are formidable expressions[24].

The primary and secondary constraints *together* are again subdivided into "first-class" and "second-class" constraints. First-class constraints are those that have zero Poisson brackets with all other constraints, while second-class constraints have some non-vanishing Poisson bracket with another second-classe constraint. Examples of second-class constraints are the ones telling that the Dirac elec tron wave function ψ is the canonical conjugate to ψ^*, while ψ^* is the canonical conjugate to ψ[15]; or also the relations telling in vector-meson theory[23] that some of the field variables are so-called "derived variables"[37]. My own work of 1940 tells that such derived variables should not be quantized independently, but should be treated as functions of the variables *from which* they are derived. Dirac[28] arrives at this same result in a more sophisticated way. For this purpose he introduces a modification of the classical Poisson brackets, which others recently have called "Dirac brackets"[15,34]. Please do not confuse *these* Dirac brackets[15,34], with Dirac's bra's and kets[38] used in quantum-mechanical transformation theory for indicating state vectors and labels on matrix elements.

The advantage of Dirac's modified Poisson brackets is that, if they instead of the conventional Poisson brackets are changed into quantum-mechanical commutators, this leads *automatically* to the proper quantization of derived variables.

The secon-class constraints then may be considered *strong* equations. The work of Bergmann and Goldberg[34] has *further* justified this interpretation. They redefined the Dirac bracket, *formally* independent of the subdivision of constraints into first-class and second-class ones, but under the condition that as Hilbert operators in quantum theory one does not consider just every field variable, but only those combinations of them which generate a mapping of the physical subspace of Hilbert space on itself. By "physical subspace" I mean here that part of Hilbert space where all constraints are satisfied.

This excludes from being Hilbert variables not only all constraints, but also what one may call their canonical conjugates. Let me give an example. In electrodynamics, P_4 is a primary constraint variable, and $(\mathrm{div}\,\underset{\sim}{E} - 4\pi\rho)$ is a secondary one. From regular quantization we therefore exclude not only P_4 and div E, but also their conjugates Φ and div $\underset{\sim}{A}$. Thus, only the *transverse* waves in $\underset{\sim}{A}$ and in $\underset{\sim}{E}$ are quantized in the usual way. This amounts exactly to the so-called "gauge-independent" quantum electrodynamics proposed in my work with Lomont a few years ago[36].

Bergmann suggests a similar treatment for the quantization of the gravitational field[31,34]. As there are 10 pairs of canonical variables, four ϕ-constraints, and four X-constraints, this will leave only two pairs of canonical variables to be quantized.

One can easily predict the consequence of this for a linear approximation of the Theory in flat space. If in the Fourier expansion of the field one uses Kramers' decomposition of the transverse waves into left and right-hand circularly polarized waves[39], for each of the two indices of the space-space part of the g^{kl}-field, then only left-left and right-right components are to be quantized[17]. All other components of $g^{\mu\nu}$ should be considered functions of the matter field or be eliminated by a change of variables, as Φ and div $\underset{\sim}{A}$ in gauge-independent quantum electrodynamics[38].

Let me conclude with a few sentences about the present outlook on the quantum theory of gravitation. The trouble is that the constraints are so complicated that nobody has yet succeeded in proposing new variables which make the constraints to canonical variables which then could be equated to zero wherever they occur. In the work on the rigorous Einstein theory, nobody seems to get beyond writing down some commutation relations, and nothing practical has been done with the latter. The only known method of introducing occupation-number operators, and thence obtaining a representation for creation and annihilation operators, as yet seems to be that used by Gupta[17,22], which splits off, from the total Hamiltonian, its larger part which is bilinear in the field. It may be necessary to follow such an approximation method for obtaining immediate results in the formulation of a quantum-gravitational perturbation theory. This appoximation method should then also be applied to the discussion of the constraints, and for defining in successive approximations the variables for a "gauge-independent" treatment of the theory[38,34]. During such work one should keep aware continuously of the results already obtained in the rigorous theory, which at each stage should be approximated by the expansion method.

For most any *un*interesting application, the expansion method will suffice. A critical question will be whether its classical analogue will be able to yield the advance of the planetary perihelion and the bending of light rays correctly. The most ***interesting*** application of the theory of gravitation, to the theory of elementary particles, however, is likely to requiere a more rigorous treatment.

DISCUSSION

QUESTION:

It was asked, to what extent the linear theory with its many constants satisfies the principle of equivalence of heavy and inertial mass.

ANSWER:

By use of the condition imposed on the constants, that in first approximation the planets shall describe Kepler orbits, the acceleration of a freely falling body may be written as

$$a_o = g\,[\,1 + (f/2 - 1)\,v^2/c^2\,] - f\,v\,(v\cdot g)/c^2,$$

if f is the numerical constant appearing in the formula fGM/c^2R for the bending of a light ray passing at a distance R from the center of the sun of mass M. For $f = 4$, this is the Einsteinian result. The value of the second interaction constant does not matter as long as it satisfies the "Kepler" condition. (It does not even affect the constant f.) A beam of particles traveling horizontally a distance L is deflected by $a_o t^2/2 = g_o L^2/2v^2$ instead of by $g\,L^2/2v^2$. The difference $(f/2 - 1)\,g\,L^2/2c^2$ is too small for measurement except for astronomical L-values; for instance, it amounts to between 10 and 20 Ångström units for L= 5 kilometers. Thus, particles of all velocities seem to fall equally fast.

QUESTION:

What about nucleons traveling with speed about one tenth of the speed of light inside a nucleus? Would not they make atoms with more intranuclear motion drop faster than say hydrogen? It is known from experiments that very accurately the gravitational acceleration is the same for all matter.

ANSWER:

This seems a serious argument. It should be investigated what effect the binding forces have on the gravitational acceleration of such compound matter.

FURTHER ANSWER ADDED BY INVESTIGATION IN SEPTEMBER:

Consider two apposite charges e_1 and e_2 circling around their common center of mass. In absence of gravity, this motion may be described by coordinates ξ_1 and ξ_2 as functions of a time τ, with $d\xi_1/d\tau = \omega \times \xi_1$, $d^2\xi_2/d\tau^2 = -\omega^2 \xi_2$, $|\xi_1| + |\xi_2| = |\xi_d|$, where $\xi_d = \xi_2 - \xi_1$ is the mutual distance of the particles. Trial solutions for the helical motion $x_1(t_1)$ and $x_2(t_2)$ of these two particles in presence of the gravitational field with potential $h = GM/c^2R \approx h_0 + g \cdot x/c^2$ are obtained from $\xi_1(\tau)$ and $\xi_2(\tau)$ by the transformation

$$t_1 = (1+S\,h_1)\,\tau \quad , \quad t_2 = (1+S\,h_2)\,\tau \quad ,$$

$$x_1 = \xi_1(1 + D\,h_1) + k\,g\,\xi_1^2/2c^2 + b_1\,g\,t_1^2/2 \quad ,$$

$$x_2 = \xi_2(1 + D\,h_2) + k\,g\,\xi_2^2/2c^2 + b_2\,g\,t_2^2/2 \quad ,$$

where $h_1 = h_0 + g \cdot x_1/c^2$, etc, and where S, D, K, b_1, b_2 are yet undetermined constants to be adjusted later.

The equations of motion for a charged particle in the gravitational field (of which the square is neglected systematically) turn out to give for the acceleration

$$a = a_0 + (e/m_\perp)\{(E + v \times B/c)[1 + h(1 - f - f v^2/2c^2(1 - v^2/c^2)] -$$

$$- v(E.v/c^2)[1 + h - f h v^2/2c^2(1 - v^2/c^2)]\} ,$$

with a_0 given above, and with $m_\perp = m_0/\sqrt{1 - v^2/c^2}$.

For simplicity's sake we have systematically neglected all magnetic, retardation, and other radiative effects, except for the essential contribution $-e\,b\,g/c^2r$

to $\underset{\sim}{E}$ due to the acceleration $b\underset{\sim}{g}$ of the source introduced by our trial solution. (We neglect the much larger radial acceleration of the charge e, which is not the gravitational effect for which we are looking.) In this crude approximation, Maxwell's equation for $\underset{\sim}{E}$ at the point $\underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \underset{\sim}{r}$, caused by the charge e_1 at $\underset{\sim}{x}_1$, are solved by

$$\underset{\sim}{E}_1(2) \approx e_1(\underset{\sim}{r}/r^3)\left[1 - f(h_o - \underset{\sim}{g}.\underset{\sim}{x}_1/c^2)/2 - (f\underset{\sim}{g}/2 + b_1\underset{\sim}{g}).\underset{\sim}{r}/2c^2\right] + \\ + (e_1/r)\left[f\underset{\sim}{g}/2 - b_1\underset{\sim}{g}\right]/2c^2 .$$

We insert this $\underset{\sim}{E}_1(2)$ containing b_1 in the equation for $\underset{\sim}{a}_2$ in the approximation mentioned, and similarly $\underset{\sim}{E}_2(1)$ containing b_2 is substituted for $\underset{\sim}{E}$ in the equation for $\underset{\sim}{a}_1$. We omit the $\underset{\sim}{v} \times \underset{\sim}{B}$ term and the terms with $\dot{\underset{\sim}{E}} v^2$ or $\underset{\sim}{E}.\underset{\sim}{v}\underset{\sim}{v}$ in the equation of motion. We attempt to solve our equations for $\underset{\sim}{a}_1$ and for $\underset{\sim}{a}_2$ by our trial solution; by the transformations given, we first express all $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ in terms of $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2$, thence also $\underset{\sim}{r} = \underset{\sim}{x}_2 - \underset{\sim}{x}_1$, while we express $\underset{\sim}{v}_1 = (d\underset{\sim}{x}_1/d\tau)/(dt_1/d\tau)$, $\underset{\sim}{a}_2 = (d\underset{\sim}{v}_2/d\tau)/(dt_2/d\tau)$, etc., all in terms of $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2$, and $\underset{\sim}{\omega}$. We choose the radii $|\xi_1|$ and $|\xi_2|$ of our trial solution in such a way that

$$e_1 e_2 \underset{\sim}{\xi}_d/\xi_d^3 = m_{\perp 2}\omega^2 \underset{\sim}{\xi}_2 = + m_{\perp 1}\omega^2 \underset{\sim}{\xi}_1 .$$

We then find that for small values of t our trial solution solves the equations of motion, *if* we choose the constants according to

$$k = -D = (f/2 - 1) , \quad S = b_1 = b_2 = 1 .$$

This shows that the particles describe a helical motion, in which the " center of rotation " of each particle has an acceleration $b\underset{\sim}{g} = \underset{\sim}{g}$ (and NOT equal to $\underset{\sim}{a}_o$). Therefore, rotating electric dipoles fall all with the same gravitational acceleration $\underset{\sim}{g}$, independent of the value of ω, that is, of the amount of internal kinetic energy. This result is obtained here for arbitrary value of f, so that no further restriction is imposed on the constants of the linear theory of gravitation by the postulate of the validity of the equivalence principle.

For $f=4$ we obtain the (linearized) Einstein case. In this case, our transformation from $\underset{\sim}{\xi}, \tau$ to $\underset{\sim}{x}, t$ just amounts to the transformation from a freely falling local inertial frame of reference to the laboratory frame of reference according to the rules of the theory of general relativity, and our solution for $\underset{\sim}{E}$ in that case is obtained by this general coordinate transformation from the Coulomb field without retardation in that local inertial system.

It is trusted or hoped that a calculation using half-retarded, half-advanced fields and including magnetic effects and Ev^2 terms, will lead to essentially similar results.

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