# The spectra masses for heavy pentaquark using generalized fractional of the extended Nikiforov-Uvaro method

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Received 9 September 2023; accepted 21 January 2024

The Nikiforov-Uvarov method is an efficient technique for solving of heavy diquark systems. It has been used to derive analytic-exact energy eigenvalues and eigenfunctions in fractional forms, which are useful in describing such systems. The potentials employed including the Cornell potential, harmonic potential, and spin-spin interaction; have been updated with respect to previous studies. Mass spectra of heavy pentaquarks were also calculated and compared to previous studies. The present results exhibit good experimental data agreement and are improved. We deduce that the fractional models contribute greatly to the heavy pentaquark masses.

*Keywords:* Nikiforov-Uvarov method; heavy diquark systems.

DOI: https://doi.org/10.31349/RevMexFis.70.030801

## 1. Introduction

In 1962, Okun [1] introduced the idea of hadrons. In separate developments, Gell-Seminal Mann [2] and Zweig [3, 4] evaluated the quark model, that described common mesons (q$\bar{q}$) as [5] and baryons (qqq) as Refs. [6- 8]. Tetraquarks (qq$\bar{q}\bar{q}$) and pentaquarks (q$\bar{q}$qqq) are two examples of the exotic hadrons that are composed of four or more quarks, were initially proposed in [2], but [9] the first supposed discoveries of exotic hadrons were not manufactured till the start of 21 st century. Although the majority of ground-state mesons and baryons are experimentally fully defined, a few recently discovered states are in inquire about since it is unknown which quark content and/or spin/parity they contain [10-14] for a study of potential exotic states. The hidden-charm pentaquarks are the most notable states of pentaquarks in recent years. In 2015, two hidden-charm pentaquark states, Pc(4380) and Pc(4450), were discovered by the LHCb Collaboration in the $J/\psi p$ invariant mass spectrum of $\Lambda_b^0 \to J/\psi + p + K^-$ [15]. The LHCb Collaboration confirmed their results four years later, and the $Pc(4450)$ was separated into the $Pc(4440)$ and $Pc(4457)$ states [16].

Up till now, an array of theories have been put forth for the characteristics of pentaquarks. A diquark-diquark-antiquark description of pentaquarks, for example, [17] uses two interpolating currents to determine the mass of the $E^{--}$ state using the QCD sum rules. The perturbative chiral quark model was used to examine the mass spectrum of $J^p = 3/2$ pentaquarks [18]. The authors in Ref. [19] determined the masses of pentaquarks using a modified mass formula that was applied to the masses of baryons. Some characteristics of qqqq$\bar{q}$ pentaquark states, like their magnetic moments and masses, which discovered with a complete description of these particles in a constituent quark model [20]. A formula was provided by Karliner and Lipkin [21] to obtain the masses of pentaquark states. Charmonium-pentaquark states were clarified using the diquark-triquark model [22]. The main characteristics correspond to the experimental findings found in [23] and are depended on the antiquark-diquark-diquark scheme. Based on the idea that the QQqq$\bar{q}$ pentaquarks are composite features of two diquarks and one antiquark, a hypothetical model is used to study them. This model contains Cornell potential, which is represented by the equation $V_{\text{Cornell}} = (-a/r) + br + c$ [24]. The pentaquark model was constructed by the authors utilizing a number of theories, despite the same potential being employed [25].

According to [26], the author solved a derivative Caputo fractional Schrödinger wave equation using the quantitative characteristic of the classical nonrelativistic Hamiltonian. A common research topic in applied sciences, fractional-order derivatives are essentially a natural extension of ordinary derivatives [27, 28]. The Nikiforov-Uvarov approach has been used to study the fractional radial Schrödinger equation [29], and provides an analytical derivation of the eigenstate solutions for the Woods-Saxon potential, harmonic oscillator potential, and Hulthen potential. The estimated bound state of the N-dimensional fractional Schrödinger equation was used by Das *et al.* [30] to calculate the mass spectra of quarkonia. The fractional Schrödinger equation [31] is obtained by using the Schrödinger equation in the normal space and a fractional derivative of the Jumarie type in a one-dimensional infinite potential. By using two reliable analytical techniques, the conformable space-time fractional Benney-Luke equation was determined [32].

The energy eigenvalues and related eigenfunctions for the DFP were computed by the author as a formula of the fractional parameter for any vibrational and rotational quantum number values in N-dimensional space [33]. The fractional Zakharov-Kuznetsov problem could be obtained via the conformable derivative and the Riccati technique [34]. Using the conformable fractional Nikiforov-Uvarov approach, it illustrates that to solve the fractional radial Schrödinger equation analytically for the hot medium interaction potential [35]. In order to get obvious solutions to fractional differential equations, the author [36] also offered a new generalized definition of the fractional derivative which offers advantages over other earlier definitions.

Given that a pentaquark is the bound state of two diquarks and an antiquark, we calculate the masses of pentaquarks in the ground state in this work. The fundamental concept that underlies the description of the diquark is the union of any two quarks into a colorful quasi-bound state. First, utilizing this method allows us to talk about the possibility of applying the diquark principle in this situation. Second, to determine more brief knowledge for the masses, expressions. We used the generalized fractional Schrödinger equation with the potential energy of quark interaction the Cornell potential, the harmonic potential, and the spin-dependent potential. To our best knowledge, the fractional Schrödinger equation is not considered in other works for calculating pentaquarks masses. Our numerical results for the ground masses for the various pentaquarks models with spin $(1/2)^-$, $(3/2)^-$ and $(5/2)^-$ are presented. Our results are consistent with the outcomes of other researchs.

The paper is organized as follows: In Sec. 2, the extended Nikiforov-Uvarov (ENU) method is briefly used to study the generalized fractional derivative. In Sec. 3, the generalized fractional Schrödinger equation (GFD-SE), which will be obtained using the ENU, will be created by applying the generalized fractional derivative (GFD) concept to the Schrödinger equation for the current potential. In Sec. 4, both the heavy diquark system and the pentaquark system will be utilized for examining the method, and the findings will be presented. At the end where the conclusion is stated.

## 2. The generalized fractional derivative with the extended Nikiforov-Uvarov method (GFD-ENU)

The extended Nikiforov-Uvarov (ENU) method is a generalization of the Nikiforov-Uvarov approach. As seen in Ref. [37], both are typically employed in quantum physics to acquire the eigenvalues and eigenfunctions of the Schrödinger or Dirac equations in addition to any other equations which have to be translated into a hypergeometric form for review. The applicability of the approach was effectively demonstrated in a few physical situations after the NU was generalized in Ref. [38] to the conformable fractional derivative. This section aims to expand the ENU within the GFD construction. Take into account the following generalized fractional differential equation in the standard form [33];

$$D^{GFD}[D^{GFD}[\psi(s)]] + \frac{\tilde{\tau}(s)}{\sigma(s)} D^{GFD}[\psi(s)] + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\ \psi(s) = 0, \tag{1}$$

where $\tilde{\tau}(s)$, $\sigma(s)$, and $\tilde{\sigma}(s)$ are polynomials with degrees of no more than second, third and fourth, respectively. BY using GFD [36], we express

$$D^{GFD}[\psi(s)] = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha-1)}\ s^{1-\alpha}\ \acute{\psi}(s), \tag{2}$$

$$D^{GFD}[D^{GFD}[\psi(s)]] = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha-1)}\right)^2 \times \left((1-\alpha)\, s^{1-2\alpha}\ \acute{\acute{\psi}}(s) + s^{2-2\alpha}\ \acute{\psi}(s)\right). \tag{3}$$

where $0 < \alpha \le 1$, $0 < \beta \le 1$, $\acute{\psi}(s)$ in first derivative of $\psi(s)$ by using Eqs. (2), (3), we get;

$$\acute{\acute{\psi}}(s) + \frac{(1-\alpha)\, s^{1-\alpha}\, \sigma(s) + \tilde{\tau}(s)}{s^{1-\alpha}\, \sigma(s)}\ \acute{\psi}(s) + \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha-1)}\right)^{-2} \frac{\tilde{\sigma}(s)}{s^{2-2\alpha}\sigma^2(s)} \psi(s) = 0. \tag{4}$$

by describing the parameters for fractional;

$$\tilde{\tau}_f(s) = (1-\alpha)s^{1-\alpha}\sigma(s) + \tilde{\tau}(s), \qquad \sigma_f(s) = s^{1-\alpha}\sigma(s)$$

$$\tilde{\sigma_f}(s) = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha-1)}\right)^{-2} \tilde{\sigma}(s). \tag{5}$$

The generalized fractional extended Nikiforov-Uvarov (ENU) equation is obtained in standard form,

$$\acute{\acute{\psi}}(s)\ +\ \frac{\tilde{\tau_f}(s)}{\sigma_f(s)}\ \acute{\psi}(s)\ +\ \frac{\tilde{\sigma_f}(s)}{\sigma_f^2(s)}\ \psi(s) = 0. \tag{6}$$

We use the next transformation to determine the solution to Eq. (1);

$$\psi(s)\ =\ \phi(s)\, X(s), \tag{7}$$

It transforms Eq. (6) into a hypergeometric equation, where $\phi(s)$ and $X(s)$ are hypergeommetric functions.

$$\sigma_f(s)\ \acute{\acute{X}}(s) + \tau_f(s)\ \acute{X}(s) + \lambda_f(s)\ X(s) = 0, \tag{8}$$

where $\phi(s)$ satisfy

$$\frac{\acute{\phi}(s)}{\phi(s)} = \frac{\pi_f(s)}{\sigma_f(s)}, \tag{9}$$

$$\lambda_f(s) - \acute{\pi_f}(s) = K(s), \tag{10}$$

and the hypergeometric type function $X(s)$ is a polynomial function that satisfies the Rodrigues relation.

$$X_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n}(\sigma_f^n(s)\rho(s)), \tag{11}$$

with the normalization constant is $B_n$, the weighting function is $\rho(s)$, and the condition are achieved.

$$\acute{\sigma_f}\rho + \acute{\rho}\ \sigma_f = \tau_f \rho. \tag{12}$$

The definition of the function $pi_f(s)$ is

$$\pi_f(s) = \frac{\acute{\sigma_f}(s) - \tilde{\tau_f}(s)}{2} \pm \sqrt{\left(\frac{\acute{\sigma_f}(s) - \tilde{\tau_f}(s)}{2}\right)^2 - \tilde{\sigma_f}(s) + K(s)\sigma_f(s)}. \quad (13)$$

When $\pi(s)$ is a second degree polynomial, it is an essential to identify $K(s)$ in order to calculate $\pi(s)$. The $\lambda_n(s)$ function also uses the relation.

$$\lambda_n(s) = -\frac{n}{2}\,\acute{\tau_f}(s) - \frac{n(n-1)}{6}\,\acute{\sigma_f}(s). \quad (14)$$

where

$$\tau_f(s) = \tilde{\tau_f}(s) + 2\pi_f(s). \quad (15)$$

We determine the energy eigenvalues by solving Eqs. (10) and (14) together.

## 3. The quark model

The Schrödinger equation describing two particles interacting with symmetric potential illustrates how quark-antiquark systems interact in the 3-dimensional space Refs. [39].

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right] R_{n,l}(r) = -2\mu[E - V(r)]\; R_{n,l}(r). \quad (16)$$

where $l$ is the angular momentum and $\mu = m_1m_2/(m_1+m_2)$ represents the reduced mass for the diquark. The wave function consists of the following:

$$R_{n,l}(r) = r^{-1}\psi_{n,l}(r), \quad (17)$$

and we utilize the harmonic, spin-spin interaction, and Cornell potential as our potential models.

$$V(r) = V(r) + V_{\text{spin}}(r), \quad (18)$$

$$V(r) = ar^2 + br + \frac{c}{r}, \quad (19)$$

$$V_{\text{spin}}(r) = \eta e^{-\sigma^2 r^2}, \qquad \eta = \frac{A\sigma^3 S_1.S_2}{2\pi^{3/2}m_1m_2}. \quad (20)$$

We use the basic approximation to solve the following equation analytically while taking into account that $r < 1$ fm. The following potential equation results from the function approximation $e^{-\sigma^2 r^2} \approx 1 - \sigma^2 r^2 + \ldots$

$$V(r) = \eta + +br + (a - \eta\sigma^2)r^2 + \frac{c}{r}. \quad (21)$$

The parameter in this case determines how the smeared delta should behave, and we use the formula $\sigma = 1.209$ GeV [24]. $A_c = 7.920$ and $A_b = 3.087$ [25] are connected to the strong constant coupling $\alpha_s$ in the one gluon exchange approximation, respectively.

The spins of the interacting particles in a spin-spin interaction are $S_1$ and $S_2$. $S_1.S_2 = (1/2)[S(S+1) - S_1(S_1+1) - S_2(S_2+1)]$, S is the total spin.

Two contributions are often present when two colored objects interact. One of them depends on the $q\bar{q}$ potential found in quantum physics through the gauge/string duality method, while the other one defines the interaction between spins. Color interaction is the name given to this potential, which was identified using a virtual model based on a Cornell potential [24].

Equation (16) becomes as the following form

$$\left[\frac{d^2}{dr^2} + 2\mu(E - V(r)) - \frac{l(l+1)}{r^2}\right]\psi_{n,l}(r) = 0. \quad (22)$$

Equation (22) is then expressed in the form of a dimensional fractional by taking the values $r = z/A$, $\grave{\mu} = \mu/A$ and $\grave{E} = E/A$ (where $A = 1$ GeV);

$$D^{GFD}[D^{GFD}[\psi_{n,l}(z)]] + \left(2\grave{\mu}\left[\grave{E} - \frac{V(z)}{A}\right] - \frac{l(l+1)}{z^{2\alpha}}\right)\;\psi_{n,l}(z) = 0. \quad (23)$$

We use the formula $w = \Gamma(\beta)/\Gamma(\beta - \alpha - 1)$. For the solution of the fractional Schrödinger equation (23), use the formulas, $\grave{a} = (a - \eta\sigma^2/A)$, $\grave{b} = b/A$ and $\grave{c} = c/A$. Equation (3) is applied.

$$\frac{d^2\psi_{n,l}(z)}{dz^2} + \frac{1-\alpha}{z}\;\frac{d\psi_{n,l}(z)}{dz} + \frac{1}{w^2}\,[2\grave{\mu}\,(\grave{E} - \grave{\eta})\,z^{2\alpha-2} - 2\grave{\mu}\,\grave{a}\,z^{4\alpha-2} - 2\grave{\mu}\,\grave{b}\,z^{3\alpha-2} - 2\grave{\mu}\,\grave{c}\,z^{\alpha-2} - \frac{l(l+1)}{z^2}]\,\psi_{n,l}(z) = 0. \quad (24)$$

Equation (24) is reduced to utilizing the radial fractional wave function $\psi_{n,l}(z) = z^{-\alpha}\phi_{n,l}(z)$,

$$\acute{\acute{\phi_{n,l}}}(z)_{n,l} + \frac{1-\alpha}{z}\;\acute{\phi_{n,l}}(z) + \frac{1}{z^2}[-c_1\,z^{4\alpha} - c_2\,z^{3\alpha} - c_3\,z^{2\alpha} - c_4\,z^{\alpha} - c_5]\;\phi_{n,l}(z) = 0, \quad (25)$$

where

$$c_1 = \frac{2\grave{\mu}\,\grave{a}}{A^3 w^2}, \qquad c_2 = \frac{2\grave{\mu}\,\grave{b}}{A^2 w^2}, \qquad c_3 = -\frac{2\grave{\mu}\,(\grave{E}-\grave{\eta})}{w^2}, \qquad c_4 = \frac{2\grave{\mu}\,\grave{c}}{w^2}, \qquad c_5 = \frac{l(l+1)}{w^2}. \tag{26}$$

Using the ENU- GFD approach, and by contrasting Eqs. (6) and (25), we discover

$$\tilde{\tau} = (1-\alpha), \qquad \sigma_f = z, \qquad \bar{\sigma_f} = -c_1 z^{4\alpha} - c_2 z^{3\alpha} - c_3 z^{2\alpha} - c_4 z^{\alpha} - c_5. \tag{27}$$

Then we have

$$\pi_f(z) = \frac{\alpha}{2} \pm \sqrt{c_1 z^{4\alpha} + c_2 z^{3\alpha} + c_3 z^{2\alpha} + c_4 z^{\alpha} + \bar{c_5} + z\,K(z)}. \tag{28}$$

where $\bar{c_5} = c_5 + (\alpha/4)$, we obtain a linear function $K(z) = (Az^{2\alpha-1} + Bz^{\alpha-1})$, which causes the function under the root in the previous formula to be quadratic $(Qz^{2\alpha} + Pz^{\alpha} + F)^2$ So,

$$\pi_f(z) = \frac{\alpha}{2} \pm (Q\,z^{2\alpha} + P\,z^{\alpha} + F) \tag{29}$$

We get the subsequent solutions after comparing the values of the coefficients of Eqs. (28) and (29).

$$Q = \pm\sqrt{c_1}, \qquad P = \frac{\pm c_2}{2\sqrt{c_1}}, \qquad F = \pm\sqrt{\bar{c_5}}, \qquad A = \frac{c_2^2}{4c_1} \pm 2\sqrt{c_1\bar{c_5}} - c_3, \qquad B = \pm\frac{c_2}{\sqrt{c_1\bar{c_5}}} - c_4. \tag{30}$$

Thus

$$\tau_f(z) = 1 \pm 2\left(Q\,z^{2\alpha} + P\,z^{\alpha} + F\right). \tag{31}$$

By Eq. (10), we find

$$\lambda(z) = Az^{2\alpha-1} + Bz^{\alpha-1} + 2Q\,\alpha\,z^{2\alpha-1} + \alpha\,P\,z^{2\alpha-1}, \tag{32}$$

and by using Eq. (14), we find

$$\lambda_n(z) = -n\left(2Q\,\alpha\,z^{2\alpha-1} + \alpha\,P\,z^{2\alpha-1}\right). \tag{33}$$

The energy eigenvalue of Eq. (25) is given by Eqs. (32) and (33).

$$E_{n,l} = -A^2 w^2 \left( \frac{b^2}{4(a-\eta^2)} - \sqrt{\frac{2(a-\eta^2)}{\mu}} \left[ \alpha(n+1) + \sqrt{\frac{l(l+1)}{w^2} + \frac{\alpha^2}{4}} \right] \right) + \eta. \tag{34}$$

The expression for the function $\phi(z)$ in Eq. (9) appears to be this:

$$\phi(z) = z^{\frac{F\alpha}{2}}\, e^{(\frac{P}{\alpha} z^{\alpha} + \frac{Q}{2\alpha} z^{2\alpha})}, \tag{35}$$

where Eq. (30) provides P, Q, and F, and Eq. (12) provides the function $\rho(z)$;

$$\rho(z) = z^{2F}\, e^{(2P\alpha z^{\alpha} + \frac{Q}{\alpha} z^{2\alpha})}, \tag{36}$$

and then the function $X_n(z)$ is

$$X_n(z) = B_n\, z^{-2F}\, e^{(\frac{-2P}{\alpha} z^{\alpha} - \frac{Q}{\alpha} z^{2\alpha})}\, \frac{D^n}{Dz^n}\left[\, z^{2F+n}\, e^{(\frac{2P}{\alpha} z^{\alpha} + \frac{Q}{\alpha} z^{2\alpha})}\, \right], \tag{37}$$

then $\psi_{nl}(z) = \phi(z)\,X(z)$ the fractional radial eigen-function get as;

$$\psi_{n,l}(z) = N_{nl}\;\; z^{\frac{\alpha F}{2} - 2F - \alpha}\;\, e^{(\frac{-P}{\alpha} z^{\alpha} - \frac{Q}{2\alpha} z^{2\alpha})}\;\, \frac{D^n}{Dz^n}\left[z^{2F+n}\;\, e^{(\frac{2P}{\alpha} z^{\alpha} + \frac{Q}{\alpha} z^{2\alpha})}\right]. \tag{38}$$

For each $n = 0, 1, 2, 3, ...$, we will determine the eigenvalue energy and related eigen-function.

TABLE I. The diquark masses(in GeV).

| Diquark | M (our) | $M_{Exp}$ | Error % | Diquark | M (our) | $M_{Exp}$ | Error % |
|---|---|---|---|---|---|---|---|
| $[bu]_{s=0}$ | 5.28 | 5.279 | 0.012 | $[cu]_{s=0}$ | 1.87 | 1.864 | 0.32 |
| $\{bu\}_{s=1}$ | 5.324 | 5.325 | 0.019 | $\{cu\}_{s=1}$ | 2.422 | 2.42 | 0.08 |
| $[bb]_{s=0}$ | 9.396 | 9.389 | 0.07 | $[cc]_{s=0}$ | 2.95 | 2.98 | 0.99 |
| $\{bb\}_{s=1}$ | 9.46 | 9.461 | 0.01 | $\{cc\}_{s=1}$ | 3.094 | 3.096 | 0.06 |

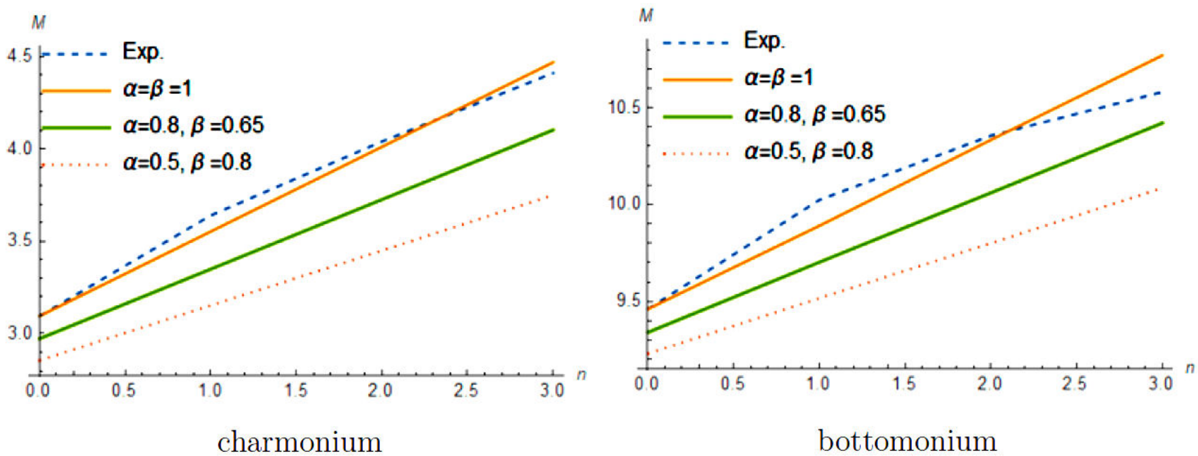


FIGURE 1. The S-states mass spectra for charmonium and bottomonium (in GeV) at different fractional orders.

## 4. Result and discussion

### 4.1. Diquark

In this part, we calculate the heavy diquark masses in 3-dimensional using equation (39), as given in Table I. The following notation considered that the $[Qq]$ and $\{Qq\}$ diquarks have spins of 0 and 1, respectively.

$$M = m_1 + m_2 + E_{n,l}, \tag{39}$$

where $E_{nl}$ is determined by Eq. (34). The potential parameters a, b, and c are obtained by fitting the mass experimental data simultaneously. Using the quark masses $m_c = 1.459$ GeV, $m_{u,d} = m_q = 0.323$ GeV, $m_b = 4.783$ GeV Ref. [40].

According to Eqs. (34) and (39), respectively, Fig. 1 presents the S-states mass spectra for charmonium and bottomonium with the different fractional orders $\alpha$ and $\beta$. When the fractional order decreases, the projected masses accord well with the experimental results, compared to the classical example at $\alpha = \beta = 1$. It also implies that these factors are crucial for producing accurate predictions of particle masses from theoretical calculations. The tables below provide further evidence of this theory, showing how varying values of both parameters can lead to different results in terms of accuracy compared with experimentally measured values; as they decrease so too does the discrepancy between prediction and reality indicating that these fractional orders are indeed playing an important part in our understanding here.

The diquark is a bound state of two quarks, and its size is slightly larger than the total of its constituent quarks as shown in Table I. To calculate the binding energy, we use $E = M - m_1 - m_2$, where $M$, $m_1$ and $m_2$ are respectively the mass of diquark system, first quark, and second quark. The predicted values for the diquark masses in our work are compared with their experiment data and calculated the error of them that consistently smaller by about $1\%$.

In Table II, with fractional parameters $\alpha = 0.79$ and $\beta = 0.65$, we can calculate charmonium spectrum mass in different states; these results are better than those reported in recent Refs [39, 41, 42]. The calculated values for ground states 1S and 1P were close to experimental data which indicates that our calculation was accurate. In conclusion, it can be said that by using fractional parameters such as $\alpha = 0.79$, and $\beta = 0.65$. We could accurately predict charmonium spectrum masses from their corresponding binding energies. This helps us understand how particles interact at subatomic levels while also providing an insight into particle physics research more generally. In order to give an acceptable level of precision when compared to other studies, we additionally determined the total error for all states as follows: total error = $\sum_{\text{states}} |(M_{\text{Our}} - M_{\text{Exp}})/nM_{\text{Exp}}|\%$ where $M_{\text{Our}}$ is the predicted mass and $M_{\text{Exp}}$ is the experimental mass, and $n$ is the number of states. The overall error for the charmonium mass of $1.95\%$ obtained in the present study is shown in Table II, and it is less than the total error reported in earlier studies. The asymptotic iteration approach with Cornell potential is used in Ref. [39] for solving the Schrödinger equation. The

TABLE II. Mass spectra of charmonium at $\alpha = 0.79$, $\beta = 0.65$, and comparison with other works (in GeV).

| state | $M_{\text{our}}$ | Ref. [39] | Ref. [41] | Ref. [42] | Ref. [43] | Ref. [44] | Exp. | Error of state |
|---|---|---|---|---|---|---|---|---|
| 1S | 3.094 | 3.096 | 3.096 | 3.239 | 3.096 | 3.096 | 3.0780 | .064 |
| 1P | 3.53 | 3.214 | 3.57 | 3.372 | 3.344 | 3.255 | 3.52 | .28 |
| 2S | 3.55 | 3.412 | 3.58 | 3.646 | 3.786 | 3.504 | 3.64 | 2.47 |
| 1D | 4.01 | 3.686 | 4.05 | 3.604 | 3.769 | 3.686 | 3.76 | 6.75 |
| 2P | 4.003 | 3.773 | 4.04 | 3.779 | 4.034 | 3.779 | 3.9 | 2.64 |
| 3S | 4.004 | 4.275 | 4.04 | 4.052 | 4.27 | 4.04 | 4.04 | 0.89 |
| 4S | 4.44 | 4.865 | 4.51 | 4.459 | 4.621 | 4.269 | 4.41 | 0.68 |
| Total Error | 1.95 | 5.19 | 2.38 | 2.52 | 3.3 | 2.87 | | |

TABLE III. Mass spectra of bottomonium at $\alpha = 0.8$, $\beta = 0.65$, and comparison with other works (in GeV).

| state | $M_{\text{our}}$ | Ref. [39] | Ref. [41] | Ref. [42] | Ref. [43] | Ref. [44] | Exp. | Error of state |
|---|---|---|---|---|---|---|---|---|
| 1S | 9.46 | 9.460 | 9.46 | 9.495 | 9.444 | 9.51 | 9.46 | 0 |
| 1P | 9.899 | 9.492 | 9.85 | 9.657 | 9.711 | 9.862 | 9.9 | 0.01 |
| 2S | 9.896 | 10.023 | 9.84 | 10.023 | 9.946 | 10.627 | 10.023 | 1.26 |
| 1D | 10.25 | 9.551 | 10.23 | 10.161 | 10.161 | 10.214 | 10.16 | 0.88 |
| 2P | 10.335 | 10.038 | 10.22 | 10.26 | 10.213 | 10.944 | 10.266 | 0.67 |
| 3S | 10.33 | 10.585 | 10.21 | 10.355 | 10447 | 11.726 | 10.355 | 0.24 |
| 4S | 10.77 | 11.148 | 10.58 | 10.579 | 10.949 | 12.834 | 10.58 | 1.79 |
| Total Error | 0.65 | 2.86 | 0.69 | 1.37 | 1.07 | 6.98 | - | |

Schrödinger equation with Cornell potential and harmonic potential has been solved with the generalized fractional derivative in Ref. [41]. In Ref. [44], an analytical exact iteration method is used to determine the radial Schrödinger equation in fractional form (at $\alpha = 0.97$).

In Table III, we achieve an acceptable result by utilizing the fractional parameters $\alpha = 0.8$ and $\beta = 0.65$ to obtain the mass spectra of bottomonium. We used Eq. (32) and the values of the experimental data for the ground states 1S and 1P to determine the potential parameters. Utilizing these characteristics, we predicted more states for bottomonium. Additionally, we point out that there is a good agreement between our findings and those of the current Refs. [39, 41–44]. According to Table III, we found a total error for bottomonium mass of 0.65% in our study, which is lower than the total error found in those works.

### 4.2. Pentaquark

Only heavy states containing at least one heavy particle, such as a charm or bottom quark, allow for the extension of the one-gluon-exchange approximation and the application of an immediate potential. We determine the masses of pentaquarks that include at least one heavy quark as a result. Additionally, we only take into account states where each state's orbital angular momentum is $l = 0$. This method allows us to explore how diquarks form bound states and obtain more concise information for mass expressions.

In order to accurately calculate the masses of these pentaquarks using this model, it is necessary to take into account various factors such as spin configurations, flavor and color combinations between them that is another type of charge or attraction could affect their binding energy levels, and quark flavor describes a certain type of positive or negative partial charge. The union of any two quarks to form a colored quasi-bound state is the physical theory behind the diquark's explanation, so we assume the pentaquark to be the bound state of two heavy diquarks and antiquarks. To do so requires careful consideration when creating wave functions for each possible combination including all possible permutations before applying methods such as the variational principle or Rayleigh-Ritz technique in order to determine their energies accurately. Additionally, since there are multiple possibilities due to different types of particles involved (*i.e.*, light/heavy quark), it is important that any calculations are done also include contributions from other interactions like gluon exchange between them too; otherwise, results may not be accurate enough for practical applications like predicting properties related to particle production rates, etc... We obtain the numerical results for ground state masses for the pentaquarks model stated in Fig. 2 and with spin $(1/2)^-$, $(3/2)^-$ and $(5/2)^-$.

TABLE IV. The masses of pentaquark at $J^p = (\frac{3}{2})^-$ in GeV.

| Pentaqurk | M (our) | M (other) |
|---|---|---|
| $[\mathrm{cu}]_{s=0}[\mathrm{cu}]_{s=1}\bar{u}$ | 5.36 | 4.64 [24] |
| | | 4.33 [25] |
| | | 4.21 [40] |
| | | 3.873 [45] |
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=0}\bar{u}$ | 5.55 | 4.57 [24] |
| | | 3.882 [45] |
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=0}\bar{c}$ | 6.29 | 5.75 [25] |
| | | 7.17 [40] |
| | | 6.047 [48] |
| $[\mathrm{cu}]_{s=0}[\mathrm{cu}]_{s=1}\bar{c}$ | 6.42 | - |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=0}\bar{u}$ | 7.505 | 6.867 [46] |
| $[\mathrm{cc}]_{s=0}[\mathrm{cc}]_{s=1}\bar{u}$ | 7.57 | - |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=0}\bar{c}$ | 8.42 | 8.42 [47] |
| $[\mathrm{cc}]_{s=0}[\mathrm{cc}]_{s=1}\bar{c}$ | 8.43 | - |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=0}\bar{u}$ | 11.899 | 11.19 [24] |
| | | 11.17 [25] |
| | | 11.12 [40] |
| $[\mathrm{bu}]_{s=0}[\mathrm{bu}]_{s=1}\bar{u}$ | 11.203 | 11.16 [24] |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=0}\bar{b}$ | 16.03 | 15.95 [25] |
| | | 16.86 [40] |
| | | 15.97 [48] |
| $[\mathrm{bu}]_{s=0}[\mathrm{bu}]_{s=1}\bar{b}$ | 16.045 | - |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=0}\bar{u}$ | 21.22 | 19.68 [46] |
| $[\mathrm{bb}]_{s=0}[\mathrm{bb}]_{s=1}\bar{u}$ | 21.43 | - |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=0}\bar{b}$ | 24.73 | 25.179 [47] |
| $[\mathrm{bb}]_{s=0}[\mathrm{bb}]_{s=1}\bar{b}$ | 24.78 | - |

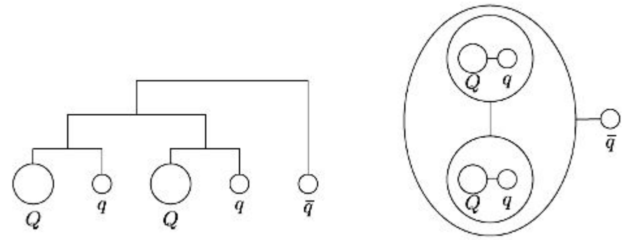

FIGURE 2. The form of pentaquark QqQq $\bar{q}$.

TABLE V. The masses of pentaquark at $J^p = (3/2)^-$ in GeV.

| Pentaquark | M (our) | M (others) |
|---|---|---|
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=1}\bar{u}$ | 5.32 | 4.62 [24] |
| | | 4.76 [25] |
| | | 5.35 [40] |
| | | 3.835 [45] |
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=0}\bar{u}$ | 5.607 | 4.72 [24] |
| | | 4.87 [25] |
| | | 5.58 [40] |
| | | 3.855 [45] |
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=0}\bar{c}$ | 6.42 | 6.15 [25] |
| | | 7.12 [40] |
| | | 5.922 [48] |
| $[\mathrm{cu}]_{s=1}[\mathrm{cu}]_{s=1}\bar{c}$ | 6.34 | 6.75 [25] |
| | | 7.27 [40] |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=0}\bar{u}$ | 7.373 | 6.76 [46] |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=1}\bar{u}$ | 8.41 | - |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=0}\bar{c}$ | 8.366 | 8.426 [47] |
| $[\mathrm{cc}]_{s=1}[\mathrm{cc}]_{s=1}\bar{c}$ | 9.32 | - |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=0}\bar{u}$ | 11.739 | 11.18 [24] |
| | | 11.47 [25] |
| | | 11.24 [40] |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=1}\bar{u}$ | 12.02 | 11.18 [24] |
| | | 11.47 [25] |
| | | 11.24 [40] |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=0}\bar{b}$ | 16.03 | 16.01 [40] |
| | | 15.88 [48] |
| $[\mathrm{bu}]_{s=1}[\mathrm{bu}]_{s=1}\bar{b}$ | 16.44 | 16.02 [40] |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=0}\bar{u}$ | 20.89 | 19.68 [46] |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=1}\bar{u}$ | 20.204 | - |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=0}\bar{b}$ | 24.76 | 25.178 [47] |
| $[\mathrm{bb}]_{s=1}[\mathrm{bb}]_{s=1}\bar{b}$ | 23.28 | - |

As illustrated in Fig. 2, the mass of pentaquarks with two heavy diquarks is calculated, with each diquark containing at least one heavy quark. The mass of heavy pentaquarks at $[(1/2)^-, (3/2)^-$, and $(5/2)^-]$ spin getting from the four-quark state to one antiquark interaction is measured using Eq. (39) for a diquark-diquark system with the considered potential. where M is the pentaquark mass and $m_1$ and $m_2$ are the masses of two diquarks and an antiquark, respectively. The masses of pentaquarks are shown in Tables IV, V, and VI, with $l = 0$ set in all cases, the states having negative parity, and our results compared to Refs. [24, 25, 40, 45–48].

Table IV display the pentaquark masses based on the spin $(1/2)^-$ and negative parity, based on the results of Table I. Table V reports the identical findings but for pentaquarks based on the spin $(3/2)^-$. The spin $(5/2)^-$ pentaquark results are also included in Table VI. Pentaquarks with spin $(1/2)^-$ range in mass from $5.36$ to $24.78$ GeV, whereas those with spin $(3/2)^-$ and $(5/2)^-$ range in mass from $5.32$ to 24.76 and $6.04$ to 23.22 GeV, respectively. The predicted values for the pentaquark masses in this work are consistently larger by about $120$ MeV compared to the values in Refs. [24, 25, 40]. The complete classification of ground state hidden charmm compact pentaquarks have been provided [45]

TABLE VI. The masses of pentaquark at $J^p = (52)^-$ in GeV.

| Pentaquark | M (our) | M (other) |
|---|---|---|
| $[cu]_{s=1}[cu]_{s=1}\bar{u}$ | 6.04 | 4.75 [24] |
| | | 4.87 [25] |
| | | 4.85 [40] |
| $[cu]_{s=1}[cu]_{s=1}\bar{c}$ | 6.585 | 7.09 [25] |
| | | 7.51 [40] |
| $[cc]_{s=1}[cc]_{s=1}\bar{u}$ | 9.056 | - |
| $[cc]_{s=1}[cc]_{s=1}\bar{c}$ | 9.14 | 8.429 [47] |
| $[bu]_{s=1}[bu]_{s=1}\bar{u}$ | 12.62 | 11.22 [24] |
| | | 11.69 [25] |
| | | 11.18 [40] |
| $[bu]_{s=1}[bu]_{s=1}\bar{b}$ | 16.47 | 16.17 [25] |
| | | 17.17 [40] |
| $[bb]_{s=1}[bb]_{s=1}\bar{u}$ | 18.813 | - |
| $[bb]_{s=1}[bb]_{s=1}\bar{b}$ | 23.22 | 25.17 [47] |

and their mass spectrum are studied, and their masses are smaller by $150$ MeV compared to our result. The author of Ref. [46] studies the pentaquarks with four heavy quarks that have the normal $QQQQ\bar{q}$ structure, inside the model of chromomagnetic interaction. The predicted values for our pentaquark masses are consistently larger by about $150$ MeV compared to the values in Ref. [46]. In Ref. [47], the chiral quark model and quark delocalization color screening model are used to systematically examine fully heavy pentaquarks. The predicted values for the heavy pentaquark masses in theis reference are consistently larger by about $90$ MeV compared to our values. The author examined the mass splittings for pentaquark states with the structure $QQQq\bar{q}$ in a chromomagnetic model in Ref. [48], and their masses are smaller by $20$ MeV compared to our result.

## 5. Conclusion

An effective method for resolving the Schrödinger equation with the harmonic potential, the spin-spin interaction, and the Cornell potential is the generalized fractional Nikiforov-Uvarov extended method. The associated energy eigenvalues and wave functions, which are dependent on the fractional parameters $\alpha, \beta$, are obtained. Graphical illustrations have been used to show how the fractional parameter affects the charmomium and bottomonium masses. We observed that decreasing the fractional parameters $\alpha, \beta$ greatly improves the agreement between the predicted mass values and the experimental results. As a result, we obtain the conclusion that the diquark masses are better constrained at lower fractional parameter values than they would be in the classical case at $\alpha, \beta$.

We also noticed that the addition of the spin-spin interaction and the harmonic force to the potential found in research [41] led to improvement in the result and a reduction in the total error rate from $2.38\%$ to $1.95\%$ in charmomium masses, and bottomonium mass from $0.69\%$ to $0.65\%$. Both pentaquark and heavy diquark systems were used to assess the method. To know that the pentaquark has not previously been investigated under the fractional and this potential, and we proved in our research the closeness of our findings with previous studies and their improvement. Our results indicated an acceptable level of agreement when compared to other commonly utilized models. The pentaquark and heavy meson masses are well described by the examination of the analytical solution of the fractional radial Schrödinger equation with the current potential. We hope to extend this model under extreme conditions similar to those described in Refs. [49-52].

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