

A lepton model with nearly Cobimaximal mixing

J. C. Gómez-Izquierdo

*Centro de Estudios Científicos y Tecnológicos No 16, Instituto Politécnico Nacional,
Pachuca: Ciudad del Conocimiento y la Cultura, Carretera Pachuca Actopan km 1+500,
San Agustín Tlaxiaca, Hidalgo, México.
e-mail: cizquierdo@ipn.mx*

A. E. Pozas Ramírez

*Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional,
Edificio 9, 07738, Ciudad de México, México,
e-mail: apozasr1700@alumno.ipn.mx*

Received 6 October 2023; accepted 12 March 2024

Cobimaximal mixing predicts $\pi/4$ and $3\pi/2$ for the atmospheric angle and the Dirac CP-violating phase, respectively. These values are in tension with the neutrino global fits. If this pattern was behind the lepton mixings, then it would have to be broken. In that case, in this paper, we explore the S_3 flavor symmetry within the $B - L$ gauge model where the aforementioned scheme comes from the neutrino sector but the charged lepton contribution breaks the well known predictions so the mixing observables as well as the m_{ee} mass can be accommodated quite well according to the available data. Notably, the predicted regions for the Dirac CP-violating phase would allow us to test the model in future experiments.

Keywords: Cobimaximal; lepton mixings; Dirac.

DOI: <https://doi.org/10.31349/RevMexFis.70.040801>

1. Introduction

The experimental evidence has shown neutrinos have mass, there is more matter than antimatter and dark matter undoubtedly exists. These observations have motivated us to look for answers beyond the Standard Model (SM) since it is not the last theory, even though its tremendous success, and extra ingredients should be added to explain the aforementioned subtle issues.

The Higgs physics, which is currently in developing, will reveal many properties of the discovered Higgs boson at LHC in 2012 [1,2]. All of this will play an important role to have a complete understanding of the Higgs sector that might be linked with fundamental problems [3,4]. In this line of thought, the Higgs and neutrino sector could be related indirectly by the type I see-saw mechanism [5-11] which utilizes a Dirac mass that comes from the Higgs mechanism. Additional connection might have to do with the peculiar pattern that exhibits the PMNS mixing matrix [12,13], in which considering the scheme of three species of neutrinos, it is possible to observe large entry values, as well as $|\mathbf{U}_{\mu i}| \approx |\mathbf{U}_{\tau i}|$ ($i = 1, 2, 3$) [14,15]. The latter fact seems to indicate us the presence of a symmetry, for instance $\mu - \tau$ reflection symmetry [16] that can be identified with Cobimaximal pattern [17-33] whose predictions on the Dirac phase, atmospheric and reactor angles are $3\pi/2$, $\pi/4$ and $\theta_{13} \neq 0$, respectively. From the model building point of view, this kind of pattern may be achieved by means of non-abelian discrete groups [34-41] but one may require extra scalar, fermion or exotic fields. Up to now, there is one Higgs doublet, so that a well motivated

framework is to include many Higgses, that transform in a non-trivial way under the discrete group, to shape the fermion mass matrices and therefore their mixings. This kind of extensions possess a rich phenomenology [42-50] such that flavored leptogenesis is a viable way to explain the excess of matter over antimatter. These theoretical ideas can be tested in the current or future experiments [51].

Once the neutrino masses and mixing angles are added to the well established quark ones, the flavor puzzle [52-54] is awaiting for a compelling solution. In this sense, the flavor symmetries might be a route to find out the underlying physics in the flavor sector and a lot of progress has been made in this direction. For example, appealing patterns [34-41] have been proposed to accommodate the available experimental data. Nonetheless, there is too much work to realize in order to embed the ideal discrete symmetry in the adequate theoretical framework to describe all the observables.

With this in mind, we will recover the well known S_3 [34] symmetry, the permutation of the three objects, which has been implemented in different frameworks [55-75] for studying masses and mixings. There are still strong motivations to study on this symmetry, for instance, the hierarchy among the fermion masses is understood by its irreducible representations and it may come from string theories [54,76]. On the other hand, the $B - L$ gauge model [77-81] is a minimal extension of the SM where the type I see-saw mechanism [5-11] is realized by the inclusion of three right-handed neutrinos (N), which are required by the $SM \otimes U(1)_{B-L}$ gauge group to cancel anomalies. These neutral fermions acquire their mass similarly to the quarks and charged leptons do in the

SM, this is, a Higgs mechanism is realized by the presence of $B - L$ charged singlet field (ϕ) which develops a vacuum expectation value (vev). Eventually, the light neutrinos, that take place in the oscillations, get tiny masses. In addition to neutrino masses and mixings, a rich phenomenology as leptogenesis, dark matter and inflation can be found in the $B - L$ model [82-86].

Motivated by the above issues, we embed the S_3 symmetry in the $B - L$ model so that the family number is extended to the scalar sector. Then, three Higgs doublets and singlets (charged under $B - L$) are included to generate the mixings. Nonetheless, the scalar ($(H_1, H_2) \sim \mathbf{2}, H_3 \sim \mathbf{1}_S$) and lepton ($l_1 \sim \mathbf{1}_S, (l_2, l_3) \sim \mathbf{2}$) families are not treated in the same footing under the irreducible representation of the S_3 symmetry. The main purpose to realize it, in the scalar sector, is to use the results on the allowed alignments that scalar potential with three Higgs doublets together with S_3 permits [87-91]. Although many scalar fields have been added, the number of Yukawa couplings are reduced substantially by imposing an extra symmetry (Z_2). Along with this, the assignation makes it possible to identify the Cobimaximal pattern, in the effective neutrino sector, by assuming real Dirac and Majorana Yukawa couplings. Whereas the charged lepton mass matrix ends up being complex due to the charged Yukawa couplings are assumed to be complex. In this benchmark, we calculate explicitly the PMNS matrix that is mainly controlled by the Cobimaximal one while the charged lepton breaks the well known predictions on the atmospheric angle (θ_{23}) and Dirac CP-violating phase (δ_{CP}). Notably, the PMNS mixing matrix, and consequently, the aforementioned observables depend on two free parameters, this feature makes the model completely different from other flavored ones [92-96]. An analytical and numerical study on the atmospheric angle and Dirac CP-violating phase allows to constrain the allowed region of values for the free parameters. In consequence, our findings are as follows: the former observable can be on the upper or lower octant for the normal and inverted hierarchy. Besides this, for the δ_{CP} phase, the predicted region is more constrained for the inverted hierarchy in comparison to the normal case, these facts would allow to test the model by confronting our results with future experiments [97-99]. Taking into account the $|m_{ee}|$ effective neutrino mass, this is controlled totally by the reactor and solar angle, so this is consistent with the experimental data.

The letter is organized as follows. The $B - L$ model is revisited briefly in Sec. 2. The S_3 discrete symmetry is embedded in the $B - L$ model, the masses and mixings are obtained in Sec. 3. In Sec. 4, we perform explicitly the PMNS mixing matrix as well as the deviation to the atmospheric angle and the Dirac CP-violating phase. The findings are highlighted through some scattered plots. In addition, in Sec. 5, the $|m_{ee}|$ effective neutrino is calculated numerically. Finally, we end up giving concluding remarks in Sec. 6.

2. B-L Model

One of the gauge models, that takes into account the missing neutrino mass term, is the well known Baryon minus Lepton (B-L) one [77-81] where the inclusion of three right-handed neutrinos (N) are required by the gauge group. These neutral fermions acquire their mass similarly to the quarks and charged leptons do in the SM framework, this is, a Higgs mechanism is realized by the presence of singlet field (ϕ) which develops a vacuum expectation value (vev). Eventually, the light (active) neutrinos, that take place in the oscillations, get tiny mass due to the type I see-saw mechanism [5-11].

In the minimal version, under the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge group, the matter content has the usual quantum numbers. In Table I, we just show the quantum numbers under the $U(1)_{B-L}$ group.

Hence, apart from the SM Lagrangian, one has to add extra terms. This is

$$\mathcal{L}_{BL} = \mathcal{L}_{SM} - y^D \bar{L} \tilde{H} N - \frac{1}{2} y^N \bar{N}^c \phi N - V(H, \phi), \quad (1)$$

with

$$V(H, \phi) = \mu_{BL}^2 \phi^\dagger \phi + \frac{\lambda_{BL}}{2} (\phi^\dagger \phi)^2 - \lambda_{H\phi} (H^\dagger H) (\phi^\dagger \phi), \quad (2)$$

where $\tilde{H}_i = i\sigma_2 H_i^*$ and σ_2 is the second Pauli matrix.

In this framework, there are two scales of the spontaneous symmetry breaking, the first one is associated to the $U(1)_{B-L}$ group and the breaking scale is larger than the electroweak scale, $\phi_0 \gg v$. The latter associates to $SU(2)_L$.

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \phi \rangle = \frac{\phi_0}{\sqrt{2}}. \quad (3)$$

In the standard basis, the fermion mass term is given by

$$-\mathcal{L}_Y = \bar{\ell}_{iL} (\mathbf{M}_\ell)_{ij} \ell_{jR} + \frac{1}{2} \bar{\nu}_{iL} (\mathbf{M}_\nu)_{ij} \nu_{jL}^c + \frac{1}{2} \bar{N}_i^c (\mathbf{M}_R)_{ij} N_j + h.c., \quad (4)$$

where the type I see-saw was realized.

In general, the above mass matrices are complex, and they do not possess any pattern which help us find out the mixing matrices that take place in PMNS mixings one. Therefore, we have the need to use discrete symmetries as guidance to explain the peculiar patterns.

TABLE I. Minimal $B - L$ model.

Matter	Quarks	Leptons	Higgs	ϕ
B-L	1/3	1	0	-2

3. Lepton sector

As we already commented, the scalar sector of the $B - L$ model is augmented such that three Higgs doublets (H_i) and singlets (ϕ_i) are considered. Then, for leptons the assignment under the flavor symmetry is given as follows: the first family is put in a $\mathbf{1}_S$ singlet; the second and third families live in a $\mathbf{2}$ doublet. On the other side, the scalar sector is assigned as the usual studios have realized this is, the first and second family are put together within a $\mathbf{2}$ and the third one belongs to $\mathbf{1}_S$. The main reason to keep the typical assignment in the scalar sector is to take into account the results on the \mathbf{S}_3 scalar potential with three Higgs where the aforementioned assignment was utilized and an exhaustive classification about the vev's alignments was released in Refs. [88-91].

TABLE II. Flavored $B - L$ model. Here, $I = 1, 2$ and $J = 2, 3$.

Matter	H_I, L_J, e_{JR}, N_J	L_1, e_{1R}, N_1	H_3, ϕ_3	ϕ_I
\mathbf{S}_3	2	$\mathbf{1}_S$	$\mathbf{1}_S$	2
\mathbf{Z}_2	1	-1	1	-1

On the other hand, some Yukawa couplings are forbidden by imposing the \mathbf{Z}_2 symmetry. In this manner, the Dirac and charged lepton mass matrices end up being almost diagonal, then the right-handed neutrino mass matrix will provide the mixings.

Table II shows explicitly the full assignment for the matter in the current model.

The Yukawa mass term that respects the $\mathbf{S}_3 \otimes \mathbf{Z}_2$ flavor symmetry and the gauge group, is given by

$$\begin{aligned}
 -\mathcal{L}_Y = & y_1^e \bar{L}_1 H_3 e_{1R} + y_2^e [(\bar{L}_2 H_2 + \bar{L}_3 H_1) e_{2R} + (\bar{L}_2 H_1 - \bar{L}_3 H_2) e_{3R}] + y_3^e [\bar{L}_2 H_3 e_{2R} + \bar{L}_3 H_3 e_{3R}] + y_1^D \bar{L}_1 \tilde{H}_3 N_1 \\
 & + y_1^D \bar{L}_1 \tilde{H}_3 N_1 + y_2^N [(\bar{L}_2 \tilde{H}_2 + \bar{L}_3 \tilde{H}_1) N_2 + (\bar{L}_2 \tilde{H}_1 - \bar{L}_3 \tilde{H}_2) N_3] + y_1^N \bar{N}_1^c \phi_3 N_1 + y_2^N [\bar{N}_1^c (\phi_1 N_2 + \phi_2 N_3) \\
 & + (\bar{N}_2^c \phi_1 + \bar{N}_3^c \phi_2) N_1] + y_3^N [\bar{N}_2^c \phi_3 N_2 + \bar{N}_3^c \phi_3 N_3] + h.c.
 \end{aligned} \tag{5}$$

From Eq. (5), the mass matrices have the following form

$$\begin{aligned}
 \mathbf{M}_e &= \begin{pmatrix} y_1^e \langle H_3 \rangle & 0 & 0 \\ 0 & y_3^e \langle H_3 \rangle + y_2^e \langle H_2 \rangle & y_2^e \langle H_1 \rangle \\ 0 & y_2^e \langle H_1 \rangle & y_3^e \langle H_3 \rangle - y_2^e \langle H_2 \rangle \end{pmatrix}; \\
 \mathbf{M}_D &= \begin{pmatrix} y_1^D \langle \tilde{H}_3 \rangle & 0 & 0 \\ 0 & y_3^D \langle \tilde{H}_3 \rangle + y_2^D \langle \tilde{H}_2 \rangle & y_2^D \langle \tilde{H}_1 \rangle \\ 0 & y_2^D \langle \tilde{H}_1 \rangle & y_3^D \langle \tilde{H}_3 \rangle - y_2^D \langle \tilde{H}_2 \rangle \end{pmatrix}; \\
 \mathbf{M}_R &= \begin{pmatrix} y_1^N \langle \phi_3 \rangle & y_2^N \langle \phi_1 \rangle & y_2^N \langle \phi_2 \rangle \\ y_2^N \langle \phi_1 \rangle & y_3^N \langle \phi_3 \rangle & 0 \\ y_2^N \langle \phi_2 \rangle & 0 & y_3^N \langle \phi_3 \rangle \end{pmatrix}.
 \end{aligned} \tag{6}$$

Before finishing this section, we would like to add a comment about the scalar potential in the scenario of three Higgs doublets with \mathbf{S}_3 symmetry (3HD-S3). As it is well known, the scalar potential study becomes mandatory to get a realistic scenario that intends to explain the mixing patterns. In this direction, works on the minimization of the scalar potential, vev's alignments, mass spectrum and phenomenology can be found in Refs. [87,100-106]. Besides this, exhaustive studios have released significant results on the minimization of the scalar potential and alignments that are allowed [88-91] in the 3HD-S3 framework. Remarkably, those have provided a larger list of alignments which must have great impact on the gauge sector, fermion and scalar masses. Having commented that, in our model, we will take advantage of the previous results, so it is not necessary to analyze again the potential. The reason is the following: the full potential is given by $V_{B-L} = V(H) + V(\phi) + V(H, \phi)$, then we make a strong

assumption, this is, the Higgses and singlets do not interact between them so one can treat them separately in consequence $V(H)$ and $V(\phi)$ share the same structure under the flavor symmetry as a result of this we can use the analysis realized in Ref. [88-91]. A partial study on the V_{B-L} potential was carried out in Ref. [107].

3.1. Masses and mixings

One of the allowed alignments for the vev's is given by $\langle H_1 \rangle = v_1$, $\langle H_2 \rangle = iv_2$ and $\langle H_3 \rangle = v_3$ [88-91]. Hence, we will utilize those in the Dirac neutrinos and charged lepton. It is worth mention that it has not been yet explored in the quark sector and this is a work in progress. In the Majorana sector, the right-handed neutrinos get their mass through

the ϕ scalars so that we will also consider the alignment $\langle\phi_2\rangle = \langle\phi_1\rangle$ which is also a solution of the scalar potential as was shown in Ref. [87].

Having chosen the alignments, we have

$$\begin{aligned}\mathbf{M}_e &= \begin{pmatrix} a_e & 0 & 0 \\ 0 & c_e + ib_e & d_e \\ 0 & d_e & c_e - ib_e \end{pmatrix}, \\ \mathbf{M}_D &= \begin{pmatrix} a_D & 0 & 0 \\ 0 & c_D - ib_D & d_D \\ 0 & d_D & c_D + ib_D \end{pmatrix}, \\ \mathbf{M}_R &= \begin{pmatrix} a_R & b_R & b_R \\ b_R & c_R & 0 \\ b_R & 0 & c_R \end{pmatrix},\end{aligned}\quad (7)$$

where

$$\begin{aligned}a_e &= y_1^e v_3, & c_e &= y_3^e v_3, & b_e &= y_2^e v_2, & d_e &= y_2^e v_1, \\ a_D &= y_1^D v_3, & c_D &= y_3^D v_3, & b_D &= y_2^D v_2, & d_D &= y_2^D v_1, \\ a_R &= y_1^N \langle\phi_3\rangle, & b_R &= y_2^N \langle\phi_1\rangle, & c_R &= y_3^N \langle\phi_3\rangle.\end{aligned}\quad (8)$$

3.1.1. Charged lepton mixings

In this brief section, let us obtain the mixing matrix that takes place in the PMNS one. To do this, we assume that the charged Yukawa couplings are complex. Then, \mathbf{M}_e mass matrix is written as

$$\mathbf{M}_e = \begin{pmatrix} a_e & 0 & 0 \\ 0 & A_e & d_e \\ 0 & d_e & B_e \end{pmatrix},\quad (9)$$

with $A_e = c_e + ib_e$ and $B_e = c_e - ib_e$. The above matrix is diagonalized by $\mathbf{U}_{eL} = \mathbf{P}_e \mathbf{O}_e$ and $\mathbf{U}_{eR} = \mathbf{P}_e^\dagger \mathbf{O}_e$ such that

$\hat{\mathbf{M}}_e = \text{Diag.}(m_e, m_\mu, m_\tau) = \mathbf{U}_{eL}^\dagger \mathbf{M}_e \mathbf{U}_{eR}$. Explicitly, we obtain

$$\begin{aligned}\mathbf{O}_e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_e & \sin \theta_e \\ 0 & -\sin \theta_e & \cos \theta_e \end{pmatrix}, \\ \mathbf{P}_e &= \text{Diag.}(e^{i\eta_e}, e^{i\eta_\mu}, e^{i\eta_\tau}).\end{aligned}\quad (10)$$

Along with this, $\eta_e = \arg.(a_e)/2$, $\eta_\mu = \arg.(A_e)/2$, $\eta_\tau = \arg.(B_e)/2$ and $\eta_\mu + \eta_\tau = \arg.(b_e)$. At the same time, we have

$$\cos \theta_e = \sqrt{\frac{m_\tau - |A_e|}{m_\tau - m_\mu}}, \quad \sin \theta_e = \sqrt{\frac{|A_e| - m_\mu}{m_\tau - m_\mu}}.\quad (11)$$

Notice that there is a free parameter, $|A_e|$, which has to satisfy the following constraint $m_\tau > |A_e| > m_\mu$.

3.1.2. Neutrino mixings

According to the previous section, we considered $\langle\phi_1\rangle = \langle\phi_2\rangle$. For this reason, in the Majorana sector, the free parameters are reduced in \mathbf{M}_R , and the inverse matrix is parametrized as

$$\mathbf{M}_R^{-1} = \begin{pmatrix} \mathcal{X} & -\mathcal{Y} & -\mathcal{Z} \\ -\mathcal{Y} & \mathcal{W} & \mathcal{Z} \\ -\mathcal{Y} & \mathcal{Z} & \mathcal{W} \end{pmatrix},\quad (12)$$

where we have defined $\mathcal{X} = c_R^2/|\mathbf{M}_R|$, $\mathcal{Y} = b_R c_R/|\mathbf{M}_R|$, $\mathcal{W} = (a_R c_R - b_R^2)/|\mathbf{M}_R|$ and $\mathcal{Z} = b_R^2/|\mathbf{M}_R|$. Here, $|\mathbf{M}_R|$ denotes the determinant of \mathbf{M}_R . With all of this, the effective neutrino mass matrix, $\mathbf{M}_\nu = -\mathbf{M}_D \mathbf{M}_R^{-1} \mathbf{M}_D^T$, is given by

$$\mathbf{M}_\nu = \begin{pmatrix} \mathcal{X} a_D^2 & -\mathcal{Y} a_D [c_D + d_D - ib_D] & -\mathcal{Y} a_D [c_D + d_D + ib_D] \\ -\mathcal{Y} a_D [c_D + d_D - ib_D] & \mathcal{W} [(c_D - ib_D)^2 + d_D^2] + 2\mathcal{Z} d_D [c_D - ib_D] & 2\mathcal{W} d_D c_D + \mathcal{Z} [b_D^2 + c_D^2 + d_D^2] \\ -\mathcal{Y} a_D [c_D + d_D + ib_D] & 2\mathcal{W} d_D c_D + \mathcal{Z} [b_D^2 + c_D^2 + d_D^2] & \mathcal{W} [(c_D + ib_D)^2 + d_D^2] + 2\mathcal{Z} d_D [c_D + ib_D] \end{pmatrix},\quad (13)$$

which can be parametrized as

$$\mathbf{M}_\nu \equiv \begin{pmatrix} A_\nu & B_\nu & B_\nu^* \\ B_\nu & C_\nu^* & D_\nu \\ B_\nu^* & D_\nu & C_\nu \end{pmatrix}.\quad (14)$$

We ought to stress that one can identify the above mass matrix with the Cobimaximal one if the Dirac Yukawa couplings and the vev's values are real. In this benchmark, the PMNS mixings are governed mainly by the neutrino sector. It is well known that \mathbf{M}_ν is diagonalized by $\mathbf{U}_\nu = \mathbf{U}_\alpha \mathbf{O}_{23} \mathbf{O}_{13} \mathbf{O}_{12} \mathbf{U}_\beta$ such that $\hat{\mathbf{M}}_\nu = \text{Diag.}(|m_1|, |m_2|, |m_3|) = \mathbf{U}_\nu^\dagger \mathbf{M}_\nu \mathbf{U}_\nu^*$ where the diagonal mass matrices are given as $\mathbf{U}_\alpha = \text{Diag.}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ and $\mathbf{U}_\beta = \text{Diag.}(1, e^{i\beta_1}, e^{i\beta_2})$ [20]. These stand for unphysical and Majorana phases, respectively. In addition, we have

$$\begin{aligned}\mathbf{O}_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \rho_{23} & \sin \rho_{23} \\ 0 & -\sin \rho_{23} & \cos \rho_{23} \end{pmatrix}, & \mathbf{O}_{13} &= \begin{pmatrix} \cos \rho_{13} & 0 & \sin \rho_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin \rho_{13} e^{i\delta} & 0 & \cos \rho_{13} \end{pmatrix}, \\ \mathbf{O}_{12} &= \begin{pmatrix} \cos \rho_{12} & \sin \rho_{12} & 0 \\ -\sin \rho_{12} & \cos \rho_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}\quad (15)$$

With $\rho_{23} = \pi/4$, $\delta = -\pi/2$, $\alpha_1 = 0 = \alpha_3$ and $\alpha_2 = \pi$; also, $\beta_1 = 0$ and $\beta_2 = \pi/2$. Thus, we reconstruct the neutrino mass matrix, $\mathbf{M}_\nu = \mathbf{U}_\nu \hat{\mathbf{M}}_\nu \mathbf{U}_\nu^T$, whose elements are

$$\begin{aligned} A_\nu &= |m_3| \sin^2 \rho_{13} + \cos^2 \rho_{13} [|m_1| \cos^2 \rho_{12} + |m_2| \sin^2 \rho_{12}], \\ B_\nu &= \frac{\cos \rho_{13}}{\sqrt{2}} [(|m_1| - |m_2|) \cos \rho_{12} \sin \rho_{12} + i \sin \rho_{13} (|m_3| - |m_1| \cos^2 \rho_{12} - |m_2| \sin^2 \rho_{12})], \\ C_\nu &= \frac{1}{2} [|m_2| (\cos \rho_{12} - i \sin \rho_{13} \sin \rho_{12})^2 + |m_1| (\sin \rho_{12} + i \cos \rho_{12} \sin \rho_{13})^2 - |m_3| \cos^2 \rho_{13}], \\ D_\nu &= \frac{1}{2} [|m_2| (\cos^2 \rho_{12} + \sin^2 \rho_{13} \sin^2 \rho_{12}) + |m_1| (\sin^2 \rho_{12} + \cos^2 \rho_{12} \sin^2 \rho_{13}) + |m_3| \cos^2 \rho_{13}]. \end{aligned} \quad (16)$$

Finally, we obtain

$$\mathbf{U}_\nu = \begin{pmatrix} \cos \rho_{13} \cos \rho_{12} & \cos \rho_{13} \sin \rho_{12} & -\sin \rho_{13} \\ \frac{1}{\sqrt{2}} (\sin \rho_{12} - i \cos \rho_{12} \sin \rho_{13}) & -\frac{1}{\sqrt{2}} (\cos \rho_{12} + i \sin \rho_{12} \sin \rho_{13}) & -i \frac{\cos \rho_{13}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} (\sin \rho_{12} + i \cos \rho_{12} \sin \rho_{13}) & -\frac{1}{\sqrt{2}} (\cos \rho_{12} - i \sin \rho_{12} \sin \rho_{13}) & i \frac{\cos \rho_{13}}{\sqrt{2}} \end{pmatrix}. \quad (17)$$

As one can realize, if $\rho_{13} = 0$, it would imply the presence of $\mu \leftrightarrow \tau$ symmetry in \mathbf{M}_ν or a $2 \leftrightarrow 3$ exchange due to the charged lepton mass matrix is not diagonal. As a result, one expects a deviation to atmospheric angle and the Dirac phase.

4. Results

Once the relevant mixing matrices were performed, the PMNS one is defined as $\mathbf{U} = \mathbf{U}_e^\dagger \mathbf{U}_\nu$ and the theoretical matrix elements are given explicitly as

$$\begin{aligned} \mathbf{U}_{11} &= (\mathbf{U}_\nu)_{11} e^{-i\eta_e}, \\ \mathbf{U}_{12} &= (\mathbf{U}_\nu)_{12} e^{-i\eta_e}, \\ \mathbf{U}_{13} &= (\mathbf{U}_\nu)_{13} e^{-i\eta_e}, \\ \mathbf{U}_{21} &= [\cos \theta_e (\mathbf{U}_\nu)_{21} - \sin \theta_e (\mathbf{U}_\nu)_{31} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \\ \mathbf{U}_{22} &= [\cos \theta_e (\mathbf{U}_\nu)_{22} - \sin \theta_e (\mathbf{U}_\nu)_{32} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \\ \mathbf{U}_{23} &= [\cos \theta_e (\mathbf{U}_\nu)_{23} - \sin \theta_e (\mathbf{U}_\nu)_{33} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \\ \mathbf{U}_{31} &= [\sin \theta_e (\mathbf{U}_\nu)_{21} + \cos \theta_e (\mathbf{U}_\nu)_{31} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \\ \mathbf{U}_{32} &= [\sin \theta_e (\mathbf{U}_\nu)_{22} + \cos \theta_e (\mathbf{U}_\nu)_{32} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \\ \mathbf{U}_{33} &= [\sin \theta_e (\mathbf{U}_\nu)_{23} + \cos \theta_e (\mathbf{U}_\nu)_{33} e^{-i\eta_\nu}] e^{-i\eta_\mu}, \end{aligned} \quad (18)$$

with $\eta_\nu \equiv \eta_\tau - \eta_\mu$ being a relative phase.

In consequence, the expressions for the mixing angles are obtained by comparing our theoretical formula with the standard parametrization of the PMNS.

$$\begin{aligned} \sin^2 \theta_{13} &= |(\mathbf{U})_{13}|^2 = \sin^2 \rho_{13}, \\ \sin^2 \theta_{23} &= \frac{|(\mathbf{U})_{23}|^2}{1 - |(\mathbf{U})_{13}|^2} = \frac{1}{2} [1 + \sin 2\theta_e \cos \eta_\nu], \\ \sin^2 \theta_{12} &= \frac{|(\mathbf{U})_{12}|^2}{1 - |(\mathbf{U})_{13}|^2} = \sin^2 \rho_{12}. \end{aligned} \quad (19)$$

In addition, the Jarlskog invariant can be performed analytically. As one can verify straightforward, using Eqs. (17)-(19), we obtain

$$\begin{aligned} \sin \delta_{CP} &= \frac{\text{Im} [(\mathbf{U})_{23} (\mathbf{U})_{13}^* (\mathbf{U})_{12} (\mathbf{U})_{22}^*]}{\frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13}}; \\ \sin \delta_{CP} &= -\frac{1}{2} \frac{\cos \rho_{12} \sin \rho_{12} \sin \rho_{13} \cos^2 \rho_{13} \cos 2\theta_e}{\cos \theta_{12} \sin \theta_{12} \cos \theta_{23} \sin \theta_{23} \sin \theta_{13} \cos^2 \theta_{13}}; \\ \sin \delta_{CP} &= -\frac{\cos 2\theta_e}{\sqrt{1 - \sin^2 2\theta_e \cos^2 \eta_\nu}}. \end{aligned} \quad (20)$$

Having obtained the above expressions, some comments are added in order. The reactor and solar angles are controlled mainly by the neutrino sector. What is more these are associated directly to the ρ_{13} and ρ_{12} parameters, respectively. Also, the atmospheric angle is deviated from $\pi/4$ and the source of such deviation is the charged lepton sector as one would expect. Also, strictly speaking the magnitude of the PMNS depends on four free parameters namely ρ_{12} , ρ_{13} , θ_e (or $|A_e|$) and the η_ν relative phase. Nevertheless, as it was already remarked, $\rho_{13} = \theta_{13}$ and $\rho_{12} = \theta_{12}$ so we end up having two free parameters, this is, $|A_e|$ and η_ν . These must be fixed by numerical study to get the PMNS values. Before making this, it is worthy pointing out extreme values for the mentioned parameters and their implications on the atmospheric angle and the Dirac CP-violating phase:

- If the η_ν effective phase is zero (or π), the atmospheric angle would lie on the upper (lower) octant and the $\delta_{CP} = -\pi/2$.
- If the η_ν effective phase is $\pi/2$, the atmospheric angle would be $\pi/4$ exactly and the δ_{CP} would be deviated from $-\pi/2$.
- If $|A_e| \approx m_\mu$, θ_{23} would be $\pi/4$ and $\delta_{CP} = -\pi/2$. Whereas, if $|A_e| \approx m_\tau$ then the θ_{23} would be $\pi/4$ and $\delta_{CP} = \pi/2$.

- If $|A_e| = (m_\mu + m_\tau)/2$ ($\theta_e = \pi/4$), then θ_{23} would lie on the upper or lower octant and $\delta_{CP} = \pi$ or 2π .

In short, the atmospheric angle and the Dirac phase are totally different of $\pi/4$ and $3\pi/2$, respectively. Hence, scattered plots will be performed to find out the full set of allowed values for the free parameters ($|A_e|$ and η_ν) that accommodates the observables up to 3σ . To make this, the charged lepton masses [16] will be considered like inputs as well as the atmospheric angle and Dirac phase. At the electroweak scale, we have

$$\begin{aligned} m_e &= 0.48307 \pm 0.00045 \text{ MeV}, \\ m_\mu &= 101.766 \pm 0.023 \text{ MeV}, \\ m_\tau &= 1728.56 \pm 0.28 \text{ MeV}. \end{aligned} \quad (21)$$

Along with this, the experimental neutrino data are given at 3σ as follows [14]

$$\begin{aligned} \sin^2 \theta_{13} &= 0.02000 - 0.02405, & \sin^2 \theta_{23} &= 0.434 - 0.610, \\ \delta_{CP}/^\circ &= 128 - 359; & \text{Normal ordering} \\ \sin^2 \theta_{13} &= 0.02018 - 0.02424, & \sin^2 \theta_{23} &= 0.433 - 0.608, \\ \delta_{CP}/^\circ &= 200 - 353, & \text{Inverted ordering.} \end{aligned} \quad (22)$$

In addition, $\sin^2 \theta_{12} = 0.271 - 0.369$ for the normal and inverted hierarchy.

From Eqs. (20) and (19), we vary arbitrarily the free parameters in their allowed range such that these fit the observables at 3σ .

$$\begin{aligned} \delta_{CP}(|A_e|(\theta_e), \eta_\nu) &= \arcsin \left[-\frac{\cos 2\theta_e}{\sqrt{1 - \sin^2 2\theta_e \cos^2 \eta_\nu}} \right], \\ \sin^2 \theta_{23}(|A_e|(\theta_e), \eta_\nu) &= \frac{1}{2} [1 + \sin 2\theta_e \cos \eta_\nu]. \end{aligned} \quad (23)$$

Having commented on that, let us show the relevant plots. In Fig. 1, we observe the set of values for the η_ν relative phase that fits the atmospheric angle. There are two regions where the angle lies on the upper and lower octant so there is a clear deviation around $\pi/4$. In addition, the allowed regions are similar for the normal and inverted hierarchy.

The $|A_e|$ parameter is constrained by the Dirac phase such that its allowed region is reduced substantially in the inverted ordering as we observe in Fig. 2. On the contrary, the region is large for the normal hierarchy and the main reason has to do with the experimental data [see Eq. (22)]. Also, we realize a large value for $|A_e|$ is needed to reach the best fit (194°).

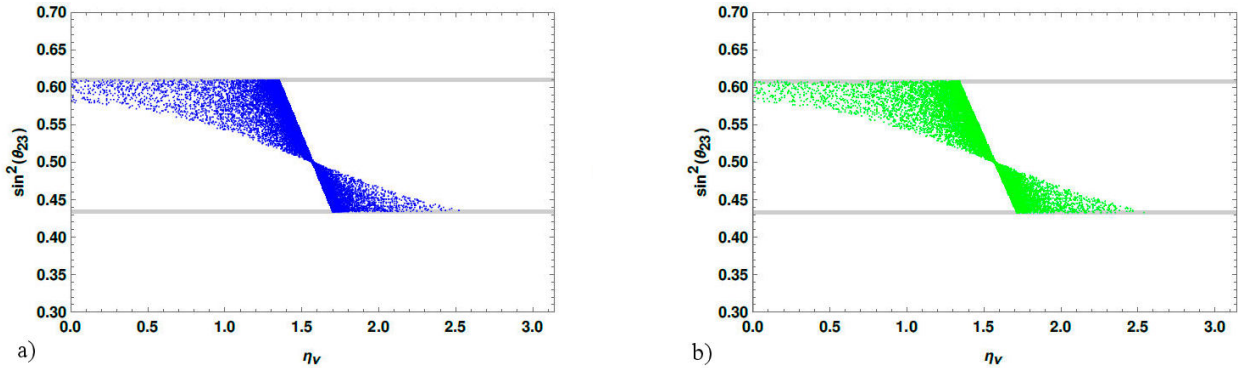


FIGURE 1. Atmospheric angle as function of the η_ν relative phase for the normal and inverted hierarchy, respectively. The solid gray line stands for the experimental data at 3σ .

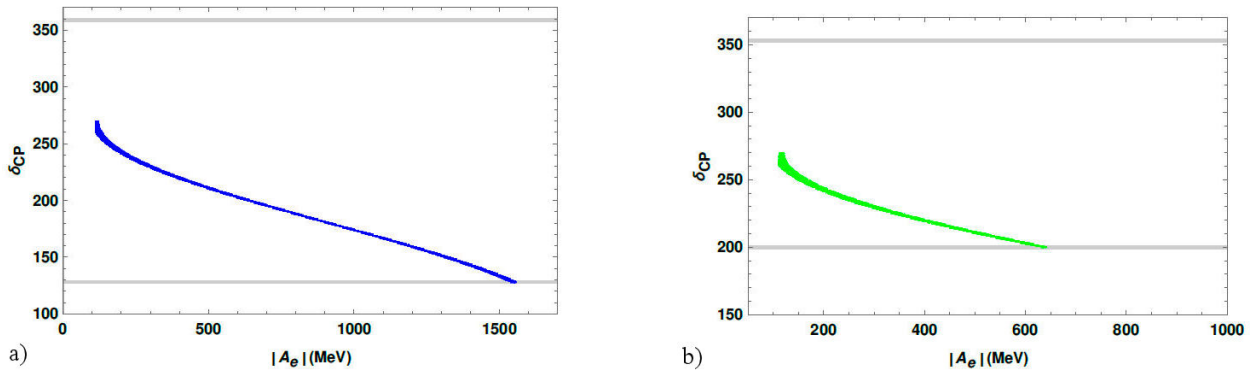


FIGURE 2. Dirac CP-violating phase as function of the $|A_e|$ parameter for the normal and inverted hierarchy, respectively. The solid gray line stands for the experimental data at 3σ .

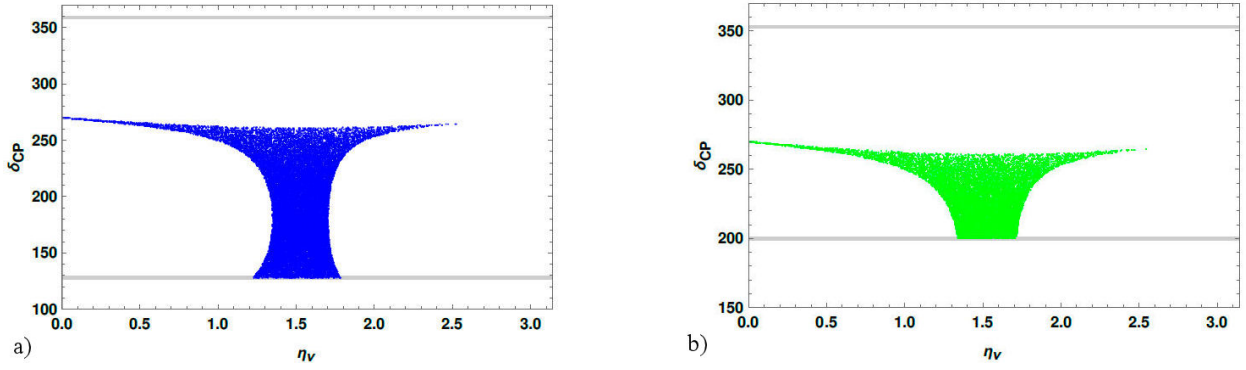


FIGURE 3. Dirac CP-violating phase as function of the η_ν parameter for the normal and inverted hierarchy, respectively. The solid gray line stands for the experimental data at 3σ .

In Fig. 3, the Dirac phase is also show as function of the η_ν relative phase. In the inverted ordering the allowed region is smaller than the normal case and the explanation resides on the experimental data at 3σ for the Dirac phase.

5. The neutrinoless double beta decay

As it is well known, the nature of neutrinos is not well established yet so those can be Dirac or Majorana fermions. The latter can be tested by means of the neutrinoless double beta decay [108-111] which occurs if and only if the neutrinos are Majorana fermions. In general, the decay width of $0\nu\beta\beta$ is written as

$$\Gamma^{0\nu} = \sum_j G_j(Q, Z) |\mathcal{M}_j F_j|^2, \quad (24)$$

where the above expression includes all the possible mechanisms whose matrix element is denoted by \mathcal{M}_j and F_j stands for a dimensionless particle physics parameter; $G(Q, Z)$ is a phase space factor that can also depend on the particle physics [108-111]. In the scenario where light active neutrinos are involved the main contribution is provided by F_ν the parameter and decay width are written as

$$\Gamma^{0\nu} = G_{01}^{0\nu} |\mathcal{M}_\nu^{0\nu}|^2 |F_\nu|^2, \quad (25)$$

with

$$F_\nu = \frac{1}{m_e} \sum_{i=1}^3 \overbrace{(\mathbf{U})_{ei}^2}^{m_{ee}} m_i, \quad (26)$$

where \mathbf{U} stands for the PMNS matrix, m_i is the active neutrino masses and m_e the electron mass. So far, there is an upper limit $|m_{ee}| < 0.06 - 0.2$ eV given by GERDA collaboration [112].

In our model, m_{ee} is not affected by the contribution of the charged lepton sector. Indeed, this is given by

$$|m_{ee}| = \left| \cos^2 \theta_{13} (|m_1| \cos^2 \theta_{12} + |m_2| \sin^2 \theta_{12}) + |m_3| \sin^2 \theta_{13} \right|. \quad (27)$$

From the neutrino oscillation data [14,15], the available information on the neutrino masses is given by the squared mass difference $\Delta m_{21}^2 = |m_2|^2 - |m_1|^2$ and $\Delta m_{31}^2 = |m_3|^2 - |m_1|^2$ ($\Delta m_{13}^2 = |m_1|^2 - |m_3|^2$) for the normal (inverted) ordering. At 3σ , we have $\Delta m_{21}^2 [10^{-5} \text{ eV}^2] = 6.94 - 8.14$ and $\Delta m_{31}^2 [10^{-3} \text{ eV}^2] = 2.47 - 2.63$ ($\Delta m_{13}^2 [10^{-3} \text{ eV}^2] = 2.37 - 2.53$). Then, fixing two neutrino masses in terms of the lightest one and the squared mass scale, we get for the normal and inverted ordering

$$\begin{aligned} |m_3|^2 &= \Delta m_{31}^2 + |m_1|^2, & |m_2|^2 &= \Delta m_{21}^2 + |m_1|^2, \\ |m_2|^2 &= \Delta m_{21}^2 + \Delta m_{13}^2 + |m_3|^2, \\ |m_1|^2 &= \Delta m_{13}^2 + |m_3|^2. \end{aligned} \quad (28)$$

Up to now, the absolute neutrino masses are still unknown however there is a bound, $\sum_{i=1}^3 m_i < 0.12$ eV, given by Planck collaboration that can be useful to determine them. In order to scan the allowed regions of values for the $|m_{ee}|$ mass, we vary θ_{12} and θ_{13} according to Eq. (22) as well as the $|m_{1,3}|$ lightest neutrino mass varies up to 0.1 eV. Our findings are shown in Fig. 4.

As one notices, the allowed regions are compatible with the GERDA phase-I data [112].

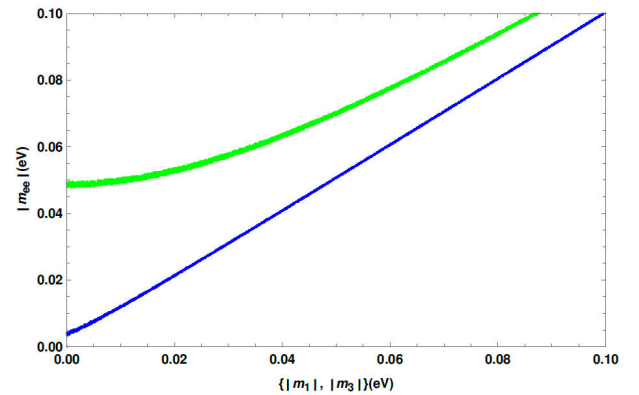


FIGURE 4. Allowed regions for $|m_{ee}|$ as function of the lightest neutrino mass.

6. Concluding remarks

Discrete symmetries have been an option to understand the peculiar lepton pattern. Too much effort has been made in order to build the model, with the ideal flavor symmetry within the suitable framework, that matches the experimental data. Nonetheless, there is no such model so far.

In this letter, we just addressed the lepton masses and mixings by implementing the $S_3 \otimes Z_2$ in the $B - L$ gauge model with enlarged scalar sector. A particular choice is realized on the scalar and lepton families under the S_3 irreducible representations, and if the Dirac, Majorana and charged lepton Yukawa couplings are real and complex, respectively. This leads to a deviation from Cobimaximal mixing pattern due to the charged leptons. According to our findings, the PMNS mixing matrix does depend on two free parameters ($|A_e|$ and η_ν) which were constrained by analyzing their effect on the atmospheric angle and the Dirac CP-violating phase that end up being deviated from $\pi/4$ and $3\pi/2$, respectively. For normal and inverted hierarchy, we found a set of values for the free parameters that accommodates the atmospheric angle with great accuracy (up to 3σ) and it can be on the lower and upper octant. We stress that there is not a clear difference between both regions of values for the η_ν free parameter. Taking into account the Dirac CP-violating phase, the numerical study allows to distinguish two regions

of values for each free parameter, this can be relevant to test the model in future experiments [97-99]. Additionally, the $|m_{ee}|$ effective neutrino mass is completely independent of the charged lepton contribution so that it was well fixed by the reactor and solar angles. As a result, the allowed regions are consistent with the available data.

Certainly, this model can be considered like one more to the market with all its limitations. In our opinion, there are still many ideas to explore by utilizing the S_3 symmetry in the context of three Higgs doublets where a rich phenomenology has not been explored with detail. For instance, an exhaustive study on the scalar and gauge sectorⁱ and its effect on the charged lepton flavor violation and leptogenesis. Consequently, we pretend to be more ambitious so that the quark sector and the above issues will be addressed in the model to have a complete study.

Acknowledgements

JCGI thanks Alma Leticia Gálves Mendoza for its valuable support at CECyT No. 16. Special mention to Rosalva Irelan Sánchez García and Edgar Geovanni Trejo Escamilla for preparing the early spanish version of the manuscript. This work was partially financed by Secretaria de Investigación y Posgrado del Instituto Politécnico Nacional under Projects 20230568 and PAPIIT IN109321.

ⁱ. A partial study was realized in Ref. [106].

1. G. Aad *et al.* (ATLAS), Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC, *Phys. Lett. B* **716** (2012) 11, <https://doi.org/10.1016/j.physletb.2012.08.020>.
2. S. Chatrchyan *et al.* (CMS), Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC, *Phys. Lett. B* **716** (2012) 30, <https://doi.org/10.1016/j.physletb.2012.08.021>.
3. G. P. Salam, L.-T.Wang and G. Zanderighi, The Higgs boson turns ten, *Nature* **607** (2022) 41, <https://doi.org/10.1038/s41586-022-04899-4>.
4. M. E. Peskin, Elementary Particle Physics Vision for EPP2024 (2023), arXiv:2302.05472 [hep-ph]
5. P. Minkowski, $\mu \rightarrow e$ gamma at a Rate of One Out of 1-Billion Muon Decays?, *Phys. Lett. B* **67** (1977) 421, [https://doi.org/10.1016/0370-2693\(77\)90435-x](https://doi.org/10.1016/0370-2693(77)90435-x).
6. T. Yanagida, Horizontal gauge symmetry and masses of neutrinos In Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, Tsukuba, Japan, 13-14 (1979).
7. M. Gell-Mann, P. Ramond and R. Slansky, Complex Spinors and Unified Theories, *Conf. Proc. C* **790927** (1979) 315, <https://doi.org/10.1142/97898128368540018>.
8. R. N. Mohapatra and G. Senjanovic, Neutrino Masses and Mixings in Gauge Models with Spontaneous Parity Violation, *Phys. Rev. D* **23** (1981) 165, <https://doi.org/10.1103/PhysRevD.23.165>.
9. R. N. Mohapatra and G. Senjanovic, Neutrino Mass and Spontaneous Parity Nonconservation, *Phys. Rev. Lett.* **44** (1980) 912, <https://doi.org/10.1103/PhysRevLett.44.912>.
10. J. Schechter and J. W. F. Valle, Neutrino Masses in $SU(2) \times U(1)$ Theories, *Phys. Rev. D* **22** (1980) 2227, <https://doi.org/10.1103/PhysRevD.22.2227>.
11. J. Schechter and J. W. F. Valle, Neutrino Decay and Spontaneous Violation of Lepton Number, *Phys. Rev. D* **25** (1982) 774, <https://doi.org/10.1103/PhysRevD.25.774>.
12. Z. Maki, M. Nakagawa and S. Sakata, Remarks on the unified model of elementary particles, *Prog. Theor. Phys.* **28** (1962) 870, <https://doi.org/10.1143/PTP.28.870>.
13. B. Pontecorvo, Neutrino experiments and the question of leptonic-charge conservation, *Sov. Phys. JETP* **26** (1968) 984.
14. P. F. de Salas *et al.*, 2020 global reassessment of the neutrino oscillation picture, *JHEP* **02** (2021) 071, [https://doi.org/10.1007/JHEP02\(2021\)071](https://doi.org/10.1007/JHEP02(2021)071).
15. I. Esteban *et al.*, The fate of hints: updated global analysis of three- flavor neutrino oscillations, *JHEP* **09** (2020) 178, [https://doi.org/10.1007/JHEP09\(2020\)178](https://doi.org/10.1007/JHEP09(2020)178).
16. Z.-Z. Xing, The $\mu - \tau$ reflection symmetry of Majorana neutrinos, *Rept. Prog. Phys.* **86** (2023) 076201, <https://doi.org/10.1088/1361-6633/acd8ce>.

17. K. Fukuura, T. Miura, E. Takasugi and M. Yoshimura, Maximal CP violation, large mixings of neutrinos and democratic type neutrino mass matrix, *Phys. Rev. D* **61** (2000) 073002, <https://doi.org/10.1103/PhysRevD.61.073002>.
18. T. Miura, E. Takasugi and M. Yoshimura, Large CP violation, large mixings of neutrinos and the $Z(3)$ symmetry, *Phys. Rev. D* **63** (2001) 013001, <https://doi.org/10.1103/PhysRevD.63.013001>.
19. E. Ma, The All purpose neutrino mass matrix, *Phys. Rev. D* **66** (2002) 117301, <https://doi.org/10.1103/PhysRevD.66.117301>.
20. W. Grimus and L. Lavoura, A Nonstandard CP transformation leading to maximal atmospheric neutrino mixing, *Phys. Lett. B* **579** (2004) 113, <https://doi.org/10.1016/j.physletb.2003.10.075>.
21. P. Chen, C.-C. Li and G.-J. Ding, Lepton Flavor Mixing and CP Symmetry, *Phys. Rev. D* **91** (2015) 033003, <https://doi.org/10.1103/PhysRevD.91.033003>.
22. E. Ma, Neutrino mixing: A_4 variations, *Phys. Lett. B* **752** (2016) 198, <https://doi.org/10.1016/j.physletb.2015.11.049>.
23. A. S. Joshipura and K. M. Patel, Generalized μ - τ symmetry and discrete subgroups of $O(3)$, *Phys. Lett. B* **749** (2015) 159, <https://doi.org/10.1016/j.physletb.2015.07.062>.
24. G.-N. Li and X.-G. He, CP violation in neutrino mixing with $\delta = -\pi/2$ in A_4 Type-II seesaw model, *Phys. Lett. B* **750** (2015) 620, <https://doi.org/10.1016/j.physletb.2015.09.061>.
25. H.-J. He, W. Rodejohann and X.-J. Xu, Origin of Constrained Maximal CP Violation in Flavor Symmetry, *Phys. Lett. B* **751** (2015) 586, <https://doi.org/10.1016/j.physletb.2015.10.066>.
26. P. Chen, G.-J. Ding, F. Gonzalez-Canales and J. W. F. Valle, Generalized $\mu - \tau$ reflection symmetry and leptonic CP violation, *Phys. Lett. B* **753** (2016) 644, <https://doi.org/10.1016/j.physletb.2015.12.069>.
27. E. Ma, Soft $A_4 \rightarrow Z_3$ symmetry breaking and cobimaximal neutrino mixing, *Phys. Lett. B* **755** (2016) 348, <https://doi.org/10.1016/j.physletb.2016.02.032>.
28. A. Damanik, Neutrino masses from a cobimaximal neutrino mixing matrix (2017), arXiv:1702.03214
29. E. Ma, Cobimaximal neutrino mixing from $S_3 \times Z_2$, *Phys. Lett. B* **777** (2018) 332, <https://doi.org/10.1016/j.physletb.2017.12.049>.
30. W. Grimus and L. Lavoura, Cobimaximal lepton mixing from soft symmetry breaking, *Phys. Lett. B* **774** (2017) 325, <https://doi.org/10.1016/j.physletb.2017.09.082>.
31. A. E. Cárcamo Hernández, S. Kovalenko, J. W. F. Valle and C. A. Vaquera-Araujo, Predictive Pati-Salam theory of fermion masses and mixing, *JHEP* **07** (2017) 118, [https://doi.org/10.1007/JHEP07\(2017\)118](https://doi.org/10.1007/JHEP07(2017)118).
32. A. E. Cárcamo Hernández, S. Kovalenko, J. W. F. Valle and C. A. Vaquera-Araujo, Neutrino predictions from a left-right symmetric flavored extension of the standard model, *JHEP* **02** (2019) 065, [https://doi.org/10.1007/JHEP02\(2019\)065](https://doi.org/10.1007/JHEP02(2019)065).
33. E. Ma, Scotogenic cobimaximal Dirac neutrino mixing from $\Delta(27)$ and $U(1)_X$, *Eur. Phys. J. C* **79** (2019) 903, <https://doi.org/10.1140/epjc/s10052-019-7440-x>.
34. H. Ishimori *et al.*, Non-Abelian Discrete Symmetries in Particle Physics, *Prog. Theor. Phys. Suppl.* **183** (2010) 1, <https://doi.org/10.1143/PTPS.183.1>.
35. W. Grimus and P. O. Ludl, Finite avour groups of fermions, *J. Phys. A* **45** (2012) 233001, <https://doi.org/10.1088/1751-8113/45/23/233001>.
36. G. Altarelli, F. Feruglio, L. Merlo and E. Stamou, Discrete Flavour Groups, theta13 and Lepton Flavour Violation, *JHEP* **08** (2012) 021, [https://doi.org/10.1007/JHEP08\(2012\)021](https://doi.org/10.1007/JHEP08(2012)021).
37. G. Altarelli, F. Feruglio and L. Merlo, Tri-Bimaximal Neutrino Mixing and Discrete Flavour Symmetries, *Fortsch. Phys.* **61** (2013) 507, <https://doi.org/10.1002/prop.201200117>.
38. S. F. King and C. Luhn, Neutrino Mass and Mixing with Discrete Symmetry, *Rept. Prog. Phys.* **76** (2013) 056201, <https://doi.org/10.1088/0034-4885/76/5/056201>.
39. S. F. King, Models of Neutrino Mass, Mixing and CP Violation, *J. Phys. G* **42** (2015) 123001, <https://doi.org/10.1088/0954-3899/42/12/123001>.
40. R. M. Fonseca and W. Grimus, Classification of lepton mixing patterns from finite flavour symmetries, *Nucl. Part. Phys. Proc.* **273-275** (2016) 2618, <https://doi.org/10.1016/j.nuclphysbps.2015.10.008>.
41. G. Chauhan *et al.*, Discrete Flavor Symmetries and Lepton Masses and Mixings, in 2022 Snowmass Summer Study (2022) arXiv:2203.08105 [hep-ph]
42. R. A. Flores and M. Sher, Higgs Masses in the Standard, Multi-Higgs and Supersymmetric Models, *Annals Phys.* **148** (1983) 95, [https://doi.org/10.1016/0003-4916\(83\)90331-7](https://doi.org/10.1016/0003-4916(83)90331-7).
43. A. Barroso, P. M. Ferreira and R. Santos, Tree-level vacuum stability in multi Higgs models, *PoS HEP* **2005** (2006) 337.
44. S. Mantry, M. Trott and M. B. Wise, The Higgs decay width in multi-scalar doublet models, *Phys. Rev. D* **77** (2008) 013006, <https://doi.org/10.1103/PhysRevD.77.013006>.
45. W. Grimus, L. Lavoura, O. M. Ogreid and P. Osland, The Oblique parameters in multi-Higgs-doublet models, *Nucl. Phys. B* **801** (2008) 81, <https://doi.org/10.1016/j.nuclphysb.2008.04.019>.
46. P. M. Ferreira and J. P. Silva, Discrete and continuous symmetries in multi-Higgs-doublet models, *Phys. Rev. D* **78** (2008) 116007, <https://doi.org/10.1103/PhysRevD.78.116007>.
47. F. J. Botella, G. C. Branco and M. N. Rebelo, Minimal Flavour Violation and Multi-Higgs Models, *Phys. Lett. B* **687** (2010) 194, <https://doi.org/10.1016/j.physletb.2010.03.014>.
48. P. M. Ferreira, L. Lavoura and J. P. Silva, Renormalization-group constraints on Yukawa alignment in multi-Higgs-doublet models, *Phys. Lett. B* **688** (2010) 341, <https://doi.org/10.1016/j.physletb.2010.04.033>.

49. A. E. Blechman, A. A. Petrov and G. Yeghiyan, The Flavor puzzle in multi-Higgs models, *JHEP* **11** (2010) 075, [https://doi.org/10.1007/JHEP11\(2010\)075](https://doi.org/10.1007/JHEP11(2010)075).
50. J. L. Diaz-Cruz and U. J. Saldaña Salazar, Higgs couplings and new signals from Flavon-Higgs mixing effects within multi-scalar models, *Nucl. Phys. B* **913** (2016) 942, <https://doi.org/10.1016/j.nuclphysb.2016.10.018>.
51. J. Gehrlein, S. Petcov, M. Spinrath and A. Titov, Testing neutrino flavor models, (2022), arXiv:2203.06219
52. F. Feruglio, Pieces of the Flavour Puzzle, *Eur. Phys. J. C* **75** (2015) 373, <https://doi.org/10.1140/epjc/s10052-015-3576-5>.
53. G. Abbas, R. Adhikari, E. J. Chun and N. Singh, The problem of flavour (2023).
54. H. P. Nilles and S. Ramos-Sanchez, The Flavor Puzzle: Textures and Symmetries (2023).
55. J. Kubo *et al.*, A minimal $S(3)$ -invariant extension of the standard model, *J. Phys. Conf. Ser.* **18** (2005) 380, <https://doi.org/10.1088/1742-6596/18/1/013>.
56. A. Mondragon, M. Mondragon and E. Peinado, Lepton masses, mixings and FCNC in a minimal $S3$ -invariant extension of the Standard Model, *Phys. Rev. D* **76** (2007) 076003, <https://doi.org/10.1103/PhysRevD.76.076003>.
57. A. Mondragon, M. Mondragon and E. Peinado, $S(3)$ -flavour symmetry as realized in lepton flavour violating processes, *J. Phys. A* **41** (2008) 304035, <https://doi.org/10.1088/1751-8113/41/30/304035>.
58. J. Barranco, F. Gonzalez Canales and A. Mondragon, Universal Mass Texture, CP violation and Quark-Lepton Complementarity, *Phys. Rev. D* **82** (2010) 073010, <https://doi.org/10.1103/PhysRevD.82.073010>.
59. D. Meloni, S. Morisi and E. Peinado, Fritzsch neutrino mass matrix from $S3$ symmetry, *J. Phys. G* **38** (2011) 015003, <https://doi.org/10.1088/0954-3899/38/1/015003>.
60. J. Kubo, Super Flavorsymmetry with Multiple Higgs Doublets, *Fortsch. Phys.* **61** (2013) 597, <https://doi.org/10.1002/prop.201200119>.
61. F. Gonzalez Canales, A. Mondragon and M. Mondragon, The $S3$ Flavour Symmetry: Neutrino Masses and Mixings, *Fortsch. Phys.* **61** (2013) 546, <https://doi.org/10.1002/prop.201200121>.
62. F. González Canales *et al.*, Quark sector of $S3$ models: classification and comparison with experimental data, *Phys. Rev. D* **88** (2013) 096004, <https://doi.org/10.1103/PhysRevD.88.096004>.
63. A. E. Cárcamo Hernández, E. Cataño Mur and R. Martinez, Lepton masses and mixing in $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ models with a $S3$ flavor symmetry, *Phys. Rev. D* **90** (2014) 073001, <https://doi.org/10.1103/PhysRevD.90.073001>.
64. A. E. Cárcamo Hernández, R. Martinez and J. Nisperuza, $S3$ discrete group as a source of the quark mass and mixing pattern in 331 models, *Eur. Phys. J. C* **75** (2015) 72, <https://doi.org/10.1140/epjc/s10052-015-3278-z>.
65. A. E. Cárcamo Hernández, I. de Medeiros Varzielas and E. Schumacher, Fermion and scalar phenomenology of a two-Higgs-doublet model with $S3$, *Phys. Rev. D* **93** (2016) 016003, <https://doi.org/10.1103/PhysRevD.93.016003>.
66. A. E. Cárcamo Hernández, I. de Medeiros Varzielas and N. A. Neill, Novel Randall-Sundrum model with $S3$ flavor symmetry, *Phys. Rev. D* **94** (2016) 033011, <https://doi.org/10.1103/PhysRevD.94.033011>.
67. A. E. Cárcamo Hernández, A novel and economical explanation for SM fermion masses and mixings, *Eur. Phys. J. C* **76** (2016) 503, <https://doi.org/10.1140/epjc/s10052-016-4351-y>.
68. A. E. Cárcamo Hernández, R. Martinez and F. Ochoa, Fermion masses and mixings in the 3-3-1 model with right-handed neutrinos based on the $S3$ flavor symmetry, *Eur. Phys. J. C* **76** (2016) 634, <https://doi.org/10.1140/epjc/s10052-016-4480-3>.
69. E. Ma and R. Srivastava, Dirac or inverse seesaw neutrino masses with $B - L$ gauge symmetry and S_3 flavor symmetry, *Phys. Lett. B* **741** (2015) 217, <https://doi.org/10.1016/j.physletb.2014.12.049>.
70. J. C. Gómez-Izquierdo, Non-minimal flavored $S3 Z2$ left-right symmetric model, *Eur. Phys. J. C* **77** (2017) 551, <https://doi.org/10.1140/epjc/s10052-017-5094-0>.
71. S.-F. Ge, A. Kusenko and T. T. Yanagida, Large Leptonic Dirac CP Phase from Broken Democracy with Random Perturbations (2018), <https://doi.org/10.1016/j.physletb.2018.04.040>.
72. D. A. Dicus, S.-F. Ge and W. W. Repko, Neutrino mixing with broken $S3$ symmetry, *Phys. Rev. D* **82** (2010) 033005, <https://doi.org/10.1103/PhysRevD.82.033005>.
73. D. Das, U. K. Dey and P. B. Pal, $S3$ symmetry and the quark mixing matrix, *Phys. Lett. B* **753** (2016) 315, <https://doi.org/10.1016/j.physletb.2015.12.038>.
74. S. Pramanick and A. Raychaudhuri, Neutrino mass model with $S3$ symmetry and seesaw interplay, *Phys. Rev. D* **94** (2016) 115028, <https://doi.org/10.1103/PhysRevD.94.115028>.
75. E. A. Garcés, J. C. Gómez-Izquierdo and F. Gonzalez-Canales, Flavored non-minimal left-right symmetric model fermion masses and mixings, *Eur. Phys. J. C* **78** (2018) 812, <https://doi.org/10.1140/epjc/s10052-018-6271-5>.
76. M. Fischer, S. Ramos-Sanchez and P. K. S. Vaudrevange, Heterotic non-Abelian orbifolds, *JHEP* **07** (2013) 080, [https://doi.org/10.1007/JHEP07\(2013\)080](https://doi.org/10.1007/JHEP07(2013)080).
77. A. Davidson, SAKATA LIKE ELECTROWEAK MODEL, *Phys. Lett. B* **83** (1979) 73, [https://doi.org/10.1016/0370-2693\(79\)90892-x](https://doi.org/10.1016/0370-2693(79)90892-x).
78. R. E. Marshak and R. N. Mohapatra, Quark-Lepton Symmetry and $B-L$ as the $U(1)$ Generator of the Electroweak Symmetry Group, *Phys. Lett. B* **91** (1980) 222, [https://doi.org/10.1016/0370-2693\(80\)90436-0](https://doi.org/10.1016/0370-2693(80)90436-0).
79. C. Wetterich, Neutrino Masses and the Scale of $B-L$ Violation, *Nucl. Phys. B* **187** (1981) 343, [https://doi.org/10.1016/0550-3213\(81\)90279-0](https://doi.org/10.1016/0550-3213(81)90279-0).

80. A. Masiero, J. F. Nieves and T. Yanagida, B^{-1} Violating Proton Decay and Late Cosmological Baryon Production, *Phys. Lett. B* **116** (1982) 11, [https://doi.org/10.1016/0370-2693\(82\)90024-7](https://doi.org/10.1016/0370-2693(82)90024-7).
81. W. Buchmuller, C. Greub and P. Minkowski, Neutrino masses, neutral vector bosons and the scale of B-L breaking, *Phys. Lett. B* **267** (1991) 395, [https://doi.org/10.1016/0370-2693\(91\)90952-M](https://doi.org/10.1016/0370-2693(91)90952-M).
82. S. Khalil, Low scale B-L extension of the Standard Model at the LHC, *J. Phys. G* **35** (2008) 055001, <https://doi.org/10.1088/0954-3899/35/5/055001>.
83. W. Emam and S. Khalil, Higgs and Z-prime phenomenology in B-L extension of the standard model at LHC, *Eur. Phys. J. C* **52** (2007) 625, <https://doi.org/10.1140/epjc/s10052-007-0411-7>.
84. S. Khalil and H. Okada, Dark Matter in B-L Extended MSSM Models, *Phys. Rev. D* **79** (2009) 083510, <https://doi.org/10.1103/PhysRevD.79.083510>.
85. T. Higaki, R. Kitano and R. Sato, Neutrino Universe, *JHEP* **07** (2014) 044, [https://doi.org/10.1007/JHEP07\(2014\)044](https://doi.org/10.1007/JHEP07(2014)044).
86. W. Rodejohann and C. E. Yaguna, Scalar dark matter in the B-L model, *JCAP* **12** (2015) 032, <https://doi.org/10.1088/1475-7516/2015/12/032>.
87. O. F. Beltran, M. Mondragon and E. Rodriguez-Jauregui, Conditions for vacuum stability in an $S(3)$ extension of the standard model, *J. Phys. Conf. Ser.* **171** (2009) 012028, <https://doi.org/10.1088/1742-6596/171/1/012028>.
88. D. Emmanuel-Costa, O. M. Ogreid, P. Osland and M. N. Rebelo, Spontaneous symmetry breaking in the S_3 -symmetric scalar sector, *JHEP* **02** (2016) 154, [Erratum: *JHEP* **08** (2016) 169], [https://doi.org/10.1007/JHEP02\(2016\)154](https://doi.org/10.1007/JHEP02(2016)154).
89. A. Kunčinas, O. M. Ogreid, P. Osland and M. N. Rebelo, S_3 -inspired three-Higgs-doublet models: A class with a complex vacuum, *Phys. Rev. D* **101** (2020) 7 075052, <https://doi.org/10.1103/PhysRevD.101.075052>.
90. W. Khater *et al.*, Dark matter in three-Higgs-doublet models with S_3 symmetry, *JHEP* **01** (2022) 120, [https://doi.org/10.1007/JHEP01\(2022\)120](https://doi.org/10.1007/JHEP01(2022)120).
91. A. Kunčinas, O. M. Ogreid, P. Osland and M. N. Rebelo, Complex S_3 -symmetric 3HDM, *JHEP* **07** (2023) 013, [https://doi.org/10.1007/JHEP07\(2023\)013](https://doi.org/10.1007/JHEP07(2023)013).
92. V. V. Vien, $3 + 1$ active-sterile neutrino mixing in B-L model with $Q_4 \times Z_4 \times Z_2$ symmetry for normal neutrino mass ordering, *Eur. Phys. J. C* **81** (2021) 416, <https://doi.org/10.1140/epjc/s10052-021-09214-5>.
93. V. V. Vien, Multiscalar B-L extension with A_4 symmetry for fermion mass and mixing with co-bimaximal scheme, *Phys. Lett. B* **817** (2021) 136296, <https://doi.org/10.1016/j.physletb.2021.136296>.
94. V. V. Vien, A non-renormalizable B-L model with $Q_4 \times Z_4 \times Z_2$ flavor symmetry for cobimaximal neutrino mixing, *Chin. Phys. C* **45** (2021) 123103, <http://doi.org/10.1088/1674-1137/ac28f2>.
95. V. V. Vien, Neutrino mass and mixing in $U(1)B - L$ extension with $\sum(18)$ symmetry, *Chin. J. Phys.* **73** (2021) 47, <https://doi.org/10.1016/j.cjph.2021.06.005>.
96. V. V. Vien, Fermion spectrum with normal neutrino mass ordering in a nonrenormalizable B-L model based on $\sum(18)_x Z_2$ symmetry, *Chin. J. Phys.* **77** (2022) 1976, <https://doi.org/10.1016/j.cjph.2021.12.017>.
97. F. An *et al.* (JUNO), Neutrino Physics with JUNO, *J. Phys. G* **43** (2016) 030401, <https://doi.org/10.1088/0954-3899/43/3/030401>.
98. B. Abi *et al.* (DUNE), Deep Underground Neutrino Experiment (DUNE), Far Detector Technical Design Report, Volume II: DUNE Physics (2020).
99. K. Abe *et al.* (Hyper-Kamiokande), Hyper-Kamiokande Design Report (2018),
100. S. Pakvasa and H. Sugawara, Discrete Symmetry and Cabibbo Angle, *Phys. Lett. B* **73** (1978) 61, [https://doi.org/10.1016/0370-2693\(78\)90172-7](https://doi.org/10.1016/0370-2693(78)90172-7).
101. J. Kubo, H. Okada and F. Sakamaki, Higgs potential in minimal $S(3)$ invariant extension of the standard model, *Phys. Rev. D* **70** (2004) 036007, <https://doi.org/10.1103/PhysRevD.70.036007>.
102. D. Emmanuel-Costa, O. Felix-Beltran, M. Mondragon and E. Rodriguez-Jauregui, Stability of the tree-level vacuum in a minimal $S(3)$ extension of the standard model, *AIP Conf. Proc.* **917** (2007) 390, <https://doi.org/10.1063/1.2751981>.
103. T. Teshima, Higgs potential in S_3 invariant model for quarklepton mass and mixing, *Phys. Rev. D* **85** (2012) 105013, <https://doi.org/10.1103/PhysRevD.85.105013>.
104. D. Das and U. K. Dey, Analysis of an extended scalar sector with S_3 symmetry, *Phys. Rev. D* **89** (2014) 095025, [Erratum: *Phys. Rev. D* **91** (2015) 039905], <https://doi.org/10.1103/PhysRevD.89.095025>.
105. E. Barradas-Guevara, O. Félix-Beltrán and E. Rodríguez-Jáuregui, Trilinear self-couplings in an $S(3)$ flavored Higgs model, *Phys. Rev. D* **90** (2014) 095001, <https://doi.org/10.1103/PhysRevD.90.095001>.
106. M. Gómez-Bock, M. Mondragón and A. Pérez-Martínez, Scalar and gauge sectors in the 3-Higgs Doublet Model under the S_3 symmetry, *Eur. Phys. J. C* **81** (2021) 942, <https://doi.org/10.1140/epjc/s10052-021-09731-3>.
107. J. C. Gómez-Izquierdo and M. Mondragón, B-L Model with S_3 symmetry: Nearest Neighbor Interaction Textures and Broken $\mu\tau$ Symmetry, *Eur. Phys. J. C* **79** (2019) 285, <https://doi.org/10.1140/epjc/s10052-019-6785-5>.
108. J. D. Vergados, The Neutrinoless double beta decay from a modern perspective, *Phys. Rept.* **361** (2002) 1, [https://doi.org/10.1016/S0370-1573\(01\)00068-0](https://doi.org/10.1016/S0370-1573(01)00068-0).
109. W. Rodejohann, Neutrino-less Double Beta Decay and Particle Physics, *Int. J. Mod. Phys. E* **20** (2011) 1833, <https://doi.org/10.1142/S0218301311020186>.
110. J. D. Vergados, H. Ejiri and F. Simkovic, Theory of Neutrinoless Double Beta Decay, *Rept. Prog. Phys.* **75** (2012) 106301, <https://doi.org/10.1088/0034-4885/75/10/106301>.

111. H. Päs and W. Rodejohann, Neutrinoless Double Beta Decay, *New J. Phys.* **17** (2015) 115010, <https://doi.org/10.1088/1367-2630/17/11/115010>.
112. M. Agostini *et al.* (GERDA), Results on Neutrinoless Double-
fi Decay of ^{76}Ge from Phase I of the GERDA Experiment, *Phys. Rev. Lett.* **111** (2013) 122503, <https://doi.org/10.1103/PhysRevLett.111.122503>.