A dynamical system approach to modified point kinetics equations

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We apply the formalism of dynamical systems and the Kosambi-Kartan-Chern theory to perform an extensive stability analysis of solutions to modified point kinetics equations, a set of two differential equations used in nuclear physics to describe the temporal behavior of a fission nuclear reactor. It is known that the stability of the system depends on reactivity, which appears in the equations as a parameter ρ, and the functional form of this parameter can describe different situations that may occur in a nuclear reactor, making the equations nonlinear or even nonautonomous in certain cases. We present a comparative study between linear stability and Jacobi stability, pointing out the usefulness of the geometrodynamical approach.

Keywords: Dynamical systems; Lyapunov stability; Jacobi stability; Kosambi-Cartan-Chern theory, reactor point-kinetics.

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1. Introduction

The point-kinetic model is a set of coupled differential equations that have been widely used to describe the temporal behavior of nuclear fission reactors and can be derived directly from the neutron transport equation, the neutron diffusion equation or by heuristic procedures [1-3]. The stability of the system has been studied using different methods [4-7]. Recently, new modifications have been formulated for this set of equations, such as Refs. [8,9]. The Modified Point Kinetics Equations (MPK) [10] arise when the derivative of the neutron density is incorporated into the derivation of the set of equations.

By considering this set of equations, we perform a stability analysis on the critical points of the system. Considering a functional form of reactivity depending on time and an integral over neutron number, we have a set of nonlinear nonautonomous equations which, due to nonautonomous nature, is difficult to analyze because the conventional techniques such as linearization about critical points, the Hartman-Grobman theorem or Lyapunov-Poincare theorem can not be applied directly. Furthermore, the system becomes integrodifferential in some cases. Nevertheless, global stability of the system can be studied by constructing an appropriate Lyapunov function $V(x,t)$ that satisfies the conditions of the Lyapunov theorem and, thereby determining the asymptotic stability of the solutions. Additionally, we employ the Kosambi-Cartan-Chern (KCC) theory to assess the Jacobi (in)stability of critical points, particularly in cases where the application of Lyapunov stability becomes intricate due to the nature of critical points or difficulties that arose for certain functional forms of reactivity ρ. Some authors refer tho the combined use of dinamycal systems theory and differential geometry as geometrodynamical approach to dynamical systems. This framework has been applied in a wide range scientific fields, and some of its earlier applications arose in biology, where Antonelli *et al.* [11] conducted a stability analysis of Tayson's equations for the cell-division cycle. The authors studied Jacobi stability of the model trough the deviation curvature tensor. In recent years, the method has gained popularity in the field of gravitation and cosmology [12], in part, due to familiarity of concepts inherent to mathematical framework used in this theories and the differential geometrical approach provided by KCC theory. Some recent investigation made comparative studies of stability using Lyapunov stability and Jacobi stability of circular orbits around black holes in modified theories of gravity [13-16].

This paper is structured as follows. Section 2 contains a brief introduction to dynamical systems and KCC theory. Section 3 provides the mathematical model considered. In Sec. 4, we present a stability analysis of MPK equations by considering some functional forms of reactivity ρ. Section 5 is devoted to a discussion of our main results, concluding that the geometrodynamical approach provides a robust framework for studying complex dynamical systems.

2. Mathematical background

The purpose of this section is to provide the essential mathematical insights for analyzing the stability of MPK equations. For dynamical systems theory, the reader is referred to classic texts such as [17-23]. For a complete description of the KCC framework, see [12,13,15,24].

The first part of this section is devoted to apply dynamical systems theory to the study of differential equations of the form

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$, $E \in R^n$ and $f : E \to \mathbb{R}^n$. Critical points x_c of Eq. (1) are those that satisfy $f(x_c) = 0$.

The critical points correspond to equilibrium solutions of the system, that is, $x(t) = x_c$ satisfies the equation for all time. The study of these points and solutions in their neighborhood is of particular interest. Through the application of linearization around the critical points, Eq. (1) can be approximated as a linear equation

$$\dot{x} = Ax, \tag{2}$$

where A is the Jacobian matrix of $f(x)$. The fundamental theorem of linear systems establishes the existence and uniqueness of solutions of this kind of systems with initial conditions $x(0) = x_0$, which is given as $x(t) = e^{tA}x_0$. As a consequence, from the eigenvalues of A we can obtain important qualitative information about the behavior of solution curves near critical points in the phase space. Some examples can be found in Refs. [22,25-28]. Generally, the process starts with a linearization around its critical points, as it is the simplest approach; however, a nonlinear analysis of the dynamical system is necessary to gain a more comprehensive understanding of the behavior of the entire system.

The stability of critical points of any nonlinear system can be analyzed from the characteristic roots of the Jacobian matrix, provided that they are distinct from zero, and its solutions can be classified in terms of nodes, focus, saddle or center behavior. The system is said to be stable if the eigenvalues of the corresponding Jacobian matrix are negative, have negative real part, or are pure imaginary; otherwise, the system is unstable, and its solutions can be classified in terms of nodes, focus, saddle, or center behavior. It should be pointed out that if the system has eigenvalues equal to zero, other methods should be used to determine stability or instability in the system, such as the Floquet theorem or the Lyapunov theorem.

Other features of nonlinear systems can be viewed by making bifurcation analysis, which consists of studying changes in the behavior of the system from the variation of parameters appearing in the equations. In the case where the independent variable appears in the equations, which for physical systems usually corresponds to time, the system is said to be nonautonomous. This kind of system should be treated carefully, as the theory of autonomous systems does not work the same. Although some of the existent theorems cover few special cases, a general theory for nonautonomous systems have not been completely developed yet [17-23,29].

Furthermore, the Kosambi-Cartan-Chern (KCC) theory is a powerful mathematical method for investigating the characteristics of dynamical systems. The KCC theory utilizes a geometric depiction of the time evolution of dynamical systems, where the solution curves are described trough techniques similar to those found in the study of geodesics in Finsler spaces. In this framework, stability of solutions of second order differential equations of the form

$$\frac{d^2x^i}{dt^2} + 2G^i(x^j, y^j, t) = 0, \quad i = 1, 2, ..., n, \tag{3}$$

is studied [30]. Here x^i are the dynamical variables defined on a smooth manifold $\mathcal{M}$, usually $\mathcal{M} = \mathbb{R}^n$. Let Ω be a subset of the Euclidian $(2n+1)-$dimensional space on which the coordinate system $(x^i, y^i, t), i = 1, 2, ..., n$, is introduced, $y^i = (dx^i/dt)$. The main idea of KCC theory is that the evolution and behavior of trajectories of the system (3) can be analyzed using tools of differential geometry.

The study of geometrical properties associated to the dynamical system defined by Eq. (3) is possible by introducing the nonlinear connection N with coefficients N^i_j defined as:

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{4}$$

The KCC-covariant differential of a vector field ξ^i is introduced trough the definition

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j\xi^j. \tag{5}$$

For $\xi^i = y^i$ is obtained

$$\frac{Dy^i}{dt} = N^i_j y^j - 2G^i = -\epsilon^i, \tag{6}$$

where ϵ^i is called the first KCC invariant.

By introducing a nonlinear connection, KCC theory provides a geometric interpretation of the evolution of a dynamical system, thereby facilitating the construction of the KCC covariant derivative of an arbitrary field χ

$$\frac{D^2\chi^i}{dt^2} = P^i_j\chi^j, \tag{7}$$

where the deviation curvature tensor is

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l\frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t}. \tag{8}$$

The Berwald connection is

$$G^i_{jl} \equiv \frac{\partial N^i_j}{\partial y^l}. \tag{9}$$

The third, fourth and fifth invariant are constructed according to definitions

$$P^i_{jk} \equiv \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right),$$

$$P^i_{jkl} \equiv \frac{\partial P^i_{jk}}{\partial y^l}, \qquad D^i_{jkl} \equiv \frac{\partial G^i_{jk}}{\partial y^l}. \tag{10}$$

In this theory, the properties of any dynamical system can be explained using the five invariants depicted, the second invariant determining the Jacobi stability of the system. In KCC theory, these five invariants are the basic mathematical quantities that describe the geometrical properties and interpretation of an arbitrary system of second-order differential equations [12,13,24].

KCC theory and its relation with chaos in 3D systems is explored by Somanath Roy *et al.* [31], showcasing the KCC theory as a powerful tool to capture nuanced dynamics. Other contributions has been made in the study of 2D systems [32]. Furthermore, some particular systems which exhibits chaotic behavior have been studied as Liu system [33], the Lorentz model [34], a Lorentz type model [35], Rössler system [36], just to name some examples.

It should be noticed that Lyapunov stability analysis (linearization, lyapunov function, center manifold) and KCC stability (Jacobi stability) differ in the type of stability that is determined by each. The former studies how solution curves evolve near critical points; it could be an attractor (repulsor), which means that solution curves are attracted (repealled) to it when $t \to \infty$. An attractor can be stable (unstable) in the sense of Lyapunov; that is, solution curves remain close starting from slightly different initial conditions and stay in a neighborhood for all t.

On the other hand, Jacobi stability is an extension of the concept of stability in geodesic flows on manifolds with Riemannian or Finslerian metrics. It examines the robustness of a second-order differential equation, where robustness is a measure of the system's and environment's sensitivity to changes in its internal parameters. From a mathematical point of view, KCC theory looks at the variations in nearby trajectories as geodesics in a Finsler space, allowing for an estimation of the perturbations that can be tolerated around the equilibrium solutions of a second-order differential equation. This is especially useful in physical applications, where determining areas with both Lyapunov and Jacobi stability is of great importance. Further discussion of the differences between these two methods can be found in Refs. [37,38]. A good correlation exists between the linear stability and Jacobi stability of critical points, nevertheless this is not always true. A critical point can be Lyapunov stable and Jacobi unstable and vice versa, and in consequence some discrepancies in the results of these two stability methods arise.

3. Mathematical model

One of the simplest ways to determine the time behavior of nuclear power in nuclear fission reactors is through the solution of point kinetics equations, which are a set of nonlinear nonautonomous coupled differential equations in the most general form. The character of nonautonomous and nonlinear is given by the functional form of the reactivity $\rho(n,t)$. Such equations can be derived from various procedures. The one that concerns us is that obtained from the neutron transport equation; this derivation needs several approximations, including weak angular dependence of the angular distribution of the neutrons, isotopic sources of neutrons, and disregarding the derivative of the neutron current density [1,2]. Recently, new modifications had been formulated for this set of equations taking into account the temporal derivative of the neutron current density for the analysis of behavior and solutions, giving as a result the Modified Point Kinetics (MPK).

TABLE I. Parameters and variables appearing in MPK model.

Parameter	Description	Value
$n(t)$	Neutron number	
$c_i(t)$	Precursor number	
$\rho(n,t)$	Reactivity	
f_D	Transport frequency	$3 \times 10^5 s^{-1}$
f_A	Neutron absorption frequency	$4.167 s^{-1}$
β	Fraction of delayed neutrons	0.007
Λ	Mean generation time	0.002s
λ_i	Decay constant	$0.0810958 s^{-1}$

The MPK equations are

$$\begin{aligned}\frac{d^2n(t)}{dt^2} + M\frac{dn(t)}{dt} &= Pn(t) \\ &+ f_D\sum_{i=1}^{6}\lambda_i c_i(t) + \sum_{i=1}^{6}\lambda_i\frac{dc_i}{dt} \\ \frac{dc(t)_i}{dt} &= \frac{\beta_i}{\Lambda}n(t) - \lambda_i c_i(t), \end{aligned} \tag{11}$$

where $M = (f_D + f_A - [1 - \beta/\Lambda])$, $P = (f_D/\Lambda)(\beta - \rho)$. For a detailed derivation, see [10]. In this investigation, the system was analyzed and solutions for constant reactivity were derived using an analytical method involving the construction of a third-order differential equation and using the Laplace transform method to solve it. This system is comparable to the traditional Point-Kinetics (PK) model, which is referred to as the "classical point kinetics" when the limit of $1/f_d$ approaches zero. Variables $n(t), c(t)$ and parameters $f_D, f_A, \beta, \Lambda, \lambda_i, \rho(n,t)$ are listed in Table I. Stability analysis for this system is performed in the next section.

4. Stability analysis

As stated earlier, stability analysis can be derived in two ways, in this section is considered first the cases that can be studied using Lyapunov stability, then some cases of nonlinear reactivity are considered using KCC theory.

The first of the Eqs. (11) is second-order and includes the derivative of precursors C. We are interested into write this equation in the form of Eq. (1) in order to apply stability analysis. From this system, the set of three first-order coupled differential equations can be constructed as follows:

Considering one group of precursors in the Eq. (11), substituting in the first equation, and taking the change of variables $x_1 = n$, $x_2 = \dot{x_1} = \dot{n}$ and $x_3 = c$, the first equation can

be stated as two coupled first order equations, and together with the second equation one obtains the following system of three coupled equations of the form of Eq. (1)

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -Mx_2 + Px_1 + f_D\lambda x_3 + \lambda\left(\frac{\beta}{\Lambda}x_1 - \lambda x_3\right),\\ \dot{x}_3 &= \frac{\beta}{\Lambda}x_1 - \lambda x_3,\end{aligned} \tag{12}$$

with initial conditions

$$\left.\frac{dc(t)}{dt}\right|_{t=0} = 0, \tag{13}$$

$$c(0) = \frac{\beta}{\lambda\Lambda}n(0) = 0, \tag{14}$$

$$n(0) = n_0. \tag{15}$$

As stated before, the stability of MPK equations and, therefore, of nuclear reactors depends on the functional form of the reactivity $\rho(n,t)$. Depending on this functional form, various cases can be considered: the system becomes linear, nonlinear and nonautonomous. Due to the complexity that exhibits the system, is suitable to analyze it using the mathematical tools described in Sec. 2.

4.1. Constant reactivity

The simplest case of MPK equations is when the reactivity has a constant value $\rho(n,t) = cte$.

4.1.1. One group of precursors

In this case, the system becomes lineal and has the form $\dot{x} = Ax$, where $x \in \mathbb{R}^3$, and A is a 3×3 matrix with constant coefficients

$$A = \begin{pmatrix} 0 & 1 & 0 \\ P + \frac{\lambda\beta}{\Lambda} & -M & \lambda(f_D - \lambda) \\ \frac{\beta}{\Lambda} & 0 & -\lambda \end{pmatrix}. \tag{16}$$

A stability analysis can be carried out by computing the correspondent eigenvalues of this matrix from the eigenvalue equation $\omega_i = det|A - \omega\mathbb{I}| = 0$:

$$-\omega(M+\omega)(\lambda+\omega) + (\lambda+\omega)N = 0, \tag{17}$$

where $N = P + [\lambda\beta/\Lambda]$ and whose solutions are

$$\omega_1 = -\lambda, \qquad \omega_{2,3} = -\frac{1}{2}(M \pm \sqrt{M^2 + 4N}). \tag{18}$$

TABLE II. Eigenvalues for the MPK system.

Reactivity	ω_1	ω_2	ω_3
$\rho = -0.5$	-0.7997577974	-255.41000694	-9925259105
$\rho = 0$	0.0000002597	-3.59852555	-99504.48257
$\rho = 0.5$	-0.08224688231	247.1062989	-99755.10515

The parameter N depends on reactivity ρ, so the solutions depends too. The eigenvalues were computed for some particular cases of reactivity covering the cases of critical, subcritical, and supercritical that can occur in a nuclear reactor.

The findings of this study are in agreement with those obtained from a comparable examination of the traditional point kinetics equations.

4.2. Nonlinear reactivity

Some authors [39] consider a nonlinear reactivity by introducing temperature feedback reactivity, in this case

$$\frac{d\rho}{dt} = \frac{d\gamma(t)}{dt} - Bn(t), \tag{19}$$

where $\gamma(t)$ is the prescribed reactivity. The dynamical system formed is of the form $\dot{x} = f(x,t)$, where $x \in \mathbb{R}^4$ and

$$f(x,t) = \begin{pmatrix} x_2 \\ Qx_1 - Mx_2 + Rx_3 - \frac{f_D}{\Lambda}\rho x_1 \\ \frac{\beta}{\Lambda}x_1 - \lambda_i x_3 \\ \frac{d\gamma(t)}{dt} - Bx_1 \end{pmatrix}, \tag{20}$$

where $R = f_D\lambda_i - \lambda_i^2$, $Q = (\beta/\Lambda)(f_d + \lambda)$.

The simplest case is when the prescribed reactivity is constant $\gamma(t) = cte$, and the equation becomes

$$f(x,t) = \begin{pmatrix} x_2 \\ Qx_1 - Mx_2 + Rx_3 - \frac{f_D}{\Lambda}\rho x_1 \\ \frac{\beta}{\Lambda}x_1 - \lambda_i x_3 \\ -Bx_1 \end{pmatrix}. \tag{21}$$

This system is nonlinear, and has infinite critical points given by $(0,0,0,x_4)^t$, however, all of this critical points are nonhyperbolic and the stability analysis is carried out using a suitable Lyapunov function. One simple function that satisfies the criteria of Lyapunov theorem

$$V(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{22}$$

This function has negative definite orbital derivative and is positively definite in a neighbourhood of $x = 0$, sufficient conditions to establish that the origin is asymptotically stable.

For different forms of prescribed reactivity, this system can be nonlinear and autonomous, or nonautonomous. some cases are considered next. As the reactivity ρ appears in the

second and fourth equation as its derivative, analyzing this problem gives place to a integro differential equation. In order to study this cases KCC theory can be applied. For this purpose, we compute the derivative of function $f(x,t)$ in Eq. (20) with respect time to obtain a equation in the form of Eq. (3), where

$$G^i = \frac{1}{2}\begin{pmatrix} \dot{x}_2 \\ (Q-\frac{f_D}{\Lambda})\dot{x}_1 - M\dot{x}_2 + R\dot{x}_3 - \frac{f_D}{\Lambda}x_1\dot{\rho} \\ \frac{\beta}{\Lambda}\dot{x}_1 - \lambda\dot{x}_3 \\ \frac{d^2\gamma}{dt^2} - B\dot{x}_1 \end{pmatrix}. \quad (23)$$

Components of the first KCC invariant are constructed using Eq. (2.)

$$\epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \dot{x}_2 \\ (Q-\frac{f_D}{\Lambda})\dot{x}_1 - M\dot{x}_2 + R\dot{x}_3 - \frac{f_D}{\Lambda}x_1\dot{\rho} \\ \frac{\beta}{\Lambda}\dot{x}_1 - \lambda\dot{x}_3 \\ \frac{d^2\gamma}{dt^2} - B\dot{x}_1 \end{pmatrix}. \quad (24)$$

In this case, the first invariant coincides with the function G^i. Deviation curvature tensor can be obtained as

$$P^i_j = \frac{1}{2}\sum_{k=1}^{n} f^i_{jk}(x^1,...,x^n)y^k + \frac{1}{4}\sum_{l=1}^{n} f^i_l(x^1,...,x^n)f^l_j(x^1,...,x^n), \quad (25)$$

where f^n are the components of $f(x,t)$. The components of this tensor different from 0 are

$$P^1_1 = \frac{Q-Sx_4}{4}; \quad P^1_2 = \frac{-M}{4}; \quad P^1_3 = \frac{R}{4}; \quad P^1_4 = \frac{3x_1}{4},$$
$$P^2_1 = -\frac{S}{2}\dot{x}_4 + \frac{1}{4}(MSx_4 - MQ + RT - BSx_1),$$
$$P^2_2 = \frac{1}{4}(Q - Sx_4 + M^2); \qquad P^2_3 = -\frac{R}{4}(M+\lambda),$$
$$P^2_4 = \frac{S\dot{x}_1}{2} - \frac{MSx_1}{4}; \qquad P^3_1 = \frac{-\lambda\beta}{4\Lambda};$$
$$P^3_2 = \frac{\beta}{a\Lambda}; \qquad P^3_3 = \frac{\lambda^2}{4},$$

where $S = f_D/\Lambda$ and

$$P = \begin{pmatrix} P^1_1 & P^1_2 & P^1_3 & P^1_4 \\ P^2_1 & P^2_2 & P^2_3 & P^2_4 \\ P^3_1 & P^3_2 & P^3_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (26)$$

The eigenvalues of this tensor indicate the Jacobi stability of the system; if they are negative, the system is stable and unstable otherwise. One of the eigenvalues of this tensor is zero, which implies that the system is Jacobi unstable.

The third, fourth and fifth invariants are constructed following the definitions (10). In the case of the third invariant we found that the only elements distinct from cero are

$$P^2_{14} = P^2_{41} = \frac{S}{3}. \quad (27)$$

The elements of the fourth invariant are partial derivatives of P^i_{jk} with respect to y^j, and since the elements of P^i_{jk} are constants, thus elements of fourth invariant are equal to 0.

On the other hand, for the fifth invariant, since elements of N^i_j depend at most of x_i, then, partial derivatives of this elements with respect to y^l are equal to 0, thus, elements of fifth invariant are equal to 0.

Under this approach of stability different types of reactivity can be considered in Eq. (12) including step ramp insertion $\gamma = at$, periodic reactivity $\gamma = \rho_0\cos(\omega t)$, and pulse type reactivity $\gamma = 4\beta e^{-2t^2}$. Deviation curvature tensor has the same form, with the last row consisting of zeros. Therefore, the eigenvalues of this tensor have similar values, including 0. By taking this into account, the instability of critical points can be determined. It should be noted that each case of prescribed reactivity leads to different critical points that appear in the set of Eqs. (12), some of them are nonhyperbolic, but all of these critical points exhibit the same behavior, that is, the presence of a zero eigenvalue in the deviation curvature tensor.

4.2.1. Periodic prescribed reactivity

Some authors consider the case where the reactivity has a periodic variation, in this case $\gamma(n,t) = \rho_0\sin(\omega t)$. Thus, the system has the form $\dot{x} = A(t)x$. In the case where only one group of precursors is considered, this system has critical points given by

$$\left(\frac{\omega\rho_0\cos(\omega t)}{B}, 0, \frac{\omega\rho_0\beta\cos(\omega t)}{\lambda\Lambda B}, \frac{Q\Lambda}{f_D} - \frac{R\beta}{B\lambda\Lambda}\right),$$

the system can be analyzed using Floquet's theorem. On the other hand, it can be studied with KCC theory.

4.2.2. Step ramp insertion

Introducing the case of step ramp insertion reactivity $\gamma = at$ in Eq. (20) gives

$$f(x,t) = \begin{pmatrix} x_2 \\ Qx_1 - Mx_2 + Rx_3 - \frac{f_D}{\Lambda}\rho x_1 \\ \frac{\beta}{\Lambda}x_1 - \lambda_i x_3 \\ a - Bx_1 \end{pmatrix}. \quad (28)$$

This system has two critical points $x_{01} = (a/B, 0, \beta/\lambda\Lambda a/B, 2\beta)^t$ and $x_{02} = (a/B, 0, \beta/\lambda\Lambda a/B, 0)^t$, both of which are nonhyperbolic.

4.2.3. Pulse type prescribed reactivity

Introducing the case of pulse type reactivity $\gamma = 4\beta e^{-2t^2}$ in Eq. (20) gives

$$f(x,t) = \begin{pmatrix} x_2 \\ Qx_1 - Mx_2 + Rx_3 - \frac{f_D}{\Lambda}\rho x_1 \\ \frac{\beta}{\Lambda}x_1 - \lambda_i x_3 \\ 16t\beta e^{-2t^2} - Bx_1 \end{pmatrix}. \quad (29)$$

This case of reactivity has critical point at $([16te^{-2t^2}/B], 0, [\beta 16te^{-2t^2}/\lambda\Lambda B], Q\Lambda/f_D - R\beta/B\lambda\Lambda)$, in which curvature tensor is evaluated, given the same functional form with a zero eigenvalue.

5. Results and discussions

The aim of this investigation is apply geometrodynamical approach provided by Lyapunov and Jacobi stability to the study of complex dynamical systems. MPK equations, due to functional form of reactivity are suitable for this analysis, allowing us to explore the scope of the theory. The conducted analysis enables a comparative examination between the "classical" point kinetics equations and the modified point kinetics equations, which is consistent with previous findings. Moreover, the KCC theory was successfully employed to identify additional characteristics, such as the Jacobi stability of the MPK system, that cannot be easily discerned through the classical dynamical systems approach. The combined methods of stability enable us to perform a deeper study of the dynamics of equations, and allowing us to gain fuller insights in the behavior of solutions.

The stability analysis in the sense of Lyapunov is performed for the case of constant reactivity and one group of precursors for the point kinetics equations coincides with a similar examination of the classical point kinetics equations. As a complementary study, in the future this results must be extended to the case of more than one precursor, and compared with other analytical methods as well numeric solutions.

Also, introducing the temperature feedback reactivity some cases can be studied using Lyapunov theorem; however, for the nonlinear reactivity feedback cases of step ramp insertion, periodic reactivity and pulse type reactivity, the stability analysis is carried out by means of the deviation curvature tensor, whose eigenvalues give information about Jacobi instability of the system. Nevertheless, to gain a complete description of behavior of solutions these results must be contrasted with numerical solutions. Furthermore, only the case with one group of precursors has been studied, while the case of two or more precursors is worth further investigation.

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