Are neutrino oscillation mixings linked to the smallness of solar neutrino scale?

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Observed reactor and atmospheric neutrino oscillation mixing values appear to be related to the neutrino scale ratio $\sqrt{\Delta m^2_{sol}/\Delta^2_{ATM}}$ in a way that suggest that the neutrino mass matrix can be expanded as a power series by using this ratio as the smallness parameter. This approach provides a simple and natural way to expose the inner hierarchies among neutrino mass terms, which amounts to also explain the solar oscillation mixing as well as solar oscillation scale. We explore a class of mass matrix textures that realize this scenario, for both normal and inverted neutrino mass hierarchies, as well as CP violation and their stability under renormalization scaling.

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1. Introduction

The flavor problem, meaning, the understanding of the observed pattern of fermion masses and flavor mixings in the Standard Model (SM) of particles at a fundamental level, remains so far as an open question whose answer might shed light on possible new physics. This question is particularly more puzzling in the case of neutrinos which are predicted to be massless on the basis of the SM, and yet the abundant data collected along the last decades by several oscillation neutrino experiments [1] undoubtedly indicates that neutrinos are rather light and mix. Most of such data is well described with the three standard neutrinos. Global fits [2] give for the two independent mass scales $\Delta m^2_{sol} = 7.42^{+0.21}_{-0.20} \times 10^{-5}$ eV^2 and $\Delta m^2_{ATM} = 2.510^{+0.027}_{-0.027}(2.490^{+0.026}_{-0.028}) \times 10^{-3}$ eV^2, corresponding to the ones that determine solar and atmospheric neutrino oscillation lengths, respectively, for the normal (inverted) hierarchy of masses. As a remainder for the reader, hierarchy refers to the relative ordering of the mass eigenvalues, m_i for $i = 1, 2, 3$, such that if $m^2_3 > m^2_1$ the mass pattern is said to be in a normal hierarchy (NH), otherwise it is known as an inverted hierarchy (IH). Theoretically, Δm^2_{sol} is identified as the squared mass difference Δm^2_{21}, where $\Delta m^2_{ij} = m^2_i - m^2_j$, whereas Δm^2_{ATM} corresponds to Δm^2_{31} ($|\Delta m^2_{32}|$), for NH (IH). Note that the sign of Δm^2_{21} is already known due the MSW effect [3, 4] that takes place within the sun, however the sign of Δm^2_{31} and therefore the hierarchy of masses is still unknown.

With massive neutrinos, mixings arise from the fact that neutrino mass terms are non diagonal in the flavor state basis which differs from the mass eigenstate basis. Both the basis are connected by a unitary matrix U_{mix} that in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) [5, 6] parameterization is written as $U_{mix} = VK$, where K is a diagonal matrix containing two Majorana phases, and V is given in terms of two real and a complex rotations, as $V = R_{23}(\theta_{23})R_{13}(\theta_{13}, \delta_{CP})R_{12}(\theta_{12})$. Same global fit analysis determines the three neutrino mixing angles to be $\sin^2\theta_{12} = \sin^2\theta_{sol} = 0.304^{+0.012(3)}_{-0.012}$, for the solar neutrino oscillations, $\sin^2\theta_{23} = \sin^2\theta_{ATM} = 0.450^{+0.019}_{-0.016}(0.570^{+0.016}_{-0.022})$ for atmospheric ones, and $\sin^2\theta_{13} = 0.02246(1)^{+0.00062(74)}_{-0.00062}$, corresponding to reactor oscillation mixings. It is worth noticing that the most recent fit from Ref. [2] changed drastically the best fit point on the atmospheric mixing for normal hierarchy, which lays now on the first octant, as compared to the previous fit in Ref. [7], where it appeared on the second octant with the value $\sin^2\theta_{ATM} = 0.573(5)^{+0.016}_{-0.020(19)}$. Yet, at three sigma level, on both fits the measured mixing regions still overlaps on basically the same area that extends on both the octants. Neutrino oscillations are also sensible to the Dirac CP phase which so far has been narrowed to be about $\delta_{CP} = 230^{+36}_{-25}$ (278^{+22}_{-30}) and within the interval $[144^o, 350^o]$ ($[194^o, 345^\circ]$) at three sigma level. All above for NH (IH). Ongoing and next generation oscillation neutrino experiments will focus on increasing the precision on the determination of this phase.

By looking up on these measured parameters it is unavoidable to notice some peculiarities that are very suggestive when they are put down together. First of all, the hierarchy among solar and atmospheric scales are so that their ratio, in central fit values, can be expressed as

$$\lambda = \sqrt{\frac{\Delta m^2_{sol}}{\Delta m^2_{ATM}}} = 0.1719(26). \tag{1}$$

On the other hand, for the reactor mixing one has

$$\tan\theta_{13} = 0.1516 \sim \mathcal{O}(\lambda). \tag{2}$$

Furthermore, the deviation of the atmospheric mixing from maximal turns out to be

$$\begin{aligned} |\tan\theta_{ATM} - 1| &= 0.0955\ (0.1513) \\ &\sim \mathcal{O}(\lambda/2)\ [\mathcal{O}(\lambda)]. \end{aligned} \tag{3}$$

As stressed by above expressions, it is striking to note that all of those parameters are about the same order of magnitude

(this peculiar numerical coincidences were early suggested in Ref. [8]). This raises the question of whether they could have a common source and exploring this possibility is the main goal of the present work. As it is clear, if the answer to such a question were on the positive, the next issue would be to understand why solar mixing, being rather large but far from maximal, is not in evident connection with the same parameter. As we shall argue here, the solution to the whole riddle could indeed lay on the smallness of the solar scale, expressed through a simple neutrino mass matrix structure that emerges when one assumes that mass terms are functions that can be (Taylor) series expanded using λ as the only smallness parameter. This comes out natural when one considers that physical masses could be subjected to loop correction effects from a yet unknown interaction characterized by a coupling parameter of order λ. In such an approach, the inner hierarchy of the mass terms provides at the zero order a maximal θ_{23} with a two degenerate neutrino spectrum, where other mixings are null. The order λ corrections would then be enough to predict all the mentioned mixing features, as well as to provide an understanding to solar mixing through the lifting of the degeneracy on the solar sector. Similar ideas had been earlier discussed on the literature. In Ref. [8], following a similar motivation, a specific mass texture was suggested. Here we are addressing the problem in a more general way, though. In Ref. [9] it was shown that θ_{12} and θ_{13} could have a common origin related to the solar scale, but one of the mixing angles, θ_{23}, was there fixed to its maximal value to stress their interest on some specific mixing matrices. In Ref. [10], on the other hand, leptonic mixing angles were related to lepton mass ratios. Nevertheless, it is desirable to have an explanation for all three mixings, and they indeed could be generated by the quantity λ, as we will show.

With this aim, this paper is organized as follows. First, in Sec. 2 we use the condition of having null reactor and solar mixings and a maximal atmospheric mixing, with a two neutrino degenerate spectrum, to search for explicit mass matrix structures that be consistent with hierarchical neutrinos, based on the Taylor series approach. Then, along Sec. 3, we parametrize the order λ contributions that raise the degeneracy, providing a $\Delta m^2_{\rm sol}$ mass gap, which, at the same time, would offer an explanation for the observed neutrino mixings patterns. There we perform the analysis for both normal and inverted hierarchies. Henceforth, in Sec. 4, we explore both analytically and numerically the possible predictions for CP violation of such scenarios, emphasizing that our approach can account for values within current observed bounds. The effect of running under renormalization group effects on the proposed mass matrix hierarchies is explored along Sec. 5. Finally, we close with some general concluding remarks, presented in Sec. 6.

2. Linking neutrino mixings to solar scale

Based on the suggestive numerical coincidences depicted by Eqs. (1) to (3), we propose that the scale ratio λ can be used to Taylor expand the general neutrino mass matrix, in order to explore the hierarchies among the neutrino mass terms, as dictated by atmospheric and reactor mixings. Thus, we write the most general Majorana mass matrix as

$$M_\nu = M_0 + \delta M(\lambda), \tag{4}$$

where M_0 stands for the zero order matrix of the expansion and $(\delta M)_{\ell\ell'}$, for $\ell, \ell' = e, \mu, \tau$, are all smooth complex functions of λ, at least of order one on it. Of course, a leading higher power on λ for some of these terms could be possible, but that should be reflected along the calculations. Hence the above approach can be make without lost of generality.

By taking the theoretical limit where Δm^2_{12} is null, as well as the right hand side of expressions (2) and (3), one ends with a neutrino spectrum with two degenerate neutrinos where $\tan\theta_{23} = \pi/4$ with not further mixings. This scenario immediately suggest that, in the theoretical limit where λ is null, the neutrino mass matrix has the simple form

$$M_0 \propto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & \sigma C \\ 0 & \sigma C & 1 \end{pmatrix}, \tag{5}$$

where σ stands for a positive (negative) sign that should be taken in the normal (inverted) hierarchy case. Here, C is a complex number with $Re\, C > 0$, and $|A| = |1 - \sigma C|$. The condition $|A| = |1 - \sigma C|$ is required in order to ensure a null Δm^2_{12}, which is needed for the zero order matrix M_0. There are symmetry groups, like A_4, which supports this kind of matrix configuration, however, the exploration of the possible models that generates the proposed matrix is beyond the scope of this work. In any case the only existing mass splitting at this limit is given by $\Delta m^2 \propto 4Re\, C$. Also, M_0 does not generate any Dirac phase. Since the A phase can be freely redefined, and taken to be zero, a single Majorana phase, ϕ, arises from the phase difference of complex eigenmasses, $1 \pm C$, which gives $\tan\phi = 2Im\, C/(1 - |C|^2)$.

Hereafter, to clearly exemplify the way our proposal works, we will concentrate our discussion to the case where $C = 1$, which means that we can take $|A| = (1 - \sigma)$ according to the hierarchy. This would restrict our analysis to truly hierarchical neutrinos. Following Eq. (4), the mass matrix should add to M_0 some smooth complex functions of λ, which by construction we assume at least proportional to λ. In this terms, the Taylor form of the neutrino mass matrix can be written as

$$M_\nu = \begin{pmatrix} \Gamma & e\lambda & a\lambda \\ e\lambda & b & \sigma c \\ a\lambda & \sigma c & 1 \end{pmatrix} m_0, \tag{6}$$

where we naturally expect the real overall mass scale $m_0 \sim \sqrt{\Delta m^2_{ATM}}/2$. The choice of the scale $m_0 \sim 2.5 \times 10^{-2}$ eV is just a matter of convenience, but it is possible to adjust the scale of the neutrino mass, which would require a slight change in the parameters. However, the goal of the work is to show that there exist matrix textures which generate the

initial conditions, *i.e.*, two degenerate neutrinos, null solar and reactor mixing and maximum atmospheric mixing. The correct observed values are obtained perturbatively. It is important to note that the model does not predict the mass scale, but it does predict the correlation of the parameters. A comprehensive model with additional fields would be needed to assess the model's falsifiability, which extends beyond the scope of this paper. It is also worth noticing that in the μ-τ block, from our construction principles, generic mass terms b and c can be expressed as $1+\delta f \approx e^{i\varphi}(1+Re\,\delta f+\mathcal{O}|\delta f|^2)$, where δf stands for the corresponding $\mathcal{O}(\lambda)$ complex correction. Hence, the overall phase turns out to be small, since $\tan\varphi \approx Im\,\delta f$. This already indicates that the phases on the first row shall be the most relevant for the analysis below. Furthermore, without loss of generality, upon rephasing of the flavor neutrino wave functions, and without altering the phases relevance, we can choose e and b to be real parameters. Others should, in general, stay complex numbers.

According to our Taylor expansion considerations, at the leading order, we can write $|\Gamma| = (1-\sigma)+d\lambda$, $b = 1+\sigma\delta_b\lambda$, and $|c| = 1-\delta_c\lambda$. Here we should assume $|a|$, d, e, δ_b, and δ_c to be at most order one numerical parameters to be fixed by the use of experimental data. Also, hereafter we denote the phases of Γ, a and c as $\phi_{\Gamma,a,c}$, respectively.

Similar textures to the above, for normal hierarchy, were presented in Refs. [8, 11]. The later was motivated within the context of $\mu-\tau$ symmetry. There, however, the corresponding corrections to m_{ee} and $m_{\mu\tau}$ where proposed to be of order λ^2, but no connection with our current motivation was made. The former, on the other hand, was similarly motivated by the numerical coincidences on the oscillation parameters, nonetheless, in there m_{ee} was assumed to be exactly λ^2 whereas $m_{e\mu,e\tau}$ were taken to be exactly λ, which also exhibits $\mu-\tau$ symmetry but leaves out solar neutrino oscillations. Such cases, of course, are contained in the present one since our approach is more general. Note also that a class of models based on $\mu-\tau$ symmetry or $\mu-\tau$ parity can also be reproduced in the present context (for a recent review of both see for instance [12]). That should be expected, since such symmetries are known to be closely attached to the smallness of θ_{13} and a large θ_{23}. Furthermore, Eq. (6) already shows some of the inner hierarchies that the observed neutrino oscillation data suggests.

3. Masses and mixing

Next, let us proceed to show that the above expression for M_ν does predict the desired spectra of masses and mixing. It would be of particular interest to verify that the right amount of solar mixing could arise from the texture. Because at the zero order $m^2_{1,2}$ are degenerated, one only needs the right amount of perturbation to split these squared masses to provide a mass gap $\Delta m^2_{21} \propto \lambda^2 m_0^2$. As for the large solar mixing, it should naturally arise due to the same kind of perturbations.

In order to estimate the mixings, we perform an approximate diagonalization of the squared hermitian matrix $H = M_\nu M_\nu^\dagger$. This has the clear advantage that H is diagonalized by a unitary transformation, such that $U^\dagger H U = M^2_{diag}$, where the RHS is given by the non negative and diagonal matrix $M^2_{diag} = diag(m_1^2, m_2^2, m_3^2)$. Up to field phase redefinitions of the neutrino basis, U^* corresponds to the U_{PMNS} mixing matrix that diagonalizes M_ν (see Refs. [13,14] for the precedents to this approach). Also, it is important to note that we work in a basis in which the charged lepton matrix is diagonal, which means that U_{PMNS} also describes the strength of the lepton flavor mixing in weak interactions. For the purpose of calculation, it becomes convenient, given the expected smallness of θ_{13} and the hierarchical nature of neutrinos, to express $U = R_{23}(\theta_{23},\varphi_1)R_{13}(\theta_{13},\varphi_2)R_{12}(\theta_{12},\varphi_3)$, where the block unitary rotations

$$R(\theta,\varphi) = \begin{pmatrix} \cos\theta & \sin\theta e^{i\varphi} \\ -\sin\theta e^{-i\varphi} & \cos\theta \end{pmatrix}, \qquad (7)$$

embedded into their three dimensional forms, are obtained by consecutively diagonalizing each indicated block on H. On this approach, Dirac CP phase on PMNS parameterization turns out to be $\delta_{CP} = \varphi_2 - \varphi_1 - \varphi_3$.

Therefore, within our present setup, H can be written as

$$H = \begin{pmatrix} \alpha & \delta & \omega \\ \delta^* & \beta & \rho \\ \omega^* & \rho^* & \gamma \end{pmatrix} m_0^2, \qquad (8)$$

where, $\alpha = |\Gamma|^2 + e^2\lambda^2 + |a|^2\lambda^2$, $\beta = b^2 + |c|^2 + e^2\lambda^2$, $\gamma = 1 + |c|^2 + |a|^2\lambda^2$, $\delta = (\Gamma e + eb + \sigma a c^*)\lambda$, $\omega = (\Gamma a^* + \sigma e c^* + a)\lambda$, and $\rho = \sigma b c^* + \sigma c + e a^* \lambda^2$. It is then easy to realize that in normal hierarchy α becomes $\mathcal{O}(\lambda^2)$, as well as the perturbative corrections that would be induced by off-diagonal matrix terms, δ and ω. This would explain solar neutrino scale without further assumptions. Such is, however, not the case for inverted hierarchy, where α gets an order one correction on λ. Here, explaining a solar scale that is of second order would require tuning some of the parameters to an order λ precision, as we will discuss below.

After some algebra, atmospheric mixing is expressed, at leading order, as

$$\tan\theta_{ATM} \approx 1 + \frac{\delta_b}{2|\cos\phi_c|}\lambda, \qquad (9)$$

irrespective of the hierarchy, whereas the reactor mixing becomes

$$\tan\theta_{13} \approx \frac{|(1-\sigma)e^{i\phi_\Gamma}(e+a^*)+(e+a)(1+\sigma e^{-i\phi_c})|}{\sqrt{8}(1+|\cos\phi_c|)}\,\lambda. \qquad (10)$$

In order to fulfill the expectation of having at most order one Taylor parameters, $|\cos\phi_c|$ should be close to one, otherwise the coefficient of λ on Eq. (9) would become too large as to account for the observable value of the mixing. Moreover, since reactor mixing in normal hierarchy vanishes for

$\phi_c = \pi$, we must avoid this branch of values. This is of course consistent with our previous expectations. In the particular case where we take $\phi_c \approx 0$, that we assume hereafter, atmospheric and reactor mixings reduce to

$$\tan\theta_{ATM} \approx 1 + \frac{\delta_b}{2}\lambda, \quad \text{and}$$
$$\tan\theta_{13} = \frac{|e+a|}{2\sqrt{2}}\,\lambda, \tag{11}$$

for both the hierarchies. Furthermore, by looking up on best fit value of the atmospheric mixing, we should notice that δ_b has to be negative for NH, in order to bring down the mixing from its zero order maximal value. The opposite should be the case for IH. From here it is straightforward to see that, at best fit point, $\delta_b = -1.11(1.753)$ and $|e+a| = 2.494(81)$ according to our initial assumptions. As an aside comment, it is obvious that above formula for atmospheric mixing would still work pretty well even if the resulting best fit point from future data were to move towards values higher than $\pi/4$, as it was in the data fit of Ref. [7], for instance.

Next, we proceed to calculate the solar neutrino scale and the solar mixing, which after a lengthy calculation get the general forms

$$\Delta m^2_{sol} \approx m_0^2\sqrt{\mu^2 + D^2}, \tag{12}$$

and

$$\tan\theta_{sol} \approx \frac{1}{x + \sqrt{1+x^2}}, \tag{13}$$

with $x = \mu/D$ and where the explicit expressions for D and μ do depend on neutrino hierarchy. From Eq. (12), it is clear that both μ and D should be required to be $\sim \mathcal{O}(\lambda^2)$, which in turn would imply that $0 < x \lesssim \mathcal{O}(1)$, just as needed to understand the large observed value of solar mixing. As a matter of fact, at best fit point data requires that $x = 0.426$. The corresponding results to each hierarchy are as follows.

(i) For *normal hierarchy* we get at the lower order

$$D \approx \frac{1}{\sqrt{2}}\left|(e-a)(\delta_b + 2\delta_c) + 2de^{i\phi_\Gamma}(e-a^*)\right|\lambda^2, \tag{14}$$

whereas,

$$\mu \approx \frac{1}{4}\left((\delta_b + 2\delta_c)^2 - 4d^2\right)\lambda^2. \tag{15}$$

(ii) For *inverted hierarchy* the situation is more challenging, since, as we have anticipated, both contributions to the solar scale, μ and D, arise at first order level, such that

$$D \approx 2\sqrt{2}\left|e(1+e^{i\phi_\Gamma}) - (a + a^*e^{i\phi_\Gamma})\right|\lambda + \mathcal{O}(\lambda^2), \tag{16}$$

and

$$\mu \approx -\left(4d + 2(\delta_b + 2\delta_c)\right)\lambda + \mathcal{O}(\lambda^2). \tag{17}$$

Thus, in order to warrant the right prediction for the solar scale, some tuning conditions must be imposed on the parameters, such that they cancel each other within the order λ contributions. From above expressions, it is straightforward to see that such conditions amount to fix

$$a \approx e - g\lambda, \qquad \text{and} \qquad d \approx -\frac{1}{2}(\delta_b + 2\delta_c) - h\lambda, \tag{18}$$

with $|g|$ and h order one numbers. This express the required level of accuracy of our approximation. It has the aside implication that ϕ_a would become small (and perhaps irrelevant for CP violation), since now $\tan\phi_a \approx -Im(g/e)\cdot\lambda$. Notice that no restrictions are made to ϕ_Γ so far, which should then become the most relevant CP phase in this case.

Using the above tuning conditions, we then get

$$D \approx \sqrt{2}\left|e\delta_b(1+e^{i\phi_\Gamma}) - 2(g + g^*e^{i\phi_\Gamma})\right|\lambda^2, \tag{19}$$

and

$$\mu \approx \left(\frac{1}{2}\delta_b^2 + 2h - |e+a|^2\right)\lambda^2. \tag{20}$$

Finally, atmospheric scale is given, for both the hierarchies as

$$\Delta m^2_{ATM} \approx (4 + (\sigma\delta_b - 2\delta_c)\lambda)\, m_0^2. \tag{21}$$

Since δ_b has been fixed by the observed value of the atmospheric mixing, in Eq. (9), the model is left with five free parameters: That is, m_0, e, δ_c and a and d for normal hierarchy, but g and h for the inverted one. Hence, there is plenty of room to accommodate the remaining four neutrino observables, $\tan\theta_{13}$, $\tan\theta_{sol}$, Δm^2_{sol} and Δm^2_{ATM}, from Eqs. (11), (12), (13) and (21). To have a feeling of the order of magnitude the model parameters should have, we can solve the system in the non CP violating case, by fixing $m_0 = \sqrt{\Delta m^2_{ATM}}/2 = 0.025\ eV$, which gives at central values $a = 0.664$, $d = 0.61$, $e = 1.829$ and $\delta_c = 1.867$ for normal hierarchy, and $e = 1.24$, $g = 0.437$, $h = 3.092$ and $\delta_c = 0.877$ for the inverted one. Of course, taking a slightly different value for m_0 would render a bit different set of solutions, but all would be of similar order, which validates our approximations. Note, also, that these values emerge for the approximated expressions we have derived above. More precise results can be obtained either from extending our analysis to include higher order corrections or by proceeding through a completely numerical calculation. We shall consider the later below.

4. CP violation

The amount of CP violation in any neutrino mass model can be estimated by using the Jarlskog invariant [15]

$$J = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta_{CP}, \tag{22}$$

which can also be written in terms of the previously defined squared hermitian matrix H as (See for instance [16])

$$J = -\frac{Im\left(H_{21}H_{32}H_{13}\right)}{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}. \tag{23}$$

Using our parameterization, we can straightforwardly express the invariant in the approximated form

$$J \approx -\left(\frac{m_0^2}{\Delta m^2_{ATM}}\right)^3 Im[\delta^*\rho^*\omega]\,\lambda^{-2}. \qquad (24)$$

It turns out that, at the leading order, $Im[\delta^*\rho^*\omega] \sim \mathcal{O}(\lambda^3)$, which means that the model could without stress accommodate a $\sin\delta_{CP} \sim \mathcal{O}(\lambda)$ which is well within the expected experimental value range. As a matter of fact, by writing down

$$Im[\delta^*\rho^*\omega]\,\lambda^{-2} \approx \kappa_0 + \kappa_1\lambda + \ldots, \qquad (25)$$

and after some calculations, we get, for normal hierarchy, $\kappa_0 = (|a|^2 - e^2)\sin 2\phi_c$ and

$$\begin{aligned}\kappa_1 = {} & \left[\delta_b(|a|^2 - e^2) + 2e^2\delta_c\right]\sin 2\phi_c \\ & + 2\cos\phi_c\Big[e|a|(\delta_b + 2\delta_c)\sin\phi_a \\ & + e^2(\delta_c - d)\sin(\phi_\Gamma + \phi_c) \\ & + |a|^2 d\sin(\phi_\Gamma + \phi_c - 2\phi_a)\Big]. \end{aligned} \qquad (26)$$

Notice that this implies that $\kappa_0 \lesssim \mathcal{O}(\lambda)$ as a consequence of the smallness of ϕ_c, just as discussed along previous sections. Furthermore, in the limit where we do neglect ϕ_c, $\kappa_0 = 0$, and thus, the CP violation phase is given by

$$\sin\delta_{CP} \approx -0.466\left(\frac{4m_0^2}{\Delta m^2_{ATM}}\right)^3 \kappa_1\,\lambda, \qquad (27)$$

where $\kappa_1 \approx 2[e|a|(\delta_b + 2\delta_c)\sin\phi_a + e^2(\delta_c - d)\sin\phi_\Gamma + |a|^2 d\sin(\phi_\Gamma - 2\phi_a)]$. It is not difficult to see that with order one parameters we can easily get an adequate value for the CP violating phase.

For inverted hierarchy we get

$$\begin{aligned}\kappa_0 = {} & 2\cos\phi_c\Big(2\big[|a|^2\sin(\phi_\Gamma + \phi_c - 2\phi_a) \\ & - e^2\sin(\phi_\Gamma + \phi_c)\big] + (|a|^2 - e^2)\sin\phi_c + 4e|a|\sin\phi_a\Big), \end{aligned}$$

and the rather complicated expression

$$\begin{aligned}\kappa_1 = {} & 2\delta_b\Big[2e|a|\sin(\phi_c - \phi_a) - |a|^2\sin(2\phi_c + \phi_\Gamma - 2\phi_a) \\ & + e^2\sin\phi_\Gamma\Big] + (e^2(2\delta_c - \delta_b) - \delta_b|a|^2)\sin 2\phi_c \\ & - 2\cos\phi_c\Big[e|a|(6\delta_c - \delta_b - 4d)\sin\phi_a \\ & + 2e|a|\delta_b\sin(\phi_a - \phi_\Gamma) + e^2(d - \delta_c)\sin(\phi_\Gamma + \phi_c) \\ & - |a|^2(d - 4\delta_c)\sin(\phi_\Gamma + \phi_c - 2\phi_a)\Big]. \end{aligned} \qquad (28)$$

In this case, the conditions imposed through Eq. (18), and the implied smallness of $\phi_{c,a}$ amount again to consider κ_0 to be of order λ. Indeed, by neglecting these phases in above expressions, they simplify into $\kappa_0 \approx -8eRe(g)\sin\phi_\Gamma\,\lambda$, and $\kappa_1 = e^2(4\delta_b + 6\delta_c)\sin\phi_\Gamma$. Hence, for inverted hierarchy we get the approximated formula

$$\sin\delta_{CP} \approx -0.4688\left(\frac{4m_0^2}{\Delta m^2_{ATM}}\right)^3 (\kappa_0 + \kappa_1\,\lambda), \qquad (29)$$

which is now controlled by the single ϕ_Γ phase, as already anticipated.

In order to explore in a more accurate way the possible predictions for the Dirac CP phase from our neutrino mass matrix proposal, we have numerically scanned the parameter space looking for value sets that reproduce the observed mixings and oscillation scales within the current level of accuracy. To this aim, we numerically solve for the eigensystem of the hermitian matrix H that is generated out of the mass matrix M_ν, in Eq. (6), by randomly assigning values to the parameters. Once the eigenvectors are numerically known, without further considerations nor approximations, we calculate the corresponding mixing angles and the Dirac CP phase as given in the PMNS parameterization, as well as the corresponding squared mass differences, and keep only those parameter sets that reproduce neutrino oscillation observables within the range of one sigma deviation for the mixings, and up to three sigma deviations for the mass scales ratio. Given that m_0 appears in Eq. (6) as a global scaling, we have found more appropriate to use the scale ratio as the discriminator rather than the scales themselves since this reduces one parameter on the analysis.

Also, in order to simplify the numerical computation we fixed our small parameter to the value $\lambda = 0.1719$ throughout the numerical analysis for both hierarchies, with the understanding that any possible change on it can always be compensated by a scaling on the other parameters. It is also worth noticing that for any given matrix H which is diagonalized by the unitary matrix U, its complex conjugated H^* would be diagonalized by U^*. Since the Dirac phase in U does correspond to the opposite phase in U^*, the parameter map for δ_{CP} should be symmetric under reflection symmetry. We use this fact to locate additional point solutions on the parameter space.

For normal hierarchy we searched the parameter space using the following parameter ranges: $|a| \in (1,2)$, $d \in (0,2)$, $e \in (0,2)$, $\delta_b \in (-1.2,-0.4)$, $\delta_c \in (0,2)$ and $\phi_{a,c,\Gamma} \in (0,2\pi)$. We show in Fig. 1 the solution sets for the e-$|a|$ and e-δ_b subspaces, where the allowed region is best depicted. Notice the apparent correlation among e and $|a|$, that was expected from the way they combine themselves to produce the reactor mixing in Eq. (10). Also, all the presented solutions were found to have a very small value of ϕ_c, as expected from our previous analytical discussions. Blue dots correspond to the values of the parameters which generate a $\Delta m^2_{\rm sol}/\Delta m^2_{\rm ATM}$ ratio within the 1σ range, while orange dots correspond to those solutions with the quotient lying in the 3σ range. The same colormap is used for all the figures presented in this work. As we can see from the plots, there is not a clear distinction between the 1σ and 3σ regions, indicating the sensibility of the mass texture to the values of the parameters regarding the prediction of neutrino mass scales. This may be an indication that certain level of tuning would be required in the underlying theory to keep masses under the expected accuracy.

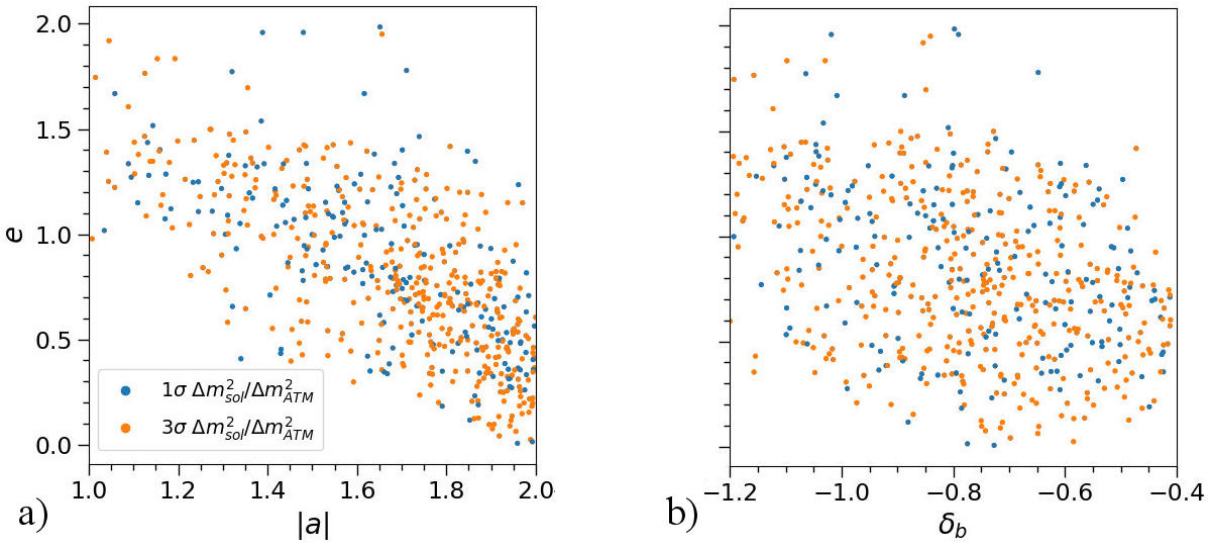

FIGURE 1. Allowed parameter space for the normal hierarchy, e is shown as a function of $|a|$ and δ_b. Blue (orange) dots generate a $\Delta m^2_{sol}/\Delta m^2_{ATM}$ ratio at 1σ (3σ) accuracy.

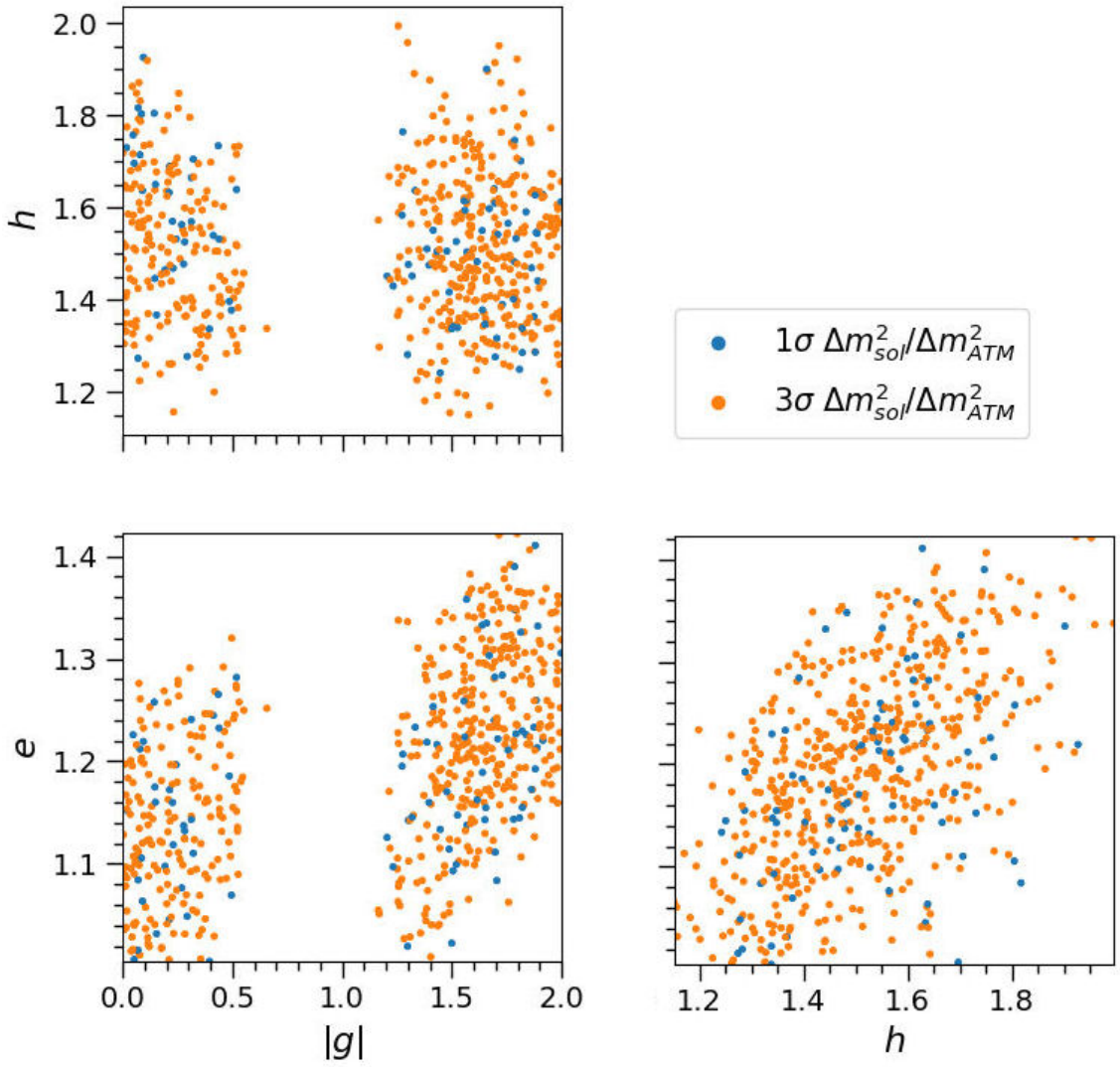

FIGURE 2. Allowed parameter values for the inverted hierarchy as seen in the e-$|g|$-h parameter subspace.

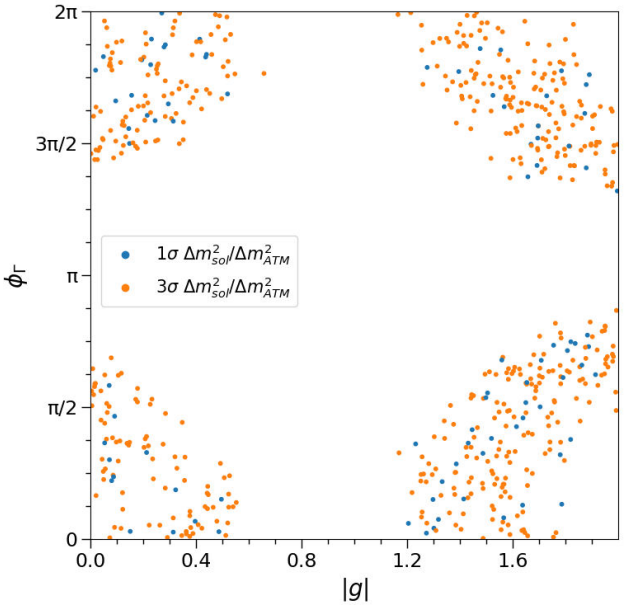

FIGURE 3. ϕ_Γ as a function of $|g|$. There are well delimited regions for the values of the model parameters which generate neutrino oscillation parameters within the required accuracy.

For the inverted hierarchy, we have explicitly introduced the tuning conditions given in Eq. (18) to focus our search. Thus, the parameters were assigned to arbitrary values within the intervals: $|g| \in (0,2)$, $h \in (1,2)$, $e \in (1,1.5)$, $\delta_b \in (1,2)$, $\delta_c \in (0,1.8)$ and $\phi_{c,g,\Gamma} \in (0,2\pi)$. Fig. 2 shows the results on the e, $|g|$ and h subspace. Unlike the parameter space for the NH case, which present just a slight correlation only between two of them, for the inverted hierarchy there is a clear correlation between all parameters. For the sake of simplicity we only show here three plots, however, δ_b and δ_c have shown a similar dependence in $|g|$ as that for e and h.

In Fig. 3 we show the phase ϕ_Γ as a function of $|g|$. This phase is of particular interest because our results were also found to allow only small values for the $\phi_{c,g}$ phases, as expected. Therefore, it is the ϕ_Γ phase the one that would determine the overall shape of the Dirac CP phase. Furthermore, as we can observe on Figs. 2 and 3, there are well delimited regions in the parameter space, although, once again, the points which generate a mass scale ratio within 1σ and 3σ ranges are mixed together. Thus, the already observed sensibility of the mass scales in NH to small variations of the parameters also appears for IH.

The predicted values of the Dirac phase that emerge from the found parameter sets that are consistent with oscillation mixings and mass scales at the given level of accuracy, are depicted in Figs. 4 and 5 for normal and inverted hierarchy, respectively. The blue (orange) band indicates the current 1σ (3σ) region for δ_{CP}, as reported in Ref. [2]. As we can observe, the model at hand has quite enough room on the parameter space to allocate a future measurement of δ_{CP} well within the current expectations. Notice that in the NH case the majority of the points lying within the 1σ range for δ_{CP} correspond approximately to $e \in (0.2, 0.8)$, whereas in the

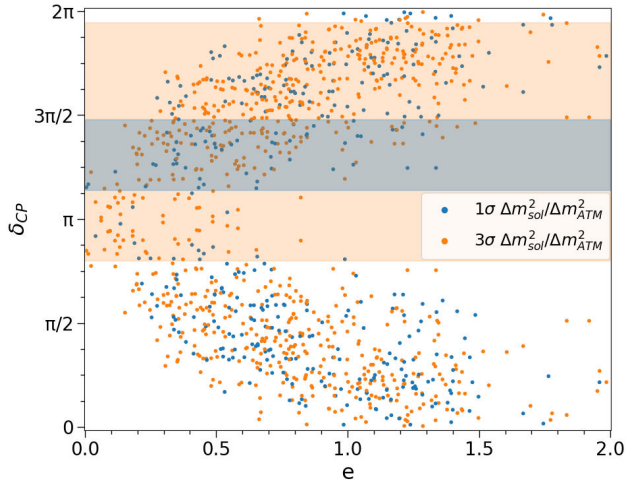

FIGURE 4. δ_{CP} predictions in terms of the e parameter for the normal hierarchy. The current allowed values are within the colored bands, as explained in the text.

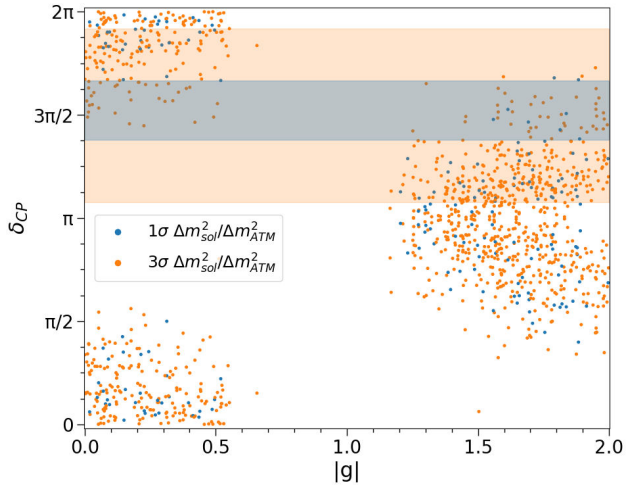

FIGURE 5. δ_{CP} predictions in terms of the $|g|$ parameter for the inverted hierarchy.

IH case, we have fewer points within the 1σ and 3σ regions, these lying approximately in the interval $g \in (0, 0.5) \cup (1.5, 2)$. Thus, indeed, the numerical analysis shows that there exist a parameter space for the proposed mass texture consistent with the 1σ values of masses and mixings in both cases. This validates the interest on our phenomenological approximation and enforces the possible link among the observables that suggested it.

5. Renormalization effects

As a final note we would like to comment on the effects that energy scaling would have over the mass matrix structure that has been derived above, by using the Taylor expansion approximation. This issue arises as a necessary analysis when one considers that any realistic flavor model of neutrino masses should rely on some new physics, either in the form of new interactions or new particles, that very likely would

emerge at some high energy Λ. Therefore, to reproduce what we measure at a low energy, Λ_0, below electroweak scale, the effect of running parameters of the theory from Λ down to Λ_0 has to be considered. This running is governed by the renormalization group equations (RGE), which, at one-loop, relate the effective neutrino mass matrix with that defined at higher energy by [17–19]

$$M_\nu(\Lambda_0) = I_\zeta \, \mathcal{I} M_\nu(\Lambda) \mathcal{I}, \tag{30}$$

where we assume that $M(\Lambda)$ has the structure given in Eq. (6). I_ζ is the scaling factor

$$I_\zeta = \exp\left[-\frac{1}{16\pi^2} \int_{\ln\Lambda_0}^{\ln\Lambda} \zeta(t)\, dt \right], \tag{31}$$

which only contributes to rescale the overall neutrino scale, but has no relevance on the renormalization of the mixing angles. Here, with the Standard Model content, $\zeta = -3g_2^2 + \lambda_H + 6y_t^2$, where g_2 stands for the $SU(2)_L$ gauge coupling, λ_H for the Higgs self coupling and y_t for the top Yukawa coupling. On the other hand, the flavour dependent components of the diagonal matrix $\mathcal{I} = Diag\{I_e, I_\mu, I_\tau\}$, are given by

$$I_\ell = \exp\left[\frac{3}{32\pi^2} \int_{\ln\Lambda_0}^{\ln\Lambda} y_\ell^2\, dt \right], \tag{32}$$

with y_ℓ the lepton Yukawa couplings. Due to charged lepton hierarchy, from which $y_e \ll y_\mu \ll y_\tau \ll 1$, the scaling factors I_ℓ would be close to unity and I_τ shall be the dominant one. Because of this, we can safely assume that $I_{e,\mu} \approx 1$ whereas $I_\tau = e^{r_\tau} \approx 1 + r_\tau$, where

$$r_\tau \approx \frac{3}{32\pi^2} y_\tau^2(\Lambda_0) \ln\left(\frac{\Lambda_0}{\Lambda}\right). \tag{33}$$

As a matter of fact, by assuming $\Lambda = \Lambda_{GUT} \sim 10^{16}$ GeV and Λ_0 the electroweak scale, one gets that $r_\tau \sim 10^{-5}$, which represent a too small correction when compared to λ, which defines the inner hierarchy of the mass matrix – actually $r_\tau \sim \mathcal{O}(\lambda^6)$–. Therefore, we would not expect RGE scaling effects to substantially alter the inner structure of the mass matrix. We can verify this by explicitly writing

$$M(\Lambda_0) = \begin{pmatrix} \Gamma_R & e_R\lambda & a_R\lambda \\ e_R\lambda & b_R & \sigma c_R \\ a_R\lambda & \sigma c_R & 1 \end{pmatrix} m_{0R}(\Lambda_0), \tag{34}$$

where the renormalized parameters stand for $\Gamma_R \approx \Gamma(1 - 2r_\tau)$, $a_R \approx a(1 - r_\tau)$, $b_R \approx b(1 - 2r_\tau)$, $c_R \approx c(1 - r_\tau)$, $e_R \approx e(1 - 2r_\tau)$ and $m_{0R}(\Lambda_0) = m_0 I_\zeta I_\tau^2$. From here it is clear that the whole effect of the RGE scaling, from SM corrections, would be rather negligible as far as the size of the parameters goes. Nevertheless, it is worth noticing that the relative scaling on c_R respect to the ones on b_R and Γ_R would produce a slight splitting among $m_{1,2}^2$ that still will be much smaller than the required by solar neutrino scale. This situation is particularly most prominent for the inverted hierarchy, although still remains under control.

6. Concluding remarks and outlook

Measured reactor mixing and the deviation of the atmospheric mixing from maximal value, as given in Eqs. (2) and (3) respectively, are about the same order of magnitude of the squared root of the ratio among neutrino oscillation scales. This curious observation could be more than just a numerical coincidence and rather be an indication of a common physical origin of those parameters. Here, we have elaborated over the idea that all these value parameters may arise from perturbative corrections, to the neutrino mass matrix, that depend on powers of a single small parameter, λ, such that mass terms can be expressed as power series on the last. Clearly, this idea would be well motivated, as well as natural, if there were hidden interactions in the neutrino sector, where loop corrections would be the ones responsible of fixing the (bare) zero order mass structure. Although we do not know yet what such interactions might be, the simplest scenario is to consider at least a single coupling parameter, which could be interpreted as λ.

In such an approximation, we have shown that the class of neutrino textures that arise at the zero order, for either normal or inverted hierarchies, are quite unique. They are required to comply with having two degenerate neutrino states and null θ_{13} and θ_{12} but a maximal θ_{23} mixing angles, which fixes the initial mass structure. Furthermore, the addition of linear corrections in λ are quite enough to generate the masses and mixings observed from neutrino oscillation experiments, using only factor coefficients that at most are of order one. The predicted CP phase is also compatible with the current experimental bounds at 1σ and 3σ accuracy. And, as we have shown, the proposed mass structures are stable under the energy scaling goberned by RGE with the SM particle content. Our results strongly support the validity of approaching mass terms with the suggested power series approximation, in particular to describe the inner hierarchies of the mass matrix. It would be interesting to explore this idea even further.

The present modeling of the neutrino mass structure, however, seems to require some level of tuning on the parameters in order to provide precise predictions on neutrino mass scales, as our numerical exploration suggests. The required tuning should be of order λ^2 or so on most parameters, though. Mixing predictions, on the other hand, are quite robust. This feature of the model may suggest the existence of further correlations among the different mass terms, and perhaps be indicative of a non trivial interplay among the parameters of the possible underlying theory. It would be interesting to explore this possibility in an extended way once a complete theoretical realization of the present idea could be provided.

The texture suggested from our series expansion proposal may be obtained in models with $\mu - \tau$ symmetry, or $\mu - \tau$

parity, although we have not explored this possibility any further. The zero order mass matrix in the case of normal hierarchy seems consistent with models with a L_e symmetry. This may suggest the kind of flavor symmetries to further explore in search for a complete model realization of our proposal. In particular, models with S_3, A_4 or other discrete flavor symmetries which may hold a doublet fermion representation could explain the structure in the $2-3$ sector, which has almost universal mass terms before λ perturbations are introduced.

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