Time-dependent conserved operators for Schrödinger equation with constant electromagnetic field and quantization of resistance

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Two systems are studied: the first one involves a charged particle under the influence of a constant electric field, and the second one involves a charged particle under the influence of a constant electromagnetic field. For both systems, it is possible to find time-dependent conserved operators that can be used to derive time-dependent solutions to the complete Schrödinger equation. These conserved operators are employed to define the symmetries of the system. An argument of invariance of the wave function under the action of a unitary operator leads to the quantization of resistance and resistivity, in integer multiples of the von Klitzing's constant, for the first and second case respectively.

Keywords: Time-dependent operators; resistance quantization; Klitzing's constant.

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1. Introduction

Conserved operators in dynamical systems have proven to be valuable tools in quantum mechanics for identifying system bases. In this context, the approach involves constructing and solving an eigenvalue equation using these conserved operators. For example, in the central force problem, angular momentum is conserved and is used to determine the eigenfunctions associated with azimuthal and polar angles [1]. Another illustrative case is that of a constant magnetic field, where, depending on the chosen norm, the conservation of linear momentum is used to find the system bases, known as the Landau ansatz [2]. There are autonomous systems in which conserved operators may depend on time. In these scenarios, a similar approach can be applied: one forms and solve an eigenvalue equation with these time-dependent operators and uses the solutions to identify the system's bases. Interestingly, although if the spatial coordinates can be formally separated from the temporal coordinate in the Schrödinger equation, the solutions obtained with these operators have mixed space-time variables, preventing a separation of the temporal part. This temporal dependence of the wave function has an additional implication: when applying the energy operator, instead of obtaining an eigenvalue relation, one finds a new wave function that satisfies the full Schrödinger equation, implying that the energy operator defines the system's degeneracy. In this work, we analyze two systems exhibiting time-dependent conserved operators. The first system involves a particle under the influence of a constant electric field, while the second is subjected to a constant electromagnetic field. Moreover, the wave functions obtained for these cases have the characteristic that, it remains invariant under a unitary transformation if the resistance (in the first case) and resistivity (in the second case) are quantized in integer multiples of the Klitzing's constant [3-5]. This result aligns with experimental measurements reported by Störmer [6-8]. It should be noted that current hypotheses attempting to explain this phenomenon generally rely on a quasi-particle approach [9, 10]. However, our findings suggest that it could also be understood as a single-particle effect.

2. About time-dependent conserved operators

The Hamiltonian we are interested in is written in CGS units as

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{P}} - \frac{q}{c}\mathbf{A}\right)^2 + qU, \tag{1}$$

where m is the mass of the particle, q is the charge of the particle, c is the speed of light, $\hat{\mathbf{P}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\nabla$ is the momentum operator, $\mathbf{A} = (A_x, A_y, A_z)$ is the magnetic vector potential such that the magnetic field is given by its curl, that is $\mathbf{B} = \nabla \times \mathbf{A}$ and $U = U(x, y, z)$ is the electric potential such that the electric field is given by the negative of its gradient, $E = -\nabla U$. Given our interest in identifying time-dependent conserved operators, it is important to demonstrate that they can be used to find solutions to the Schrödinger equation. The total variation of an operator, $\hat{f}$, is given by the Heisenberg equation

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t}. \tag{2}$$

When the above expression is equals to zero, one can say that the operator is conserved, that is

$$\frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t} = 0. \tag{3}$$

Here, $\hat{f}$ is a time-dependent operator which can be used to write down an eigenvalue equation

$$\hat{f}\psi = \lambda\psi, \tag{4}$$

where λ is a scalar. Then, applying Eq. (3) to ψ we can write

$$\hat{f}(\hat{H}\psi) - \hat{H}(\hat{f}\psi) + i\hbar\frac{\partial \hat{f}}{\partial t}\psi = 0, \tag{5}$$

using Eq. (4) and $\partial_t(\hat{f})\psi = \partial_t(\hat{f}\psi) - \hat{f}\partial_t\psi$ the above expression can be rewritten as follows

$$(\hat{f} - \lambda)\left(\hat{H}\psi - i\hbar\frac{\partial\psi}{\partial t}\right) = 0. \tag{6}$$

The above equality can be satisfied for the following reasons: 1) because $\hat{H}\psi - i\hbar\partial_t\psi$ is an element of the kernel of the operator $\hat{f} - \lambda$ defined as the set of all squareintegrable functions in $\mathbb{R}^3$ that satisfies the previous expression,

$$\text{Ker}(\hat{f} - \lambda) = \left\{\Phi \in L^2(\mathbb{R}^3) \mid (\hat{f} - \lambda)\Phi = 0\right\}. \tag{7}$$

2) Because ψ satisfies the Schrödinger equation

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{8}$$

3. One-dimensional constant electric field

The simplest example where this approach can be used is when we work with a constant force in one dimension. In this case, the force is produced by the electric field, and the electric potential is given by $V(x) = -q\mathscr{E}x$ where $\mathscr{E}$ is a constant representing the electric field intensity. The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\hat{p}_x^2\psi - q\mathscr{E}x\psi, \tag{9}$$

for this case, it is known that the solutions for the stationary equation are the Airy function [11]. However, using Eq. (2) one can realize that this system has the following conserved operators

$$\hat{f} = \hat{p}_x - q\mathscr{E}t, \tag{10}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}. \tag{11}$$

Therefore one can write down the eigenvalue equation for the operator (10),

$$\hat{p}_x\psi - q\mathscr{E}t\psi = -q\mathscr{E}\delta t\psi, \tag{12}$$

where, for simplicity, the eigenvalues were written as $-q\mathscr{E}\delta t$ such that δt is a constant. Solving the above equation, it can be found that

$$\psi(x,t) = \mathcal{C}(t)\exp\left(i\frac{q\mathscr{E}}{\hbar}(t - \delta t)x\right), \tag{13}$$

where $\mathcal{C}(t)$ is functions that depends only on time. To determine this function, we substitute the above wavefunction into Eq. (9) to obtain the following equation

$$i\hbar\frac{\partial\mathcal{C}}{\partial t} = \frac{q^2\mathscr{E}^2}{2m}(t - \delta t)^2\mathcal{C}, \tag{14}$$

having the following solution

$$\mathcal{C}(t) = \exp\left(-i\frac{q^2\mathscr{E}^2}{6m\hbar}(t - \delta t)^3\right), \tag{15}$$

and the solution can be written as

$$\psi(x,t) = N\exp\Bigg(-i\frac{q^2\mathscr{E}^2}{6m\hbar}(t - \delta t)^3 + i\frac{q\mathscr{E}}{\hbar}(t - \delta t)x\Bigg), \tag{16}$$

where N is a normalization constant. At this point, one can search for a simplest form of the solution, in a sense that the above wave function is just a unitary transform of the new function. The conserved operators in Eq. (10) and Eq. (11) can be used to define a couple of unitary operators

$$\hat{U}_x = \exp\left(-i\frac{\delta x}{\hbar}(\hat{p}_x - q\mathscr{E}t)\right), \tag{17}$$

$$\hat{U}_t = \exp\left(i\frac{\delta t}{\hbar}\hat{E}\right), \tag{18}$$

where δx is a constant. These operators define the symmetries of the system such that the complete Schrödinger equation remains invariant under a unitary transformation, *i.e.*,

$$\hat{H} - \hat{E} = \hat{U}_x^\dagger(\hat{H} - \hat{E})\hat{U}_x = \hat{U}_t^\dagger(\hat{H} - \hat{E})\hat{U}_t. \tag{19}$$

Note that the wave function in Eq. (16) can be written as

$$\psi(x,t) = \hat{U}_t N\exp\left(-i\frac{q^2\mathscr{E}^2}{6m\hbar}t^3 + i\frac{q\mathscr{E}}{\hbar}tx\right), \tag{20}$$

which satisfies

$$\hat{H}\psi(x,t) - \hat{E}\psi(x,t) = 0, \tag{21}$$

multiplying the above equality by $\hat{U}_t^\dagger$, using Eq. (19) and the fact that $\hat{U}_t$ is unitary, we have

$$\hat{H}\varphi(x,t) - \hat{E}\varphi(x,t) = 0, \tag{22}$$

where

$$\varphi(x,t) = \hat{U}_t^\dagger\psi = N\exp\left(-i\frac{q^2\mathscr{E}^2}{6m\hbar}t^3 + i\frac{q\mathscr{E}}{\hbar}tx\right). \tag{23}$$

And this is the simplest form of the searched wave function. However, there is still another important characteristic of this solution to be analyzed, and is its degeneracy.

Since the operator in Eq. (11) is conserved, $[\hat{E};\hat{H}] = 0$, the following equality is satisfied

$$\hat{E}(\hat{H}\varphi) = \hat{H}(\hat{E}\varphi), \tag{24}$$

using the fact that φ satisfies Eq. (9), the previous equality can be written as

$$i\hbar\frac{\partial(\hat{E}\varphi)}{\partial t} = \hat{H}(\hat{E}\varphi), \tag{25}$$

it proves that

$$\hat{E}\varphi(x,t) = \left(\frac{q^2\mathscr{E}^2}{2m} - q\mathscr{E}x\right)\varphi, \tag{26}$$

is another solution of Eq. (9). This characteristic can be generalized to any $j \in \mathbb{N}$ applications of the operator $\hat{E}$, since $[\hat{E}^j, \hat{H}] = 0$ one can prove that

$$i\hbar\frac{\partial(\hat{E}^j\varphi)}{\partial t} = \hat{H}(\hat{E}^j\varphi). \tag{27}$$

Hence, we have a countable set of solutions. Defining each one of these functions as $\varphi^j(x,t) = \hat{E}^j\varphi$ such that $\varphi^0 = \varphi$, the following commutator can be calculated

$$[\hat{f}, \hat{E}^{j+1}] = i\hbar q\mathscr{E}(j+1)\hat{E}^j, \tag{28}$$

and applying it to φ we have that each one of these functions satisfies the following eigenvalue equation

$$\hat{f}\hat{E}\varphi^j = i\hbar q\mathscr{E}(j+1)\varphi^j, \tag{29}$$

It is important to note that the eigenvalues for the above equation are imaginary. Despite the fact that the operators in Eq. (10) and Eq. (11) are Hermitian on their own, the product of two Hermitian operators is not necessarily Hermitian. Hence, $\hat{f}\hat{E}$ can have imaginary eigenvalues.

The general solution of this system can be written, as a superposition of the wave function Eq. (23) and its degeneration, that is

$$\Psi(x,t) = \sum_{j=0}^{\infty} c_j\hat{E}^j\varphi(x,t). \tag{30}$$

where c_j are constants. Due to the degeneracy, it can be difficult to work with the general solution, but there is a specific selection of constants in the linear combination that can simplify the wave function. If the constants are written as

$$c_j = \frac{1}{j!}\frac{\delta t^j}{(i\hbar)^j}, \tag{31}$$

the general solution can be written as

$$\Psi(x,t) = \varphi(x, t-\delta t), \tag{32}$$

where $(x, t-\delta t) = \psi(x,t)$ defined by Eq. (16). Even though one might assume that the general solution can be obtained straightforwardly by solving Eq. (12) and Eq. (14), it is important to note that this is just a particular case of the general solution. The possibility that different selections of constants c_j will lead to different effects of the degeneracy must not be discarded.

Finally, an important property of this system can be deduced when the wave function in Eq. (32) is invariant under a unitary transformation of the operator Eq. (17), that is

$$\hat{U}_x\Psi(x,t) = \exp\left(i\frac{q\mathscr{E}}{\hbar}\delta x\delta t\right)\Psi(x,t), \tag{33}$$

which leads to the condition that

$$\frac{q\mathscr{E}}{\hbar}\delta x\delta t = 2\pi n, \quad n \in \mathbb{N}. \tag{34}$$

To interpret the above quantity, we rewrite it as follows

$$\frac{q^2}{\hbar}\mathscr{E}\delta x\frac{\delta t}{q} = 2\pi n, \tag{35}$$

then we note that $q^2/\hbar$ is the inverse of Klitzing's constant [3], $\mathscr{E}\delta x$ is the voltage, and $q/\delta t$ is the current of a single particle. Therefore, redefining Plank's constant as $h = 2\pi\hbar$ and rearranging

$$R = \frac{h}{q^2}n, \tag{36}$$

where R is the resistance of the system. This means that the resistance produced by the particle is quantized in integer multiples of the von Klitzing constant.

4. Constant electromagnetic field

To consider the magnetic field in Hamiltonian Eq. (1), it is necessary to define a gauge to work with. In this case, Landau's gauge will be used to describe a positive magnetic field of constant intensity B along z-direction. Such a gauge leads to a vector potential of the form $\mathbf{A} = B(-y, 0, 0)$. When this gauge is used, typically the electric potential is selected such that it depends on the same variable as the gauge (in this case y), because this selection keeps the momentum $\hat{p}_x$ conserved [2,12,13]. However, in this work, an alternative selection of electric potential is considered, that is $V = -q\mathscr{E}x$. From now on we are only interested in the dynamics of the particle in the $x - y$ plane, therefore we set $p_z = 0$. Therefore, the Hamiltonian is written as

$$\hat{H} = \frac{1}{2m}(\hat{p}_x + m\omega_c y)^2 + \frac{1}{2m}\hat{p}_y^2 - q\mathscr{E}x, \tag{37}$$

where $\omega_c = qB/mc$. This system has the following conserved operators

$$\hat{\pi}_x = \hat{p}_x - q\mathscr{E}x, \tag{38}$$

$$\hat{\pi}_y = \hat{p}_y + m\omega_c x, \tag{39}$$

and

$$\hat{E} = i\hbar\frac{\partial}{\partial t}. \tag{40}$$

As it can be seen, a time-dependent term added to the momentum $\hat{p}_x$ guarantees the conservation of the operator. To find the solutions of this system, it is necessary to begin by writing down the eigenvalue expression of the operators in Eq. (38) and Eq. (39) as follows

$$\hat{p}_x\psi - q\mathscr{E}t\psi = -q\mathscr{E}\delta t\psi, \tag{41}$$

$$\hat{p}_y\bar{\psi} + m\omega_c x\bar{\psi} = m\omega_c\delta x\bar{\psi}, \tag{42}$$

where δt and δx are constants. Solving Eq. (41), we get the following expression

$$\psi(x,y,t) = \mathcal{C}(y,t)\exp\left(i\frac{q\mathscr{E}}{\hbar}(t-\delta t)x\right), \tag{43}$$

and the function $\mathcal{C} = \mathcal{C}(y;t)$ can be determined by substituting the above function into the Schrödinger equation using the Hamiltonian Eq. (37)

$$i\hbar\frac{\partial\mathcal{C}}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\mathcal{C}}{\partial y^2} + \frac{m\omega_c^2}{2}\left(y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right)^2\mathcal{C}. \tag{44}$$

Since the information of the spectrum is still encoded in the above equation, one can search for solutions of the form

$$\mathcal{C}(y,t) = \exp\left(-i\frac{E}{\hbar}t\right)\mathcal{C}_2(y,t), \tag{45}$$

where E is the spectrum of the system and $\mathcal{C}_2 = \mathcal{C}_2(y,t)$ is determined by substituting the above expression into the equation Eq. (44). Making the change of variable by

$$\xi = \sqrt{\frac{m\omega_c}{\hbar}}\left(y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right), \tag{46}$$

one has that

$$\left(\frac{2E}{\hbar\omega_c}\mathcal{C}_2 + i\frac{2q\mathscr{E}}{m\omega_c^2}\sqrt{\frac{m\omega_c}{\hbar}}\frac{\partial\mathcal{C}_2}{\partial\xi}\right) = -\frac{\partial^2\mathcal{C}_2}{\partial\xi^2} + \xi^2\mathcal{C}_2. \tag{47}$$

To solve this equation, one can use the Fourier transform defined as

$$\mathscr{F}\{\phi\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ik\xi}\phi d\xi, \tag{48}$$

which gives the harmonic oscillator equation in the space k, having the eigenvalues

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{1}{2m}\frac{q^2\mathscr{E}^2}{\omega_c^2}. \tag{49}$$

Performing the inverse Fourier transform on the solution in the Fourier space and knowing that the Fourier transform of a harmonic oscillator is another harmonic oscillator, the solution in the original space (y,t) is given by

$$\mathcal{C}_2 = \exp\left(-i\frac{q\mathscr{E}}{\hbar\omega_c}\left[y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right]\right) \times \varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left[y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right]\right), \tag{50}$$

where the function φ_n is the solution of the harmonic oscillator equation defined as follows

$$\varphi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp(-\xi^2)H_n(\xi), \tag{51}$$

where H_n are the Hermite polynomials. Hence, the wave function can be written as

$$\psi_n = \exp\left(-i\frac{q\mathscr{E}}{\hbar\omega_c}\left[y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right] + i\frac{q\mathscr{E}}{\hbar}(t-\delta t)x\right) \times \exp\left(-i\frac{E_n}{\hbar}t\right)\varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left[y + \frac{q\mathscr{E}}{m\omega_c}(t-\delta t)\right]\right). \tag{52}$$

Once again, the wave function is time-dependent and therefore is a solution of the complete Schrödinger equation instead of the stationary one.

To proceed to study the degeneracy, it is helpful to find a simplified form of the solution. The conserved operators in Eq. (38), Eq. (39), and Eq. (40) can be used to construct a set of unitary operators that define the symmetries of the system. These unitary operators are

$$\hat{U}_x = \exp\left(-i\frac{\delta x}{\hbar}\hat{\pi}_x\right), \tag{53}$$

$$\hat{U}_y = \exp\left(-i\frac{\delta y}{\hbar}\hat{\pi}_y\right), \tag{54}$$

and

$$\hat{U}_t = \exp\left(i\frac{\delta t}{\hbar}\hat{E}\right), \tag{55}$$

such that

$$\hat{H} - \hat{E} = \hat{U}_i^\dagger(\hat{H} - \hat{E})\hat{U}_i, \quad i = x, y, t. \tag{56}$$

Then, note that Eq. (52) can be written as $\psi_n = \hat{U}_t\phi_n$ where

$$\phi_n = \exp\left(-i\frac{E_n}{\hbar}t - i\frac{q\mathscr{E}}{\hbar\omega_c}\left[y + \frac{q\mathscr{E}}{m\omega_c}t\right] + i\frac{q\mathscr{E}}{\hbar}tx\right) \times \varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left[y + \frac{q\mathscr{E}}{m\omega_c}t\right]\right). \tag{57}$$

A similar situation to the previous example happens here. It is not possible for the above wave function to share basis with all the conserved operators at the same time.

Instead, the operators in Eq. (39) and Eq. (40) are generators of solutions for the time-dependent Schrödinger equation. This statement can be proved as follows: since the operator in Eq. (39) is conserved, it follows that

$$\hat{H}\hat{\pi}_y\phi_n = \hat{\pi}_y\hat{H}\phi_n, \tag{58}$$

Knowing that Eq. (57) satisfies the time-dependent Schrödinger equation and that Eq. (39) does not depend on time, the above expression can be written as

$$\hat{H}(\hat{\pi}_y\phi_n) = i\hbar\frac{\partial}{\partial t}(\hat{\pi}_y\phi_n), \tag{59}$$

that is $\hat{\pi}_y\phi_n$ is another solution of the time-dependent Schrödinger equation. This can be generalized to $j \in \mathbb{N}$ applications of the operator $\hat{\pi}_y$ to show that $\hat{\pi}_y^j\phi_n$ are solutions for the system. Also, this applies for the operator in Eq. (40) showing that $\hat{E}^j\phi_n$ are also solutions of the time-dependent equation. Hence, the following functions can be defined $f_n^j = \hat{\pi}_y^j\phi_n, g_n^j = \hat{E}^j\phi_n$ where all of them satisfy the complete Schrödinger equation.

Now, it is worthwhile to analyze a characteristic of the linear combination of the wave function Eq. (57) plus its degeneration

$$\Psi(x,y,t) = \sum_{n,j',j} c_{n,j',j}\hat{E}^{j'}\hat{\pi}_y^j\phi_n(x,y,t). \qquad (60)$$

Once again, this expression can be simplified by a specific choice of constants $c_{n,j',j}$

$$c_{n,j',j} = c_n\frac{1}{j!}\frac{\delta t^{j'}}{(i\hbar)^{j'}}\frac{(-1)}{j!}\frac{\delta y^j}{(i\hbar)^j}, \qquad (61)$$

then

$$\Psi(x,y,t) = \exp\left(-i\frac{m\omega_c}{\hbar}x\delta y\right) \times \sum_n \phi_n(x,y-\delta t,t-\delta t). \qquad (62)$$

Hence, applying the unitary operator $\hat{U}_x$ to the above expression one obtains the expression

$$\hat{U}_x\Psi = \exp\left(i\frac{q\mathscr{E}}{\hbar}\delta x\delta t\right)\exp\left(i\frac{m\omega_c}{\hbar}\delta x\delta y\right)\Psi. \qquad (63)$$

If the invariant condition $\hat{U}_x\Psi = \Psi$ is proposed to be satisfied, this implies that the phases must be quantized

$$\frac{q\mathscr{E}}{\hbar}\delta x\delta t = 2\pi l, \qquad (64)$$

$$\frac{m\omega_c}{\hbar}\delta x\delta y = 2\pi k, \qquad (65)$$

where $l,k \in N$. The first one was already obtained in the previous case, it implies the quantization of the resistance along the x axis, which is

$$R_x = \frac{h}{q^2}l. \qquad (66)$$

On the other hand, the second quantization condition is related to the quantization of the magnetic flux through the area $A = \delta x\delta y$, since it can be rewritten as follows

$$B\delta x\delta y = \frac{h}{q}\delta x\delta y, \qquad (67)$$

in MKS units, where $h = 2\pi\hbar$. The implications of the second quantization condition can be seen when the electric current, defined as

$$\mathbf{J} = \frac{iq\hbar}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) - \frac{q^2}{mc}\mathbf{A}\Psi^*\Psi, \qquad (68)$$

along the y-direction is calculated obtaining the expression

$$J_y = -\mathscr{E}\frac{q^2}{\hbar}\frac{\hbar}{m\omega_c}\mid\Psi\mid^2, \qquad (69)$$

note that the negative sign only defines the direction where the particle is moving along the y-axis. Therefore, the magnitude of the conductivity is $\sigma_y = \mid J_y\mid/\mathscr{E}$ and the resistivity is defined as the inverse of this quantity, that is

$$\rho_y = \frac{\hbar}{q^2}\frac{m\omega_c}{\hbar}\frac{1}{\mid\Psi\mid^2}. \qquad (70)$$

Then, the average of the above expression can be calculated inside an area $A = \delta x\delta y$ giving the following result

$$\langle\Psi\mid\rho_y\mid\Psi\rangle = \frac{\hbar}{q^2}\frac{m\omega_c}{\hbar}\delta x\delta y, \qquad (71)$$

Hence, the invariance condition in Eq. (65) implies that the resistivity along the y-direction must be quantized in integer multiples of the Klitzing's constant

$$\langle\Psi\mid\rho_y\mid\Psi\langle = \frac{\hbar}{q^2}k. \qquad (72)$$

Note that his expression is similar to the one found experimentally in Ref. [6].

Continuing with the analysis, the solution of Eq. (42) gives the expression

$$\bar{\psi} = \mathscr{D}(x,t)\exp\left(-i\frac{m\omega_c}{\hbar}(x-\delta x)y\right), \qquad (73)$$

where $\mathscr{D} = \mathscr{D}(x,t)$ is a function to be determined. Substituting it into the time-dependent Schrödinger equation with the Hamiltonian Eq. (37) and rearranging, one obtains the harmonic oscillator equation in the x-axis displaced by the quantity $\delta x + q\mathscr{E}/m\omega_c^2$. This equation has the following solutions

$$\mathscr{D} = e^{-i(E'_n/\hbar)t}\varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left[x - \delta x - \frac{q\mathscr{E}}{m\omega_c^2}\right]\right), \qquad (74)$$

where the harmonic oscillator was defined by Eq.(51) and the eigenvalues are defined as

$$E'_n = \hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\frac{q^2\mathscr{E}^2}{\omega_c^2} - \delta x\delta\mathscr{E}. \qquad (75)$$

An interesting peculiarity can be noted at this point. Comparing the energies in Eq. (49) with the above expression, we note that the spectrum differs by a term of $\delta xq\mathscr{E}$. This discrepancy is due to the fact that the solution obtained is a unitary transform of $\hat{U}_x^\dagger$, that is, defining the function $\bar{\phi}_n = \hat{U}_x^\dagger\bar{\phi}$ one gets the following solution

$$\bar{\phi}_n = \exp\left(-i\frac{E_n}{\hbar}t - i\frac{m\omega_c}{\hbar}xy\right) \times \varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left[x - \frac{q\mathscr{E}}{m\omega_c}\right]\right), \qquad (76)$$

which has the eigenvalues defined by Eq. (49). Of course, the displacement dependence of the energy in Eq. (75) represents the existence of a continuous degeneration of the spectrum, which could be eliminated by performing the previous unitary transformation. However, there exists a second discrete degeneration of the system given by the operators in Eq. (38) and Eq. (40) such that any $j \in \mathbb{N}$ applications of them will give a new solution of the system. These new solutions are denoted as $\bar{f}_n^j = \hat{\pi}_x^j \bar{\phi}_n$, $\bar{g}_n^j = \hat{E}^j \bar{\phi}_n$.

The linear combination of the Eq. (76) with its degeneration can be written as

$$\bar{\Psi}(x, y, t) = \sum_{n,j',j} \bar{c}_{n,j',j} \hat{E}^{j'} \hat{\pi}_x^j \bar{\phi}_n(x, y, t). \quad (77)$$

Finally, the general solution for this system can be written as a linear combiantion of the functions Eq. (60) and Eq. (77)

$$\Phi = \sum_{n,j',j} c_{n,j',j} \hat{E}^{j'} \hat{\pi}_y^j \bar{\phi}_n + \sum_{n,j',j} \bar{c}_{n,j',j} \hat{E}^{j'} \hat{\pi}_x^j \bar{\phi}_n. \quad (78)$$

5. Conclusions

Two systems were analyzed: one with a constant electric field and another with a constant electromagnetic field. For both cases, it was shown that it is possible to find time-dependent conserved operators, which were used to find solutions to the time-dependent Schrödinger equation. The degeneracy of the systems was analyzed, where the conserved operators act as generators of solutions of the partial differential equation. These conserved operators were used to define the symmetries of the complete Schrödinger equation, and an invariant proposition of the wave function under a unitary transformation leads to the quantization of the resistance (or resistivity) in integer multiples of the von Klitzing's constant. This is similar to the experimental results found in Refs. [6-8]. Finally, it should be noted that Tao and Wu have already pointed out that the degeneracy of the system is important in order to have a quantized resistance [14], which matches with the results presented in this work.

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