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A4×Z2×Z4 flavor symmetry model for neutrino oscillation phenomenology

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We propose a Standard Model (SM) extension which aims to explain the most recent experimental data on neutrino oscillation. Beside A_4, two Abelian symmetries Z_2 and Z_4 are supplemented to prevent some Yukawa terms to get the desired mass matrices and then give predictions for the neutrino oscillation parameters in agreement with the most recent experimental data on neutrino oscillation in 3σ range. The model provide a predictive relation between the solar and reactor neutrino mixing angles and gives possible prediction on the Dirac CP phase and two Majorana phases as well as the effective neutrino mass being in agreement with the most recent constraints.

Keywords: Neutrino mass and mixing; extensions of electroweak Higgs sector; non-standard-model neutrinos; right-handed neutrinos; discrete symmetry.

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1. Introduction

The current neutrino data (see, for example, Ref. [1], as shown in Table I) has promoted various theoretical efforts to understand neutrino flavor mixing and small non-degenerate mass.

In the three-neutrino scheme, the neutrino oscillation probability is described by six parameters, including two neutrino mass squared splittings[i] Δ^2_{21} and $|\Delta^2_{31}|$; three neutrino mixing angles θ_{12}, θ_{13} and θ_{23}; and one Dirac CP-violation phase δ. Presently, although the absolute value of two mass squared splittings, the solar mixing angle θ_{12} and the reactor mixing angle θ_{13} have now been determined with high accuracy [1], three other quantities including the sign of Δ^2_{31}, the octant of atmospheric neutrino mixing angle θ_{23} and the Dirac CP violation phase δ are still unknown. On the other hand, the sign of Δ^2_{31} provides two neutrino mass hierarchies, namely, normal hierarchy (NH) with $m_1 < m_2 < m_3$ and inverted hierarchy (IH) with $m_3 < m_1 < m_2$. Although some of the analysis seem favor the normal hierarchy [2–4], the most recent result from KamLAND-Zen provides a clue of the Majorana nature of neutrinos in the inverted mass ordering [5].

Discrete symmetries are useful tools for explaining the observed neutrino data in which A_4 symmetry has been used in different works (see for instance Refs. [6, 7] and the references therein). However, the previous works include non minimal scalar sectors with many $SU(2)_L$ doublets or/and alot of singlets and and most of them have not mentioned the mass hierarchy problem which are significant differences with our current work. The minimal scalar sector is an important feature of the model to distinguish it from previous works; thus, it would be necessary to construct an A_4 flavor model with less scalar content compared to the mentioned works.

A_4 is a group of even permutations of four objects which is known as the tetrahedron group. It has four irreducible representations including three one-dimensional representations $\underline{1}$, $\underline{1}'$ and $\underline{1}''$, and one three-dimensional representation. We will work in the T-diagonal basis, in which the tensor product of the triplets is given by [8]:

TABLE I. Neutrino oscillation parameters [1]. The second column is for NH and the third column is for IH.

Parameters	Best-fit point(3σ)	Best-fit point(3σ)
$\frac{\Delta^2_{21}\,[\mathrm{meV}^2]}{10}$	$7.50\,(6.94, 8.14)$	$7.50\,(6.94, 8.14)$
$\frac{\lvert\Delta^2_{31}\rvert\,[\mathrm{meV}^2]}{10^3}$	$2.55\,(2.47, 2.63)$	$2.45\,(2.37, 2.53)$
s^2_{12}	$0.318\,(0.271, 0.369)$	$0.318\,(0.271, 0.369)$
s^2_{23}	$0.574\,(0.434, 0.610)$	$0.578\,(0.433, 0.608)$
$\frac{s^2_{13}}{10^{-2}}$	$2.200\,(2.00, 2.405)$	$2.225\,(2.018, 2.424)$
δ/π	$1.08\,(0.71, 1.99)$	$1.58\,(1.11, 1.96)$

$$\underline{3}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{pmatrix}\times\underline{3}\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \mathbf{y}_3\end{pmatrix}=(\mathbf{x}_1\mathbf{y}_1+\mathbf{x}_2\mathbf{y}_3+\mathbf{x}_3\mathbf{y}_2)_{\underline{1}}$$
$$+(\mathbf{x}_3\mathbf{y}_3+\mathbf{x}_1\mathbf{y}_2+\mathbf{x}_2\mathbf{y}_1)_{\underline{1}'}+(\mathbf{x}_2\mathbf{y}_2+\mathbf{x}_1\mathbf{y}_3+\mathbf{x}_3\mathbf{y}_1)_{\underline{1}''}$$
$$+\frac{1}{3}\begin{pmatrix}2\mathbf{x}_1\mathbf{y}_1-\mathbf{x}_2\mathbf{y}_3-\mathbf{x}_3\mathbf{y}_2\\ 2\mathbf{x}_3\mathbf{y}_3-\mathbf{x}_1\mathbf{y}_2-\mathbf{x}_2\mathbf{y}_1\\ 2\mathbf{x}_2\mathbf{y}_2-\mathbf{x}_1\mathbf{y}_3-\mathbf{x}_3\mathbf{y}_1\end{pmatrix}_{\underline{3}_s}+\frac{1}{2}\begin{pmatrix}\mathbf{x}_2\mathbf{y}_3-\mathbf{x}_3\mathbf{y}_2\\ \mathbf{x}_1\mathbf{y}_2-\mathbf{x}_2\mathbf{y}_1\\ \mathbf{x}_3\mathbf{y}_1-\mathbf{x}_1\mathbf{y}_3\end{pmatrix}_{\underline{3}_a}. \quad (1)$$

The remainder of our paper is organized as follows. Section 2 is the description of the model. Neutrino masses and mixings is presented in Sec. 3. Section 4 is devoted for the numerical analysis as well as the effective neutrino mass parameters. Some conclusions are given in Sec. 5. Appendix A provides the Yukawa terms prevented by the model symmetries and Appendix B presents the scalar potential of the model.

2. The model

In the considered model, the SM is supplemented by three discrete symmetries A_4, Z_2 and Z_4, *i.e.*, the full symmetry

TABLE II. The assignment of leptons and scalars under A_4, Z_2 and Z_4.

Field	ψ_L	l_{1R}	l_{2R}	l_{3R}	ν_R	H_1	H_2	ϕ	φ	ρ	η
A_4	$\underline{3}$	$\underline{1}$	$\underline{1}''$	$\underline{1}'$	$\underline{3}$	$\underline{1}$	$\underline{1}$	$\underline{3}$	$\underline{3}$	$\underline{1}$	$\underline{1}'$
Z_2	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
Z_4	i	i	i	i	1	1	1	1	i	i	i

of the model is $G_{\rm SM} \times A_4 \times Z_2 \times Z_4 \equiv$ G. Besides, three right-handed neutrinos, one $SU(2)_L$ doublet and four singlet scalars are added to the SM^{ii}. Under A_4 symmetry, three left-handed leptons (ψ_L) and three right-handed neutrinos (ν_R) are assigned in $\underline{3}$ while right-handed charged leptons $l_{1,2,3R}$ are assigned in $\underline{1}, \underline{1}''$ and $\underline{1}'$, respectively. The particle content of the model is given in Table II.

The invariant Yukawa terms, up to five-dimension, is as follows:

$$\begin{aligned}-\mathcal{L}_{\rm clep} &= \frac{h_1}{\Lambda}(\overline{\psi}_L\phi)_1(H_2 l_{1R})_1 + \frac{h_2}{\Lambda}(\overline{\psi}_L\phi)_{1'}(H_1 l_{2R})_{1''} \\ &+ \frac{h_3}{\Lambda}(\overline{\psi}_L\phi)_{1''}(H_1 l_{3R})_{1'} + \frac{3x_1}{\Lambda}(\overline{\psi}_L\nu_R)_{3_s}(\widetilde{H}_2\varphi)_3 \\ &+ \frac{2x_2}{\Lambda}(\overline{\psi}_L\nu_R)_{3_a}(\widetilde{H}_2\varphi)_3 + \frac{y_1}{\Lambda}(\overline{\psi}_L\nu_R)_1(\widetilde{H}_2\rho)_1 \\ &+ \frac{y_2}{\Lambda}(\overline{\psi}_L\nu_R)_{1''}(\widetilde{H}_2\eta)_{1'} + \frac{M}{2}(\overline{\nu}^C_R\nu_R)_{\underline{1}} + h.c, \quad (2)\end{aligned}$$

where the terms $h_{1,2,3}$, $x_{1,2}$ and $y_{1,2}$ are Yukawa-like couplings, M is the Majorana bare mass term of the right-handed neutrino and Λ is the cut-off scale. The factors "3" and "2" in the numerators of the terms in the second row of Eq. (2) is just for convenience. It is noted that φ does not contribute to the charged lepton mass matrix while ϕ does not contribute to the neutrino mass matrices. Namely, the charged-lepton masses can be generated from the couplings of $\bar{\psi}_L l_{1,2,3R}$ to scalars in which, under the considered symmetries, $\bar{\psi}_L l_{1R} \sim (2, -1/2, \underline{3}, +, 1)$ and $\bar{\psi}_L l_{2,3R} \sim (2, -1/2, \underline{3}, -, 1)$, *i.e.*, the scalar doublets which respectively transform as $(2, 1/2, \underline{3}, +, 1) \equiv H_2\phi$ and $(2, 1/2, \underline{3}, -, 1) \equiv H_1\phi$ are needed to construct invariant terms which generate the charged-lepton mass matrix. On the other hand, the Majorana neutrino masse s can be produced by the couplings of $\bar{\nu}^c_R\nu_R$ to scalars where $\bar{\nu}^c_R\nu_R \sim (\mathbf{1}, 0, \underline{1} + \underline{1}' + \underline{1}'' + \underline{3}_s + \underline{3}_a, +, 1)$, *i.e.*, $\underline{1}$ as one of results of the tensor product of two A_4 triplets corresponding to $(\bar{\nu}^c_R\nu_R)_{\underline{1}} \sim (\mathbf{1}, 0, \underline{1}, +, 1)$ is invariant under all the considered symmetries which contributes to the entries "11", "23" and "32" of the Majorana neutrino mass matrix M_R. Furthermore, the Dirac neutrino masses arise from the coupling of $\bar{\psi}_L\nu_R$ to scalars where $\bar{\psi}_L\nu_R \sim (2, 1/2, \underline{1} + \underline{1}' + \underline{1}'' + \underline{3}_s + \underline{3}_a, +, -i)$, *i.e.*, the scalar doublets which transform as $(2, -1/2, \underline{1} + \underline{1}' + \underline{1}'' + \underline{3}_s + \underline{3}_a, +, -i)$. For the given scalar fields H_1, H_2 and ϕ, all Yukawa terms up to five-dimensions are prevented by one (or some) of the considered symmetries (see Table III). We thus additionally introduce one $SU(2)_L$ singet put in $\underline{3}$ of A_4 (denoted as φ) which can combine with H_2 to form invariant terms including $(\overline{\psi}_L\nu_R)_{3_s}(\widetilde{H}_2\varphi)_3$ and $(\overline{\psi}_L\nu_R)_{3_a}(\widetilde{H}_2\varphi)_3$. With the VEV $\langle\varphi\rangle = (v_\varphi,\ v_\varphi,\ v_\varphi)$, these terms contribute to all the entries of the Dirac neutrino mass matrix M_D. However, due to the properties of the $\underline{3} \times \underline{3}$ tensor product of A_4, the efective neutrino matrix obtained via Type-I seesaw mechanism owns two degenerate masses which is ruled out by experimental data [1]. Hence, two additional scalars (ρ and η) are introduced to eliminate the neutrino mass degeneracy.

Besides, there exist five-dimensional terms which are invariant under all the considered symmetries, including $(1/2\Lambda)\big(\phi^2\big)_{\underline{1}/\underline{1}'/\underline{1}''/\underline{3}_{s,a}}\big(\overline{\nu}^C_R\nu_R\big)_{\underline{1}/\underline{1}''/\underline{1}'/\underline{3}_{s,a}}$, $(1/2\Lambda)\big(\varphi^*\rho + \varphi\rho^*\big)_{\underline{3}}\big(\overline{\nu}^C_R\nu_R\big)_{\underline{3}_{s,a}}$, $(1/2\Lambda)\big(\varphi^*\eta + \varphi\eta^*\big)_{\underline{3}}\big(\overline{\nu}^C_R\nu_R\big)_{\underline{3}_{s,a}}$, $(1/2\Lambda)\big(\rho^*\eta\big)_{\underline{1}'}\big(\overline{\nu}^C_R\nu_R\big)_{\underline{1}''}$ and $(1/2\Lambda)\big(\rho\eta^*\big)_{\underline{1}''}\big(\overline{\nu}^C_R\nu_R\big)_{\underline{1}'}$ which contribute to the Majorana neutrino mass matrix. However, the couplings corresponding to $\big(\overline{\nu}^C_R\nu_R\big)_{\underline{3}_a}$ are vanished due to the antisymmetry of ν_{iR} and ν_{jR} under 3_a as a consequence of the tensor product of $\underline{3} \times \underline{3}$ of A_4 symmetry. Moreover, the other terms contributes to the Majorana neutrino mass matrix which corresponding to $(v_\varphi v_\rho/\Lambda)$, $v_\varphi v_\eta/\Lambda$ and $(v_\rho v_\eta/\Lambda)$ which are very small compared to M since $(v_\varphi/\Lambda) \sim (v_\rho/\Lambda) \sim (v_\eta/\Lambda) \sim 10^{-3} \div 10^{-2}$; thus, their contributions were ignored. The additional symmetries A_4, Z_2 and Z_4 play crucial roles in forbidding some Yukawa terms to get the desired form of the mass matrices which are listed in Appendix A, respectively.

As presented in Appendix B, the vacuum expectation value (VEV) of the scalars, which comes from the minimum condition of scalar potential, are given as follows:

$$\begin{aligned}&\langle H_1\rangle = \begin{pmatrix}0 & v_1\end{pmatrix}^T, \qquad \langle H_2\rangle = \begin{pmatrix}0 & v_2\end{pmatrix}^T, \\ &\langle\phi\rangle = (v_\phi,\ 0,\ 0), \qquad \langle\varphi\rangle = (v_\varphi,\ v_\varphi,\ v_\varphi), \\ &\langle\rho\rangle = v_\rho, \qquad \langle\eta\rangle = v_\eta. \qquad (3)\end{aligned}$$

In 2HDM, the electroweak symmetry breaking is performed by both two $SU(2)_L$ scalar doublets H_1 and H_2 via their non-zero VEVs v_1 and v_2. On the other hand, the scalar ϕ with the VEV $\langle\phi\rangle = (v_\phi, 0, 0)$ breaks A_4 down to Z_2 symmetry while φ with the VEV owns equal value for all three components, $\langle\varphi\rangle = (v_\varphi, v_\varphi, v_\varphi)$, breaks A_4 down to Z_3 symmetry.

The fact that the electroweak scale is low $\sqrt{v_1^2 + v_2^2} = v = 246\,$GeV, and the cut-off scale Λ is unknown and it is assumed to be a very high scale $\Lambda \in (10^{13},\ 10^{15})\,$GeV [12]. However, in a recent work [11] it is demonstrated that $\Lambda \simeq 3.8 \times 10^{13}\,$GeV in electroweak theory. Therefore, in this study, we use $\Lambda \simeq 10^{13}$ GeV for its scale:

$$v = 246\ \text{GeV}, \qquad \Lambda \simeq 10^{13}\ \text{GeV}. \qquad (4)$$

On the other hand, to solve the hierarchy problem of charged-lepton masses and the implementation of the type I seesaw mechanism that generates the smallness of the neutrino masses, the VEVs of scalar fields are required as follows:

$$v_\phi = v_\varphi = 5\times 10^{10}\ \text{GeV},\quad v_\eta = 2v_\rho = 2\times 10^{10}\ \text{GeV},$$
$$v_1 = 2.458\times 10^2\ \text{GeV},\quad v_2 = 10\ \text{GeV}. \tag{5}$$

3. Lepton masses and mixings

Using the multiplication rules of the A_4 [8], from the Yukawa terms in the first line in Eq. (2), after the scalars $H_{1,2}$ and ϕ get their VEVs as in Eq. (3), the charged-lepton mass matrix get the diagonal form,

$$M_l = \frac{v_\phi}{\Lambda}\text{diag}\,(h_1v_2, h_2v_1, h_3v_1) \equiv \left(m_e,\, m_\mu,\, m_\tau\right). \tag{6}$$

Therefore, the diagonalization matrices of charged-lepton mass matrix M_l are $U_R = U_L = \mathbf{I}_{3\times3}$ and the lepton mixing matrix depends on only that of the neutrinos, $U_{\text{lep}} = U_\nu$. From Eq. (6) we get the relation:

$$h_1 = \frac{\Lambda m_e}{v_2 v_\phi},\quad h_2 = \frac{\Lambda m_\mu}{v_\phi\sqrt{v^2 - v_2^2}},$$
$$h_3 = \frac{\Lambda m_\tau}{v_\phi\sqrt{v^2 - v_2^2}}. \tag{7}$$

As will see in Sec. 4, there exist possible regions of v_2 and v_ϕ such that the quantities h_1, h_2 and h_3 are just different from each other by one order of magnitude. Thus, the charged lepton mass hierarchy problem is naturally achievable in the model.

Similarity from the Yukawa terms in the second and third lines in Eq. (2), after the scalars H_2, φ, ρ and η get their VEVs as in Eq. (3), we obtain the Dirac and Majorana neutrino mass matrices as follows:

$$M_D = \begin{pmatrix} 2a+c & -a+b & -a-b+d \\ -a-b & 2a+d & -a+b+c \\ -a+b+d & -a-b+c & 2a \end{pmatrix},$$
$$M_R = M\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{8}$$

where the parameter M is related to the Majorana mass term of the right-handed neutrino in Eq. (2) which can be in the range of very high scale [13], and

$$a = \frac{x_1 v_2 v_\varphi}{\Lambda},\quad b = \frac{x_2 v_2 v_\varphi}{\Lambda},$$
$$c = \frac{y_1 v_2 v_\rho}{\Lambda},\quad d = \frac{y_2 v_2 v_\eta}{\Lambda}. \tag{9}$$

The effective neutrino mass matrix is obtained via the type-I seesaw mechanism, $m_\nu = M_D M_R^{-1} M_D^T$, as follows:

$$m_\nu = \begin{pmatrix} 2\alpha+\gamma+2\sigma & -\alpha+\kappa-\beta-\sigma & -\alpha+\beta-\sigma+\tau \\ -\alpha+\kappa-\beta-\sigma & -\alpha+\beta+2\sigma+\tau & 2\alpha+\gamma-\sigma \\ -\alpha+\beta-\sigma+\tau & 2\alpha+\gamma-\sigma & -\alpha-\beta+\kappa+2\sigma \end{pmatrix}, \tag{10}$$

where

$$\alpha = \frac{3a^2-b^2}{M},\quad \beta = \frac{6ab}{M},\quad \gamma = \frac{c^2}{M},\quad \kappa = \frac{d^2}{M},\quad \sigma = \frac{2ac-ad+bd}{M},\quad \tau = \frac{2cd}{M}. \tag{11}$$

Since the Yukawa-like couplings $x_{1,2}$ and $y_{1,2}$ are complex parameters, Eq. (9) implies that the parameters $\alpha, \beta, \gamma, \kappa, \sigma$ and τ are complex. As a consequence, the neutrino matrix m_ν in Eq. (10) is complex. Therefore, in order to get the real and positive neutrino masses, we define a Hermitian matrix $M_\nu^2 = m_\nu^\dagger m_\nu$ whose entries take the following form:

$$M_\nu^2 = \begin{pmatrix} B+2H & -A-F-H+iK & -A+F-H-iK \\ -A-F-H-iK & B+F-H & -A+2H+iK \\ -A+F-H+iK & -A+2H-iK & B-F-H \end{pmatrix}, \tag{12}$$

where A, B, F, H and K are defined in Appendix A.

Diagonalizing the matrix M_ν^2 in Eq. (12) we obtain:

$$\begin{cases} m_1^2 = A+B-\sqrt{3}\mathcal{X}_0,\ m_2^2 = B-2A, \\ m_3^2 = A+B+\sqrt{3}\mathcal{X}_0(\text{NH}), \\ m_1^2 = A+B+\sqrt{3}\mathcal{X}_0,\ m_2^2 = B-2A, \\ m_3^2 = A+B-\sqrt{3}\mathcal{X}_0(\text{IH}). \end{cases} \tag{13}$$

$$s_{13}^2 = \begin{cases} \frac{1}{3}+\frac{H}{\sqrt{3}\mathcal{X}_0}\ (\text{NH}), \\ \frac{1}{3}-\frac{H}{\sqrt{3}\mathcal{X}_0}\ (\text{IH}), \end{cases}\qquad t_{12}^2 = \begin{cases} \frac{1}{1-\frac{\sqrt{3}H}{\mathcal{X}_0}}\ (\text{NH}), \\ \frac{1}{1+\frac{\sqrt{3}H}{\mathcal{X}_0}}\ (\text{IH}), \end{cases} \tag{14}$$

$$t_{23}^2=\begin{cases}\dfrac{7F^2+9H^2+4K^2+4\sqrt{3}F\mathcal{X}_0}{(F-3H)^2+4K^2}\text{(NH)},\\ \dfrac{7F^2+9H^2+4K^2-4\sqrt{3}F\mathcal{X}_0}{(F-3H)^2+4K^2}\text{ (IH)},\end{cases}\tag{15}$$

$$s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}\sin\delta=\begin{cases}-\dfrac{K}{6\sqrt{3}c_{12}c_{13}^2c_{23}s_{12}s_{13}s_{23}\mathcal{X}_0}\text{(NH)},\\ \dfrac{K}{6\sqrt{3}c_{12}c_{13}^2c_{23}s_{12}s_{13}s_{23}\mathcal{X}_0}\text{(IH)}.\end{cases}\tag{16}$$

$$\alpha_{1\mathrm{M}}=\begin{cases}\arctan\left(\dfrac{\Upsilon_1\sin\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)-3K\Omega_1\cos\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)}{3K\Omega_1\sin\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)+\Upsilon_1\cos\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)}\right)\text{(NH)},\\ \arctan\left(\dfrac{3K\Omega_3\cos\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)+\Upsilon_3\sin\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)}{3K\Omega_3\sin\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)-\Upsilon_3\cos\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)}\right)\text{ (IH)},\end{cases}\tag{17}$$

$$\alpha_{3\mathrm{M}}=\begin{cases}\arctan\left(\dfrac{3K\Omega_3\cos\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)+\Upsilon_3\sin\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)}{3K\Omega_3\sin\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)-\Upsilon_3\cos\left(\frac{1}{2}\arg\left(-\frac{1}{\Omega_3}\right)\right)}\right)\text{ (NH)},\\ \arctan\left(\dfrac{\Upsilon_1\sin\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)-3K\Omega_1\cos\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)}{3K\Omega_1\sin\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)+\Upsilon_1\cos\left(\dfrac{\arctan\left(\frac{\mathrm{Re}\Omega_1}{\mathrm{Im}\Omega_1}\right)}{2}\right)}\right)\text{(IH)},\end{cases}\tag{18}$$

where

$$\mathcal{X}_0=\sqrt{F^2+3H^2+K^2},\quad \mathcal{X}_\pm=\sqrt{\frac{\mathcal{X}_0}{2\mathcal{X}_0\pm\sqrt{3}(F+H)}},\tag{19}$$

$$\Omega_{1,3}=\sqrt{3}(F+H)\pm 2\mathcal{X}_0,\quad \Upsilon_{1,3}=\sqrt{3}\left(3H^2+5F^2+2K^2\right)\pm 3\mathcal{X}_0(3F-H),\tag{20}$$

with A, B, F, H, K are given in Appendix C.

Equations (15) and (20) provide the following relations:

$$A=\frac{1}{6}(\Delta_{31}^2-2\Delta_{21}^2)\text{ (NH and IH)},\quad H=\frac{\Delta_{31}^2}{6}\left(3s_{13}^2-1\right)\text{ (NH and IH)},\tag{21}$$

$$F=\frac{\Delta_{31}^2c_{13}^2\left(t_{23}^2-1\right)}{2\left(t_{23}^2+1\right)}\text{(NH and IH)},\quad K=\frac{\Delta_{31}^2\mathcal{Y}_0}{2\left(t_{23}^2+1\right)}\text{(NH and IH)},\tag{22}$$

$$t_{12}^2=\frac{1}{2-3s_{13}^2}\text{ (NH and IH)},\quad s_\delta=-\frac{\mathcal{Y}_0}{6c_{12}c_{13}^2c_{23}s_{12}s_{13}s_{23}\left(t_{23}^2+1\right)}\times\text{(NH and IH)},\tag{23}$$

$$t_{1\mathrm{M}}=\begin{cases}\dfrac{3\mathcal{Y}_0c_\Theta+[(5-6s_{13}^2)t_{23}^2-1]s_\Theta}{3\mathcal{Y}_0s_\Theta-[(5-6s_{13}^2)t_{23}^2-1]c_\Theta}\text{(NH)},\\ \dfrac{\mathcal{Y}_0}{(2s_{13}^2-1)t_{23}^2+1}\text{(IH)},\end{cases}\quad t_{3\mathrm{M}}=\begin{cases}\dfrac{\mathcal{Y}_0}{(2s_{13}^2-1)t_{23}^2+1}\text{(NH)},\\ \dfrac{3\mathcal{Y}_0c_\Theta+[(5-6s_{13}^2)t_{23}^2-1]s_\Theta}{3\mathcal{Y}_0s_\Theta-[(5-6s_{13}^2)t_{23}^2-1]c_\Theta}\text{(IH)},\end{cases}\tag{24}$$

with

$$\mathcal{Y}_0=\sqrt{4s_{13}^2\left(t_{23}^4+1\right)-4s_{13}^4\left(t_{23}^4+t_{23}^2+1\right)-\left(t_{23}^2-1\right)^2}, \quad (25)$$

$$\Theta=\frac{1}{2}\arctan\left(\frac{2(t_{23}^2+1)-3c_{13}^2}{3c_{13}^2}\right). \quad (26)$$

It is noted that the determination of the octant of atmospheric mixing angle (θ_{23}) is still an open problem experimentally, *i.e.*, it either in the first octan ($\theta_{23}<45^\circ$, *i.e.*, $s_{23}^2<0.5$) or the second octant ($\theta_{23}>45^\circ$, *i.e.*, $s_{23}^2>0.5$). The data in Ref. [1] as shown in Table I implies $s_{23}^2\in(0.434,\,0.610)$ for NH and $s_{23}^2\in(0.433,\,0.608)$ for IH. On the other hand, the neutrino mass hierarchy depends the sign of Δm_{31}^2, *i.e.*, $\Delta m_{31}^2>0$ (or $m_1<m_2<m_3$) for normal hierarchy and $\Delta m_{31}^2<0$ (or $m_3<m_1<m_2$) for inverted hierarchy. Furthermore, expressions (23) and (25) yield the relations between lepton mixing angles and Dirac CP-violating phase, expressions (21)-(22) and (25) tell us that four model parameters A, F, H and K are expressed in terms of four observable quantities s_{23}^2, s_{13}^2, Δm_{21}^2 and Δm_{31}^2 that have now been quite well measured [1]. Namely, A depends on Δm_{21}^2 and Δm_{31}^2; H depends on Δm_{31}^2 and s_{13}^2; F and K depend on $\Delta m_{31}^2, s_{13}^2$ and s_{23}^2; $\sin\delta$ depends on s_{13}^2 and s_{23}^2, *i.e.*, A and H depend on the neutrino mass hierarchy, s_δ depends on the octant of θ_{23}, and F and K depend on both the neutrino mass hierarchy and the octant of θ_{23} which will be presented in Sec. 4.

Now, we calculate the effective neutrino mass parameter. From Eq. (13) we can rewrite three neutrino masses in terms of two neutrino mass-squared differences and B,

$$m_1=\sqrt{B-\frac{\Delta_{21}^2+\Delta_{31}^2}{3}},$$
$$m_2=\sqrt{B+\frac{2\Delta_{21}^2-\Delta_{31}^2}{3}},$$
$$m_3=\sqrt{B-\frac{\Delta_{21}^2-2\Delta_{31}^2}{3}}\ \text{(NH and IH)}, \quad (27)$$
$$\sum=\sqrt{B-\frac{\Delta_{21}^2+\Delta_{31}^2}{3}}+\sqrt{B+\frac{2\Delta_{21}^2-\Delta_{31}^2}{3}}+\sqrt{B-\frac{\Delta_{21}^2-2\Delta_{31}^2}{3}}\ \text{(NH and IH)}. \quad (28)$$

It should be noted that the analytic expression in Eq. (27) is satisfied for both NH and IH, however, the range of values Δ_{31}^2 are different from each other with respect to NH and IH [1].

Next, the effective neutrino mass governing the neutrinoless beta decay,

$$m_\beta=\left(\sum_{i=1}^3|U_{ei}|^2\,m_i^2\right)^{1/2},$$

allows us to express B in terms of three parameters Δ_{31}^2, s_{13}^2 and m_β,

$$B=\Delta_{31}^2\left(\frac{1}{3}-s_{13}^2\right)+m_\beta^2\ \text{(NH and IH)}. \quad (29)$$

Furthermore, one can express the effective neutrino mass governing the neutrinoless double beta decay, $\langle m_{ee}\rangle=|\sum_{i=1}^3 U_{ei}^2 m_i|$, in terms of five parameters s_{13}^2, s_{23}^2, Δ_{21}^2, Δ_{31}^2 and m_β as follows

$$\langle m_{ee}\rangle=\frac{\mathcal{Z}_0\sqrt{\Gamma}}{2\sqrt{3}\left\{\left[(2s_{13}^2-1)t_{23}^2+s_{13}^2\right]^2+\mathcal{Y}_0^2\right\}}\times\text{(NH and IH)}, \quad (30)$$

where

$$\mathcal{Z}_0=\sqrt{3B-\Delta_{21}^2-\Delta_{31}^2},$$
$$\Gamma=\left[2\left(t_{23}^2-s_{13}^2(2t_{23}^2+1)\right)^2+\mathcal{Y}_0^2\right]^2-\mathcal{Y}_0^4, \quad (31)$$

with $\mathcal{Y}_0$ is defined in Eq. (25).

Expressions (25), (27)-(28) and (29) tell us that B depends on three parameters $\theta_{13}, \Delta_{31}^2$ and m_β; three active neutrino masses $m_{1,2,3}$ and their sum $\sum$ depend on four parameters $\theta_{13}, \Delta_{31}^2, \Delta_{21}^2$ and m_β where $\theta_{13}, \Delta_{21}^2, \Delta_{31}^2$ are observable parameters that have now been quite well measured [1] and m_β has been constrained by experiments [16,25]. On the other hand, expressions (25), (30) and (31) imply that $\langle m_{ee}\rangle$ depends on five parameters s_{13}^2, s_{23}^2, Δ_{21}^2, Δ_{31}^2 and m_β, *i.e.*, $\langle m_{ee}\rangle$ depends on the neutrino mass hierarchy and the octant of θ_{23}. Therefore, we will considered s_{13}^2, s_{23}^2, Δ_{21}^2, Δ_{31}^2 and m_β as input parameters to determine the possible range of $B, m_{1,2,3}, \sum$ and $\langle m_{ee}\rangle$ as well as Dirac and Majorana phases which will be presented in Sec. 4.

For the quark sector, under $\mathbf{G}$ symmetry quark fields transforms as $Q_{1L}\sim(2,1/6,\underline{1},+,1)$, $u_{1R}\sim(1,2/3,\underline{1},+,-i)$, $d_{1R}\sim(1,-1/3,\underline{1},+,-i)$; $Q_{2L}\sim\left(2,1/6,\underline{1}',+,1\right)$, $u_{2R}\sim\left(1,2/3,\underline{1}',-,1\right)$, $d_{2R}\sim\left(1,-1/3,\underline{1}',-,1\right)$; $Q_{3L}\sim\left(2,1/6,\underline{1}'',-,1\right)$, $u_{3R}\sim\left(1,2/3,\underline{1}'',+,1\right)$, $d_{3R}\sim\left(1,-1/3,\underline{1}'',+,1\right)$. The up quarks masses arise from the couplings of $\bar{Q}_{iL}u_{jR}\,(i,j=1,2,3)$ to scalars and the down quarks masses arise from the couplings of $\bar{Q}_{iL}d_{jR}\,(i,j=1,2,3)$ to scalars. With the scalar fields of the model in Table II, the SM quark masses are generated by the following Yukawa terms:

$$-\mathcal{L}_q=\frac{h_1^u}{\Lambda}(\bar{Q}_{1L}u_{1R})_{\underline{1}}(\widetilde{H}_2\rho)_{\underline{1}}+h_2^u(\bar{Q}_{2L}u_{2R})_{\underline{1}}\widetilde{H}_2+h_3^u(\bar{Q}_{3L}u_{3R})_{\underline{1}}\widetilde{H}_1+\frac{h_1^d}{\Lambda}(\bar{Q}_{1L}d_{1R})_{\underline{1}}(H_2\rho)_{\underline{1}}+h_2^d(\bar{Q}_{2L}d_{2R})_{\underline{1}}H_2+h_3^d(\bar{Q}_{3L}d_{3R})_{\underline{1}}H_1+\text{h.c.} \quad (32)$$

After symmetry breaking, from Eq. (32), we obtain the quark masses as follows

$$m_u = h_1^u v_2 \left(\frac{v_\rho}{\Lambda}\right), \quad , m_c =_2^u v_2, \quad m_t = h_3^u v_1,$$
$$m_d = h_1^d v_2 \left(\frac{v_\rho}{\Lambda}\right), \quad m_s = h_2^d v_2, \quad m_b = h_3^d v_1. \quad (33)$$

With the aid of Eqs. (4) and (5), the obtained quark masses in Eq. (33) can accommodate the quark mass heirarchy [16], $m_u = 2.16$ MeV, $m_c = 1.27$ GeV, $m_t = 172.76$ GeV, $m_d = 4.67$ MeV, $m_s = 93.0$ MeV, $m_b = 4.18$ GeV provided that $h_1^u \sim h_2^u \sim h_3^u \sim h_1^d \sim 1$, $h_2^d \sim 10^{-1}$, $h_3^d \sim 10^{-2}$. The unitary matrices which couple the left-handed up-and down-quarks are unit matrices, and as a consequence, the quark mixing matrix is unity. A detailed study of quark mixing is out of scope of this work.

4. Numerical analysis and discussion

• *For the charged lepton sector*, using the observed values of the charged lepton masses [16], $m_e = 0.51099\,\mathrm{MeV}, m_\mu = 105.65837\,\mathrm{MeV}, m_\tau = 1776.86\,\mathrm{MeV}$, and v and Λ in Eq. (4), three Yukawa-like couplings h_1, h_2 and h_3 in Eq. (7) depend on two parameters v_2 and v_ϕ. We find the possible ranges for v_2 and v_ϕ with

$$v_2 \in (5\times 10^8, 10^9)\,\mathrm{eV},$$
$$v_\phi \in (8\times 10^{10}, 10^{11})\,\mathrm{GeV}, \quad (34)$$

so that

$$h_1 \in (0.051, 0.128), \quad h_2 \in (0.043, 0.054),$$
$$h_3 \in (0.722, 0.903), \quad (35)$$

which are just different from each other by one order of magnitude, *i.e.*, the charged-lepton mass hierarchy is naturally satisfied.

• *Now, we analyze the neutrino sector.* Expression (23) shows the relation between θ_{12} and θ_{13}. Since the experimental result for θ_{13} is more accurate than that of θ_{12} [1], we will determine the possible region of t_{12}^2 based on the experimental region of s_{13}^2. At $3\,\sigma$ range [1] $s_{13}^2 \in (2.000, 2.405)10^{-2}$ for NH and $s_{13}^2 \in (2.018,\ 2.424)10^{-2}$ for IH, from Eq.(23) we find the possible ranges of t_{12}^2 as follows:

$$t_{12}^2 \in \begin{cases} (0.515, 0.519) & \text{(NH)} \\ (0.516, 0.519) & \text{(IH)} \end{cases},$$
$$\text{i.e., } \theta_{12} \in \begin{cases} (35.68^\circ, 35.76^\circ) & \text{(NH)} \\ (35.68^\circ, 35.77^\circ) & \text{(IH)} \end{cases}. \quad (36)$$

Expression (21) implies that A depends on Δ_{21}^2 and Δ_{31}^2, and H depends on Δ_{31}^2 and s_{13}^2. At 3σ range [1], $\Delta_{21}^2 \in (69.40, 81.40)\,\mathrm{meV}^2$, $\Delta_{31}^2 \in (2.47, 2.63)10^3\,\mathrm{meV}^2$ and $s_{13}^2 \in (2.000, 2.405)10^{-2}$ for NH while $\Delta_{21}^2 \in (69.40, 81.40)\,\mathrm{meV}^2$, $\Delta_{31}^2 \in (-2.53, -2.37)10^3\,\mathrm{meV}^2$ and $s_{13}^2 \in (2.018,\ 2.424)10^{-2}$ for IH, we find the following possible ranges of A and H:

$$A \in \begin{cases} (384.500, 415.200)\,\mathrm{meV}^2 & \text{(NH)} \\ (-448.800, -418.100)\,\mathrm{meV}^2 & \text{(IH)} \end{cases},$$
$$H \in \begin{cases} (-412.000, -382.000)\,\mathrm{meV}^2 & \text{(NH)} \\ (366.500,\ 396.400)\,\mathrm{meV}^2 & \text{(IH)} \end{cases}. \quad (37)$$

At present, the constrain on the effective electron anti-neutrino mass m_β has been implemented by experiments. Namely, the constraint on m_β is given in Ref. [16] with $8.5\,\mathrm{meV} < m_\beta < 1100\,\mathrm{meV}$ for NH and $48\,\mathrm{meV} < m_\beta < 1100\,\mathrm{meV}$ for IH [16]. Another improved bound on m_β is [25] $m_\beta < 800$ meV. Therefore, we will consider m_β as an input parameter with

$$m_\beta \in \begin{cases} (9.508,\ 100.00)\,\mathrm{meV} & \text{(NH)}, \\ (50.061, 200.00)\,\mathrm{meV} & \text{(IH)}. \end{cases} \quad (38)$$

At 3σ range of the best-fit point of θ_{13}, Δm_{21}^2, Δm_{31}^2 taken from Ref. [1] and m_β given in Eq. (38), with the aid of Eqs. (27), (28) and (29), we find the possible ranges of $B, m_{1,2,3}$ and $\sum$ as follows

$$\begin{cases} m_1 \in (0.131, 99.640)\,\mathrm{meV}, \\ m_2 \in (8.568, 100.024)\,\mathrm{meV}, \\ m_3 \in (49.740, 112.046)\,\mathrm{meV} \quad \text{(NH)}, \\ m_1 \in (50.270, 200.100)\,\mathrm{meV}, \\ m_2 \in (50.990, 200.300)\,\mathrm{meV}, \\ m_3 \in (0.161, 194.100)\,\mathrm{meV} \quad \text{(IH)}. \end{cases} \quad (39)$$

$$\sum \in \begin{cases} (60.940, 331.600)\,\mathrm{meV} & \text{(NH)}, \\ (117.500, 594.400)\,\mathrm{meV} & \text{(IH)}. \end{cases} \quad (40)$$

$$B \in \begin{cases} (8.543\times 10^2, 1.082\times 10^4)\,\mathrm{meV}^2 & \text{(NH)}, \\ (1.714\times 10^3, 3.927\times 10^4)\,\mathrm{meV}^2 & \text{(IH)}. \end{cases} \quad (41)$$

The followings are some comments:

(1) The obtained values of θ_{12} in Eq. (36) belongs to $2\,\sigma$ range of the best-fit value taken from Ref. [1]. This proves that the relation between t_{12} and s_{13} in Eq. (23) is predictive.

(2) For NH, in the case of $m_\beta < 9.508$ meV the minimum value of the lightest neutrino mass (m_1) being complex number which is ruled out. For IH, in the case of $m_\beta < 50.061$ meV the minimum value of the lightest neutrino mass (m_3) being complex number which is ruled out. In the case of $m_\beta > 230$ meV, the maximum value of the sum of neutrino masses, for both NH and IH, is relatively large and can go beyond the limit taken from Refs. [1, 19–21]; thus, it is not preferred in this study. Furthermore, in the case of $m_\beta > 100$ meV, the maximum value of the effective neutrino mass $\langle m_{ee}\rangle$ is relatively large and can go beyond the limit taken from Refs. [22–24].

(3) The obtained result on the sum of neutrino masses as in Eq. (40) is consistent with the current experimental constraints, such as, $\sum_\nu m_\nu < 0.13\,\mathrm{eV}$ for NH, $\sum_\nu m_\nu < 0.15\,\mathrm{eV}$ for IH [1] and $\sum_\nu m_\nu < 0.12 \div 0.69\,\mathrm{eV}$ [19–21].

4.1. Higher octant of atmospheric mixing angle θ_{23}

For the higer octant of atmospheric mixing angle θ_{23}, $\theta_{23} > 45^\circ$, *i.e.*, $s^2_{23} > 0.50$; thus, experimental range of s^2_{23} is $s^2_{23} \in (0.500,\, 0.610)$ [1]. However, from the real condition of $t_{1\mathrm{M}}$ for IH and of $t_{3\mathrm{M}}$ for NH in Eq. (24), we find the possible ranges of s^2_{23} for higher octant as follows[iii]

$$s^2_{23} \in \begin{cases} (0.513,\, 0.600) & \text{(NH)}, \\ (0.505,\, 0.600) & \text{(IH)}, \end{cases}$$
$$i.e., \quad \theta_{23} \in \begin{cases} (45.74,\, 50.77)^\circ & \text{(NH)}, \\ (45.29,\, 50.77)^\circ & \text{(IH)}. \end{cases} \tag{42}$$

Expressions (22)- (23) and (25) show that F and K depend on three parameters Δ^2_{31}, s^2_{13} and s^2_{23}, and s_δ depends on s^2_{13} and s^2_{23}. At 3σ range of Δ^2_{31} and s^2_{13} taken from Ref. [1], and with the aid of Eq. (42), we find the possible ranges of F, K and s_δ as follows

$$F \in \begin{cases} (31.340, 257.700)\,\mathrm{meV}^2 & \text{(NH)}, \\ (-247.900, -11.560)\,\mathrm{meV}^2 & \text{(IH)}, \end{cases}$$
$$K \in \begin{cases} (24.200, 281.200)\,\mathrm{meV}^2 & \text{(NH)}, \\ (-273.100, -32.160)\,\mathrm{meV}^2 & \text{(IH)}, \end{cases} \tag{43}$$
$$s_\delta \in \begin{cases} (-0.993, -0.102) & \text{(NH)}, \\ (-0.999, -0.140) & \text{(IH)}, \end{cases}$$
$$i.e., \ \delta(^\circ) \in \begin{cases} (276.60, 354.20) & \text{(NH)}, \\ (270.50, 352.00) & \text{(IH)}. \end{cases} \tag{44}$$

Furthermore, expressions (24)-(26) imply that $t_{1\mathrm{M}}$ and $t_{3\mathrm{M}}$ depend on two parameters s^2_{13} and s^2_{23}. At 3σ range of s^2_{13} and s^2_{23} taken from Ref. [1], we find the possible ranges of $t_{1\mathrm{M}}$ and $t_{3\mathrm{M}}$ as follows

$$t_{1\mathrm{M}} \in \begin{cases} (-0.320, -0.023) & \text{(NH)}, \\ (-0.331, -0.033) & \text{(IH)}, \end{cases}$$
$$i.e., \ \alpha_{1\mathrm{M}}(^\circ) \in \begin{cases} (342.30, 358.70) & \text{(NH)}, \\ (341.70, 358.10) & \text{(IH)}, \end{cases} \tag{45}$$
$$t_{3\mathrm{M}} \in \begin{cases} (-161.400, -0.110) & \text{(NH)}, \\ (-0.067,\ \ 6.478) & \text{(IH)}, \end{cases}$$
$$i.e., \ \alpha_{3\mathrm{M}}(^\circ) \in \begin{cases} (270.40, 353.60) & \text{(NH)}, \\ (81.22, 356.20) & \text{(IH)}. \end{cases} \tag{46}$$

Finally, at 3σ range of the best-fit point of θ_{13}, Δ^2_{21}, Δ^2_{31} taken from Ref. [1], with the aid of Eqs. (38) and (42), we find the possible range of $\langle m_{ee}\rangle$ for the higher octant,

$$\langle m_{ee}\rangle \in \begin{cases} (2.607, 141.200)\,\mathrm{meV} & \text{(NH)}, \\ (46.850, 199.800)\,\mathrm{meV} & \text{(IH)}. \end{cases} \tag{47}$$

4.2. Lower octant of atmospheric mixing angle θ_{23}

For the lower octant of θ_{23}, $\theta_{23} < 45^\circ$, *i.e.*, $s^2_{23} < 0.50$; thus, at 3σ range $s^2_{23} \in (0.434,\, 0.500)$ for NH and $s^2_{23} \in (0.433,\, 0.500)$ for IH [1]. At 3σ range of Δ^2_{31}, s^2_{13} and s^2_{23} taken from Ref. [1], we find the possible ranges of F, K and s_δ as follows

$$F \in \begin{cases} (-170.100, 0.00)\,\mathrm{meV}^2 & \text{(NH)}, \\ (0.00, 163.600)\,\mathrm{meV}^2 & \text{(IH)}, \end{cases}$$
$$K \in \begin{cases} (183.500, 283.200)\,\mathrm{meV}^2 & \text{(NH)}, \\ (-273.400, -177.400)\,\mathrm{meV}^2 & \text{(IH)}. \end{cases} \tag{48}$$
$$s_\delta \in \begin{cases} (-1.00, -0.761) & \text{(NH)} \\ (-1.00, -0.763) & \text{(IH)} \end{cases}$$
$$i.e., \ \delta(^\circ) \in \begin{cases} (270.00, 310.50) & \text{(NH)}, \\ (270.00, 310.20) & \text{(IH)}. \end{cases} \tag{49}$$

Furthermore, at 3σ range of s^2_{13} and s^2_{23} for the lower octant of θ_{23} taken from Ref. [1], we find the possible ranges of $t_{1\mathrm{M}}$ and $t_{3\mathrm{M}}$ as follows

$$t_{1\mathrm{M}} \in \begin{cases} (-0.356, -0.287) & \text{(NH)}, \\ (-0.331, -0.290) & \text{(IH)}, \end{cases}$$
$$i.e., \ \alpha_{1\mathrm{M}}(^\circ) \in \begin{cases} (340.40, 344.00) & \text{(NH)}, \\ (341.70, 343.80) & \text{(IH)}, \end{cases} \tag{50}$$
$$t_{3\mathrm{M}} \in \begin{cases} (0.995, 9.849) & \text{(NH)}, \\ (-1.00, -0.102) & \text{(IH)}, \end{cases}$$
$$i.e., \ \alpha_{3\mathrm{M}}(^\circ) \in \begin{cases} (44.840, 84.200) & \text{(NH)}, \\ (315.00, 354.20) & \text{(IH)}. \end{cases} \tag{51}$$

Similar to the higher octant, at 3σ range of the best-fit point of s^2_{13}, s^2_{23}, Δm^2_{21} and Δm^2_{31} taken from Ref. [1], with the help of Eq. (38), we find the possible range of $\langle m_{ee}\rangle$ for the lower octant as follows

$$\langle m_{ee}\rangle \in \begin{cases} (2.606, 141.200)\,\mathrm{meV} & \text{(NH)}, \\ (46.520, 184.600)\,\mathrm{meV} & \text{(IH)}. \end{cases} \tag{52}$$

The followings are some comments:

- The predicted ranges of the Dirac CP violating phase δ in Eqs. (44) and (49) are in consistent with $3\,\sigma$ range of the best-fit value taken from Ref. [1] with $128 \leq \delta^{\circ} \leq 358$ for NH and $200 \leq \delta^{\circ} \leq 353$ provided that $s^2_{23} \in (0.513,\, 0.600)$ for NH and $s^2_{23} \in (0.505,\, 0.600)$ for IH in the higher octant of θ_{23} and $s^2_{23} \in (0.434,\, 0.500)$ for NH and $s^2_{23} \in (0.433,\, 0.500)$ for IH in the lower octant of θ_{23} while s^2_{13} belongs to $3\,\sigma$ range of the best-fit value taken from Ref. [1]. Thus, the considered model gives predictions for the neutrino mixing angles and the Dirac CP violating phase in agreement with the data on neutrino oscillation taken from Ref. [1] in 3σ range.

- The predicted Majorana phases in Eqs. (45)-(46) and (50)-(51) are acceptable because these phases have not yet been experimentally determined but are assumed to be in $[0,\, 360^{\circ}]$ [16]. The model thus provides possible predictions on two Majorana phases.

- The obtained effective neutrino mass in Eqs. (47) and (52) are in consistent with the most recent upper limits on $\langle m_{ee}\rangle$, such as, CUORE [22] $\langle m_{ee}\rangle < (75 \div 350)\,\mathrm{meV}$, Majorana Collaboration [23] $\langle m_{ee}\rangle < (113 \div 269)\,\mathrm{meV}$ and KamLAND-Zen [24] $\langle m_{ee}\rangle < (36 \div 156)\,\mathrm{meV}$. Hence, the model provides a possible prediction for the effective neutrino mass.

5. Conclusions

We have proposed a SM extension that can accommodate the most recent experimental data on neutrino oscillation. Beside A_4, two Abelian symmetries Z_2 and Z_4 are supplemented to prevent some Yukawa terms to get the desired mass matrices and then give predictions for the neutrino oscillation parameters in agreement with the most recent experimental data on neutrino oscillation in 3σ range.

The model provides a relation between the solar neutrino mixing angle θ_{12} and the reactor neutrino mixing angle θ_{13} with $\left(2 - 3s^2_{13}\right)t^2_{12} = 1$ with predicts $\theta_{12} \in (35.68, 35.76)^{\circ}$ provided that s^2_{13} belongs to $3\,\sigma$ range of the best-fit value taken from Ref. [1].

For the higher octant of θ_{23}, the considered model predicts the Dirac CP phase to be approximately range $\delta \in (270.50, 352.00)^{\circ}$ and two Majorana phases to be approximately range $\alpha_{1\mathrm{M}}(^{\circ}) \in (342.30, 358.70)$ (NH) and $\alpha_{1\mathrm{M}}(^{\circ}) \in (341.70, 358.10)$ (IH) while $\alpha_{3\mathrm{M}}(^{\circ}) \in (270.40, 353.60)$ (NH) and $\alpha_{3\mathrm{M}}(^{\circ}) \in (81.22, 356.20)$ (IH).

For the lower octant of θ_{23}, the considered model predicts the Dirac CP phase to be approximately range $\delta \in (270.00, 310.00)^{\circ}$ and two Majorana phases to be approximately range $\alpha_{1\mathrm{M}}(^{\circ}) \in (340.40, 344.00)$ (NH) and $\alpha_{1\mathrm{M}}(^{\circ}) \in (341.70, 343.80)$ (IH) while $\alpha_{3\mathrm{M}}(^{\circ}) \in (44.840, 84.200)$ (NH) and $\alpha_{3\mathrm{M}}(^{\circ}) \in (315.00, 354.20)$ (IH).

The effective Majorana mass is also predicted to be $\langle m_{ee}\rangle \in (5.044, 141.200)\,\mathrm{meV}$ (NH) and $\langle m_{ee}\rangle \in (46.850, 199.800)\,\mathrm{meV}$ (IH) for higher octant whereas $\langle m_{ee}\rangle \in (5.030, 141.200)\,\mathrm{meV}$ (NH) and $\langle m_{ee}\rangle \in (46.520, 184.600)\,\mathrm{meV}$ (IH) for lower octant.

Appendix

A. Yukawa terms prevented by the model symmetries

TABLE III. Forbidden interactions.

Yukawwa couplings	Prevented by
$(\overline{\psi}_L l_{1R})_3(\widetilde{H_2}\varphi)_3, (\overline{\psi}_L l_{2R})_3(\widetilde{H_1}\varphi)_3$	$U(1)_Y$
$(\overline{\psi}_L l_{1R})_3 H_1$	A_4
$(\overline{\psi}_L l_{1R})_3(H_1\phi)_3, (\overline{\psi}_L l_{2R})_3(H_2\phi)_3,$ $(\overline{\psi}_L l_{3R})_3(H_2\phi)_3; (\overline{\psi}_L \nu_R)_{\underline{3}}(\widetilde{H}_1\varphi)_{\underline{3}},$ $(\overline{\psi}_L \nu_R)_{\underline{1}}(\widetilde{H}_1\rho)_{\underline{1}}, (\overline{\psi}_L \nu_R)_{\underline{1}''}(\widetilde{H}_1\eta)_{\underline{1}'}$	Z_2
$(\overline{\psi}_L l_{1R})_3(H_2\varphi)_3, (\overline{\psi}_L l_{1R})_3(H_2\varphi^*)_3,$ $(\overline{\psi}_L l_{2R})_3(H_1\varphi)_3, (\overline{\psi}_L l_{2R})_3(H_1\varphi^*)_3,$ $(\overline{\psi}_L l_{3R})_3(H_1\varphi)_3, (\overline{\psi}_L l_{3R})_3(H_1\varphi^*)_3;$ $(\overline{\psi}_L \nu_R)_{\underline{3}}(\widetilde{H}_2\varphi^*)_{\underline{3}}, (\overline{\psi}_L \nu_R)_{\underline{1}}(\widetilde{H}_2\rho^*)_{\underline{1}},$ $(\overline{\psi}_L \nu_R)_{\underline{1}'}(\widetilde{H}_2\eta^*)_{\underline{1}''}$	Z_4

B. Higgs potential invariant under the model's symmetry

The total scalar potential invariant under the model's symmetry is given by[iv]:

$$\begin{aligned} V_{\mathrm{S}} &= V(H_1) + V(H_2) + V(\phi) + V(\varphi) + V(\rho) \\ &+ V(\eta) + V(H_1, H_2) + V(H_1, \phi) + V(H_1, \varphi) \\ &+ V(H_1, \rho) + V(H_1, \eta) + V(H_2, \phi) + V(H_2, \varphi) \\ &+ V(H_2, \rho) + V(H_2, \eta) + V(\phi, \varphi) + V(\phi, \rho) \\ &+ V(\phi, \eta) + V(\varphi, \rho) + V(\varphi, \eta) + V(\rho, \eta) \\ &+ V_{\mathrm{tri}} + V_{\mathrm{quart}}, \end{aligned} \quad \text{(B.1)}$$

where[v]

$$\begin{aligned} V(H_1) &= \mu^2_{1H}(H_1^\dagger H_1)_{\underline{1}} + \lambda_{1H}(H_1^\dagger H_1)_{\underline{1}}(H_1^\dagger H_1)_{\underline{1}}, \\ V(H_2) &= V(H_1 \to H_2), \\ V(\phi) &= \mu^2_{\phi}(\phi^*\phi)_{\underline{1}} + \lambda^{\phi}\left[(\phi^*\phi)_{\underline{1}}(\phi^*\phi)_{\underline{1}} + (\phi^*\phi)_{\underline{3}_s}(\phi^*\phi)_{\underline{3}_s}\right], \\ V(\varphi) &= V(\phi \to \varphi, \underline{3}_s \to \underline{1}), \\ V(\rho) &= \mu^2_{\rho}(\rho^*\rho)_{\underline{1}} + \lambda^{\rho}(\rho^*\rho)_{\underline{1}}(\rho^*\rho)_{\underline{1}}, \\ V(\eta) &= V(\rho \to \eta), \end{aligned}$$

$$\begin{aligned}
V(H_1,H_2) &= \lambda_{12H}\big[(H_1^\dagger H_1)_{\underline{1}}(H_2^\dagger H_2)_{\underline{1}} + (H_1^\dagger H_2)_{\underline{1}}(H_2^\dagger H_1)_{\underline{1}}\big],\\
V(H_1,\phi) &= \lambda_{H_1\phi}\big[(H_1^\dagger H_1)_{\underline{1}}(\phi^*\phi)_{\underline{1}} + (H_1^\dagger \phi)_{\underline{3}}(\phi^* H_1)_{\underline{3}}\big],\\
V(H_2,\phi) &= V(H_1\to H_2,\phi),\, V(H_1,\varphi) = V(H_1,\phi\to\varphi),\\
V(H_2,\varphi) &= V(H_1\to H_2,\varphi),\, V(H_1,\rho) = V(H_1,\varphi\to\rho),\\
V(H_2,\rho) &= V(H_1\to H_2,\rho),\\
V(H_1,\eta) &= \lambda_{H_1\eta}\big[(H_1^\dagger H_1)_{\underline{1}}(\eta^*\eta)_{\underline{1}} + (H_1^\dagger \eta)_{\underline{1}'}(\eta^* H_1)_{\underline{1}''}\big],\\
V(H_2,\eta) &= V(H_1\to H_2,\eta),\\
V(\phi,\varphi) &= \lambda_{\phi\varphi}\big[(\phi^*\phi)_{\underline{1}}(\varphi^*\varphi)_{\underline{1}} + (\phi^*\varphi)_{\underline{1}}(\varphi^*\phi)_{\underline{1}} + (\phi^*\varphi)_{\underline{1}'}(\varphi^*\phi)_{\underline{1}''} + (\phi^*\varphi)_{\underline{1}''}(\varphi^*\phi)_{\underline{1}'} + (\phi^*\varphi)_{\underline{3}_s}(\varphi^*\phi)_{\underline{3}_s}\\
&\quad + (\phi^*\varphi)_{\underline{3}_s}(\varphi^*\phi)_{\underline{3}_a} + (\phi^*\varphi)_{\underline{3}_a}(\varphi^*\phi)_{\underline{3}_s} + (\phi^*\varphi)_{\underline{3}_a}(\varphi^*\phi)_{\underline{3}_a}\big],\\
V(\phi,\rho) &= \lambda_{\phi\rho}\big[(\phi^*\phi)_{\underline{1}}(\rho^*\rho)_{\underline{1}} + (\phi^*\rho)_{\underline{3}}(\rho^*\phi)_{\underline{3}}\big],\\
V(\phi,\eta) &= V(\phi,\rho\to\eta),\\
V(\varphi,\rho) &= V(\phi\to\varphi,\rho),\\
V(\rho,\eta) &= \lambda_{\rho\eta}\big[(\rho^*\rho)_{\underline{1}}(\eta^*\eta)_{\underline{1}} + (\rho^*\eta)_{\underline{1}'}(\eta^*\rho)_{\underline{1}''}\big],\\
V(\varphi,\eta) &= V(\phi\to\varphi,\eta),\\
V_{\mathrm{tri}} &= \lambda_{\varphi\rho\eta}\big[(\varphi^*\varphi)_{\underline{1}''}(\rho^*\eta)_{\underline{1}'} + (\varphi^*\varphi)_{\underline{1}'}(\rho\eta^*)_{\underline{1}''}\big],\\
V_{\mathrm{quart}} &= \lambda_{H_1H_2\phi\varphi\rho}\big[(H_1^\dagger H_2)_{\underline{1}}(\phi\varphi^*)_{\underline{1}}\rho + (H_1^\dagger H_2)_{\underline{1}}(\phi\varphi)_{\underline{1}}\rho^* + (H_2^\dagger H_1)_{\underline{1}}(\phi\varphi^*)_{\underline{1}}\rho + (H_2^\dagger H_1)_{\underline{1}}(\phi\varphi)_{\underline{1}}\rho^*\big]\\
&\quad + \lambda_{H_1H_2\phi\varphi\eta}\big[(H_1^\dagger H_2)_{\underline{1}}(\phi\varphi^*)_{\underline{1}''}\eta + (H_1^\dagger H_2)_{\underline{1}}(\phi\varphi)_{\underline{1}'}\eta^* + (H_2^\dagger H_1)_{\underline{1}}(\phi\varphi^*)_{\underline{1}''}\eta + (H_2^\dagger H_1)_{\underline{1}}(\phi\varphi)_{\underline{1}'}\eta^*\big]. \qquad (2)
\end{aligned}$$

All the other terms, up to five-dimension, of three or four or five distinct scalars are vanished due to the violations under one or some of the symmetries of the model. Now, we can show that the VEVs in Eq. (3) satisfy the minimization condition of V_{S} by supposing that all the VEVs are real. The minimization condition of V_{S}, $\partial V_{\mathrm{S}}/\partial v_\Phi = 0$, $\delta^2_\Phi \sim \partial^2 V_{\mathrm{S}}/\partial v^2_\Phi > 0$ with $v_\Phi = v_{1,2},\, v_\phi,\, v_\varphi,\, v_\rho,\, v_\eta$, has the form

$$\begin{aligned}
&\mu^2_{1H}v_1 + 2v_1\big(\lambda_{1H}v_1^2 + \lambda_{12H}v_2^2 + \lambda_{H_1\eta}v_\eta^2 + \lambda_{H_1\phi}v_\phi^2 + \lambda_{H_1\rho}v_\rho^2\big)\\
&\qquad + 2v_2v_\phi v_\varphi(\lambda_{H_1H_2\phi\varphi\eta}v_\eta + \lambda_{H_1H_2\phi\varphi\rho}v_\rho) + 6\lambda_{H_1\varphi}v_1v_\varphi^2 = 0, \qquad (3)\\
&\mu^2_{2H}v_2 + 2v_2\big(\lambda_{2H}v_2^2 + \lambda_{12H}v_1^2 + \lambda_{H_2\eta}v_\eta^2 + \lambda_{H_2\phi}v_\phi^2 + \lambda_{H_2\rho}v_\rho^2\big)\\
&\qquad + 2v_1v_\phi v_\varphi(\lambda_{H_1H_2\phi\varphi\eta}v_\eta + \lambda_{H_1H_2\phi\varphi\rho}v_\rho) + 6\lambda_{H_2\varphi}v_2v_\varphi^2 = 0, \qquad (4)\\
&\mu^2_\phi v_\phi + 2v_\phi\big(\lambda_{H_1\phi}v_1^2 + \lambda_{H_2\phi}v_2^2 + \frac{26}{9}\lambda_\phi v_\phi^2 + \lambda_{\phi\eta}v_\eta^2 + \lambda_{\phi\rho}v_\rho^2\big)\\
&\qquad + 2v_1v_2v_\varphi\big(\lambda_{H_1H_2\phi\varphi\eta}v_\eta + \lambda_{H_1H_2\phi\varphi\rho}v_\rho\big) + \frac{62}{9}\lambda_{\phi\varphi}v_\phi v_\varphi^2 = 0, \qquad (5)\\
&\mu^2_\varphi v_\varphi + 2v_\rho v_\varphi(\lambda_{\varphi\rho}v_\rho + \lambda_{\varphi\rho\eta}v_\eta) + \frac{2v_1v_2v_\phi}{3}\big(\lambda_{H_1H_2\phi\varphi\eta}v_\eta + \lambda_{H_1H_2\phi\varphi\rho}v_\rho\big)\\
&\qquad + 2v_\varphi\big(\lambda_{H_1\varphi}v_1^2 + \lambda_{H_2\varphi}v_2^2 + \frac{31}{27}\lambda_{\phi\varphi}v_\phi^2 + 6\lambda_\varphi v_\varphi^2 + \lambda_{\varphi\eta}v_\eta^2\big) = 0, \qquad (6)\\
&\mu^2_\rho v_\rho + 2v_\rho\,\big(\lambda_{H_1\rho}v_1^2 + \lambda_{H_2\rho}v_2^2 + \lambda_{\phi\rho}v_\phi^2 + \lambda_\rho v_\rho^2\big)\\
&\qquad + 2\lambda_{H_1H_2\phi\varphi\rho}v_1v_2v_\phi v_\varphi + 3(2\lambda_{\varphi\rho}v_\rho + \lambda_{\varphi\rho\eta}v_\eta)v_\varphi^2 = 0, \qquad (7)\\
&\mu^2_\eta v_\eta + 2v_\eta(\lambda_\eta v_\eta^2 + \lambda_{H_1\eta}v_1^2 + \lambda_{H_2\eta}v_2^2 + \lambda_{\phi\eta}v_\phi^2)\\
&\qquad + 2\lambda_{H_1H_2\phi\varphi\eta}v_1v_2v_\phi v_\varphi + 3(2\lambda_{\varphi\eta}v_\eta + \lambda_{\varphi\rho\eta}v_\rho)v_\varphi^2 = 0, \qquad (8)
\end{aligned}$$

$$\mu_{1H}^2 + 2\big(3\lambda_{1H}v_1^2 + \lambda_{12H}v_2^2 + \lambda_{H_1\eta}v_\eta^2 + \lambda_{H_1\phi}v_\phi^2 + \lambda_{H_1\rho}v_\rho^2 + 3\lambda_{H_1\varphi}v_\varphi^2\big) > 0, \tag{9}$$

$$\mu_{2H}^2 + 2\big(3\lambda_{2H}v_2^2 + \lambda_{12H}v_1^2 + \lambda_{H_2\eta}v_\eta^2 + \lambda_{H_2\phi}v_\phi^2 + \lambda_{H_2\rho}v_\rho^2 + 3\lambda_{H_2\varphi}v_\varphi^2\big) > 0, \tag{10}$$

$$\mu_\phi^2 + 2\lambda_{H_1\phi}v_1^2 + 2\lambda_{H_2\phi}v_2^2 + \frac{26\lambda_\phi v_\phi^2}{3} + 2\lambda_{\phi\eta}v_\eta^2 + 2\lambda_{\phi\rho}v_\rho^2 + \frac{62\lambda_{\phi\varphi}v_\varphi^2}{9} > 0, \tag{11}$$

$$\mu_\varphi^2 + 2\lambda_{H_1\varphi}v_1^2 + 2\lambda_{H_2\varphi}v_2^2 + \frac{62\lambda_{\phi\varphi}v_\phi^2}{27} + 36\lambda_\varphi v_\varphi^2 + 2\lambda_{\varphi\eta}v_\eta^2 + 2v_\rho(\lambda_{\varphi\rho}v_\rho + \lambda_{\varphi\rho\eta}v_\eta) > 0, \tag{12}$$

$$\mu_\rho^2 + 2\big(\lambda_{H_1\rho}v_1^2 + \lambda_{H_2\rho}v_2^2 + \lambda_{\phi\rho}v_\phi^2 + 3\lambda_\rho v_\rho^2 + 3\lambda_{\varphi\rho}v_\varphi^2\big) > 0, \tag{13}$$

$$\mu_\eta^2 + 2\big(3\lambda_\eta v_\eta^2 + \lambda_{H_1\eta}v_1^2 + \lambda_{H_2\eta}v_2^2 + \lambda_{\phi\eta}v_\phi^2 + 3\lambda_{\varphi\eta}v_\varphi^2\big) > 0, \tag{14}$$

Equations (3)-(8) yield the following solution:

$$\mu_{1H}^2 = -2\big(\lambda_{1H}v_1^2 + \lambda_{12H}v_2^2 + \lambda_{H_1\phi}v_\phi^2 + 3\lambda_{H_1\varphi}v_\varphi^2 + \lambda_{H_1\rho}v_\rho^2 + \lambda_{H_1\eta}v_\eta^2\big) - \frac{2v_2v_\phi v_\varphi\Lambda_{\rho\eta}}{v_1}, \tag{15}$$

$$\mu_{2H}^2 = -2\big(\lambda_{12H}v_1^2 + \lambda_{2H}v_2^2 + \lambda_{H_2\phi}v_\phi^2 + 3\lambda_{H_2\varphi}v_\varphi^2 + \lambda_{H_2\rho}v_\rho^2 + \lambda_{H_2\eta}v_\eta^2\big) - \frac{2v_1v_\phi v_\varphi\Lambda_{\rho\eta}}{v_2}, \tag{16}$$

$$\mu_\phi^2 = -2\big(\lambda_{H_1\phi}v_1^2 + \lambda_{H_2\phi}v_2^2 + \frac{13}{9}\lambda_\phi v_\phi^2 + \frac{31}{9}\lambda_{\phi\varphi}v_\varphi^2 + \lambda_{\phi\rho}v_\rho^2 + \lambda_{\phi\eta}v_\eta^2\big) - \frac{2v_1v_2v_\varphi\Lambda_{\rho\eta}}{v_\phi}, \tag{17}$$

$$\mu_\varphi^2 = -2\left(\lambda_{H_1\varphi}v_1^2 + \lambda_{H_2\varphi}v_2^2 + \frac{31\lambda_{\phi\varphi}v_\phi^2}{27} + 6\lambda_\varphi v_\varphi^2 + \lambda_{\varphi\rho}v_\rho^2 + \lambda_{\varphi\eta}v_\eta^2 + \lambda_{\varphi\rho\eta}v_\rho v_\eta + \frac{v_1v_2v_\phi\Lambda_{\rho\eta}}{3v_\varphi}\right), \tag{18}$$

$$\mu_\rho^2 = -2\big(\lambda_{H_1\rho}v_1^2 + \lambda_{H_2\rho}v_2^2 + \lambda_{\phi\rho}v_\phi^2 + 3\lambda_{\varphi\rho}v_\varphi^2 + \lambda_\rho v_\rho^2\big) - \frac{\Lambda_{\phi\varphi\eta}}{v_\rho}, \tag{19}$$

$$\mu_\eta^2 = -2\big(\lambda_{H_1\eta}v_1^2 + \lambda_{H_2\eta}v_2^2 + \lambda_{\phi\eta}v_\phi^2 + 3\lambda_{\varphi\eta}v_\varphi^2 + \lambda_\eta v_\eta^2\big) - \frac{\Lambda_{\phi\varphi\rho}}{v_\eta}, \tag{20}$$

where

$$\begin{aligned}&\Lambda_{\rho\eta} = \lambda_{H_1H_2\phi\varphi\rho}v_\rho + \lambda_{H_1H_2\phi\varphi\eta}v_\eta, \qquad \Lambda_{\phi\varphi\eta} = v_\varphi(3\lambda_{\varphi\rho\eta}v_\eta v_\varphi + 2\lambda_{H_1H_2\phi\varphi\rho}v_1v_2v_\phi),\\ &\Lambda_{\phi\varphi\rho} = v_\varphi(3\lambda_{\varphi\rho\eta}v_\rho v_\varphi + 2\lambda_{H_1H_2\phi\varphi\eta}v_1v_2v_\phi).\end{aligned} \tag{21}$$

With the aid of Eq. (20), expressions (9)-(14) become

$$\begin{aligned}&\lambda_{1H} > \frac{v_2v_\phi v_\varphi\Lambda_{\rho\eta}}{2v_1^3}, \quad \lambda_{2H} > \frac{v_1v_\phi v_\varphi\Lambda_{\rho\eta}}{2v_2^3}, \quad \lambda_\phi > \frac{9v_1v_2v_\varphi\Lambda_{\rho\eta}}{26v_\phi^3},\\ &\lambda_\varphi > \frac{v_1v_2v_\phi\Lambda_{\rho\eta}}{36v_\varphi^3}, \quad \lambda_\rho > \frac{\Lambda_{\phi\varphi\eta}}{4v_\rho^3}, \quad \lambda_\eta > \frac{\Lambda_{\phi\varphi\rho}}{4v_\eta^3}.\end{aligned} \tag{22}$$

C. Explicit expressions of A, B, F, H and K

The elements of the Hermitian matrix M_ν^2 in Eq. (12) are

$$\begin{aligned}A &= 3\alpha_0^2 + \beta_0^2 + 3\sigma_0^2 + 2\alpha_0\gamma_0\cos\Delta_{\alpha\gamma} - \alpha_0\kappa_0\cos\Delta_{\alpha\kappa} - \alpha_0\tau_0\cos\Delta_{\alpha\tau} - \beta_0\kappa_0\cos\Delta_{\beta\kappa}\\ &+ \beta_0\tau_0\cos\Delta_{\beta\tau} - \gamma_0\kappa_0\cos\Delta_{\gamma\kappa} - \gamma_0\tau_0\cos\Delta_{\gamma\tau} - \kappa_0\tau_0\cos\Delta_{\kappa\tau},\\ B &= 6\alpha_0^2 + 2\beta_0^2 + \gamma_0^2 + \kappa_0^2 + 6\sigma_0^2 + \tau_0^2 + 4\alpha_0\gamma_0\cos\Delta_{\alpha\gamma}\\ &- 2\big(\alpha_0\kappa_0\cos\Delta_{\alpha\kappa} + \alpha_0\tau_0\cos\Delta_{\alpha\tau} + \beta_0\kappa_0\cos\Delta_{\beta\kappa} - \beta_0\tau_0\cos\Delta_{\beta\tau}\big),\\ F &= 3\sigma_0(2\beta_0\cos\Delta_{\beta\sigma} - \kappa_0\cos\Delta_{\kappa\sigma} + \tau_0\cos\Delta_{\sigma\tau}),\\ H &= \sigma_0(6\alpha_0\cos\Delta_{\alpha\gamma} + 2\gamma_0\cos\Delta_{\gamma\sigma} - \kappa_0\cos\Delta_{\kappa\sigma} - \tau_0\cos\Delta_{\sigma\tau}),\\ K &= 3\alpha_0\big(2\beta_0\sin\Delta_{\alpha\beta} - \kappa_0\sin\Delta_{\alpha\kappa} + \tau_0\sin\Delta_{\alpha\tau}\big) - \kappa_0\tau_0\sin\Delta_{\kappa\tau} + \beta_0(\kappa_0\sin\Delta_{\beta\kappa} + \tau_0\sin\Delta_{\beta\tau}\\ &- 2\gamma_0\sin\Delta_{\beta\gamma}) - \gamma_0(\kappa_0\sin\Delta_{\gamma\kappa} + \tau_0\sin\Delta_{\gamma\tau}).\end{aligned} \tag{C.1}$$

with

$$\Delta_{mn} = x_m - x_n, \qquad x_m = \mathrm{Arg}(m), \qquad m_0 = |m| \; \big(m, n = \alpha, \beta, \gamma, \kappa, \sigma, \tau\big). \tag{C.2}$$

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$i.$ In this study, we use the following notaions: $\Delta^2_{ij} = m^2_i - m^2_j$, $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, $t_{ij} = s_{ij}/c_{ij}$ with θ_{12}, θ_{23} and θ_{13} are respectively the solar, atmospheric and reactor angles; $s_\delta = \sin\delta$ with δ is the Dirac CP phase; $t_{1,3\mathrm{M}} = \tan\alpha_{1,3\mathrm{M}}$ with $\alpha_{1,3\mathrm{M}}$ being two Majorana phases; $s_\Theta = \sin\Theta$ and $c_\Theta = \cos\Theta$. Further, there are three Majorana phases of the form $\mathrm{diag}\left(e^{i\delta_{1M}}, e^{i\delta_{2M}}, e^{i\delta_{3M}}\right)$ which are specified by two combinations of the form $\delta_{iM} - \delta_{2M}$ instead of $\delta_{iM}\,(i = 1, 2, 3)$, i.e., $\mathrm{diag}\left(e^{i\delta_{1M}}, e^{i\delta_{2M}}, e^{i\delta_{3M}}\right)$ is reduced to $\mathrm{diag}\left(e^{i\alpha_{1\mathrm{M}}}, 1, e^{i\alpha_{3\mathrm{M}}}\right)$.

$ii.$ The considered model contains two $SU(2)_L$ doublets. See, for instance [9, 10], for a review of the 2HDM.

$iii.$ In the case of $s^2_{23} < 0.513$ or $s^2_{23} > 0.600$ for NH $\tan\alpha_{3\mathrm{M}}$ becomes complex function and in the case of $s^2_{23} < 0.505$ or $s^2_{23} > 0.600$ for IH $\tan\alpha_{1\mathrm{M}}$ becomes complex function.

$iv.$ Here, $V(\mathbf{a}\to\mathbf{x}, \mathbf{b}\to\mathbf{y}, \cdots) \equiv V(\mathbf{a}, \mathbf{b}, \cdots)\big|_{\{\mathbf{a}=\mathbf{x}, \mathbf{b}=\mathbf{y}, \cdots\}}$.

$v.$ It is noted that, with the VEV aligments in Eq. (3), $(\phi^*\phi)_{\underline{1}'} = (\phi^*\phi)_{\underline{1}''} = 0$, $(\phi^*\phi)_{\underline{3}_a} = 0$, $(\varphi^*\varphi)_{\underline{3}_s} = (\varphi^*\varphi)_{\underline{3}_a} = 0$; $(\varphi^*\varphi)_{\underline{1}''}(\varphi^*\varphi)_{\underline{1}'} = (\varphi^*\varphi)_{\underline{1}'}(\varphi^*\varphi)_{\underline{1}''}$.

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