A non-newtonian approach to geometric phase through optic fiber via multiplicative quaternions

H. Ceyhan
Department of Mathematics, Science Faculty, Ankara University, 06100 Ankara, Turkey.
e-mail: hazallceyhan@gmail.com

Z. Özdemir
Department of Mathematics, Arts and Science Faculty, Amasya University, 05189 Amasya, Turkey.
e-mail: zehra.ozdemir@amasya.edu.tr

S. Kaya
Department of Mathematics, Faculty of Engineering and Natural Sciences, Usak University, 64200 Usak, Turkey.
e-mail: semra.kaya@usak.edu.tr

İ. Gürgil
Department of Mathematics, Faculty of Engineering and Natural Sciences, Usak University, 64200 Usak, Turkey.
e-mail: ibrahim.gurgil@usak.edu.tr

Received 29 March 2024; accepted 17 June 2024

In this paper, we researched magnetic and electromagnetic curves in multiplicative Euclidean 3-space via multiplicative quaternion algebra. Firstly, we examined the geometric phase representation of the polarized light wave in the optic fiber by multiplying Frenet frame. Using the quaternionic approaches, we were able to derive the magnetic curve equations and theorems. Then, three particular instances have been illustrated with examples of electromagnetic curves and magnetic field equations. Lastly, we provided an interpretation of the findings. With the help of the results, we were able to present an alternative viewpoint on the construction of trajectories (such as circular or spiral-like ones) that do not exist uniquely in the realm of physics.

Keywords: Multiplicative calculus; non-newtonian calculus; magnetic flows; electromagnetic theory; ordinary differential equations.

DOI: https://doi.org/10.31349/RevMexFis.70.061301

1. Introduction

Alternative calculus, sometimes known as non-Newtonian calculus, was developed by Grossman and Katz in Ref. [1]. This page, which includes definitions for many different calculi (geometric calculus, anageometric calculus, etc.), made it possible to deal with various calculus. Then, Grossman researched this new subject comprehensively [2]. Studies have advanced since the introduction of these new types of calculus. Works on bigeometric and multiplicative calculus were written by Grossman and Stanley, respectively, in Refs. [3, 4]. This article makes use of multiplicative differential geometry. Multiplicative calculus was first studied by several mathematical specialties [5]. After that, authors examined multiplicative calculi in different systems in Ref. [6]. At the same time, important methods for mathematics were studied to understand the relationship with multiplicative calculi [7]. Bashirov and the others researched multiplicative calculi and their interpretations in Ref. [8]. In the light of these papers, this subject was researched in various fields, for example, physics and biomedical concerning mathematical approach [9, 10]. At the same time, Boruah and Hazarika produced two significant works on non-Newtonian calculus [11, 12]. Apart from all these publications, a comprehensive book was also published that explored multiplicative differential geometry, which we used as a basis for our paper [13]. Additionally, Georgiev and Zennir produced two more volumes that dealt with multiplicative differential calculus to build on their earlier work [14] and multiplicative analytic geometry [15]. Aslan *et al*. constructed new quaternions to integrate these findings with quaternion algebra in [16]. Outstanding special curves were examined by Aydin et al. using multiplicative differential geometry [17]. Non-Newtonian calculus was used by Nurkan and others to investigate vector characteristics [18]. Ceyhan and Özdemir researched the generalized tube surfaces according to multiplicative calculus [19]. Then, hyperbolic quaternion, one of the quaternionic structures, and its motion were investigated via non-Newtonian approach [20].

A key topic in the study of physics is electromagnetic theory. It is the first paper on topological phase study in this field, which made it possible for mathematicians to pursue this topic as a research area [21]. After the seminal work by Krastov and Oblov [22], it was believed that this topic could be studied from a geometric perspective. Ross examined the motions of the polarization plane in a highly significant paper, which the author prepared in light of these events [23]. The geometric phase, or Berry's phase, has since been used in numerous studies on a wide range of topics; this is our research area for the advancement of electromagnetic theory [24–26]. Geometry's foundational topic, Riemannian surfaces, has also been explored concerning this topic [27]. Moreover, two important areas of geometry research, complex projective space, and Kahler magnetic field have been studied [28, 29]. With all of the advancements and the capacity to study magnetic trajectories, research on the connection between magnetic curves and special curves has started [30]. Studying charged particle mobility along optic fiber has also been done [31]. Barros released two significant works that expanded our viewpoint at the same time [32, 33]. Cabrerizo has published two works demonstrating that electromagnetic theory can be examined in other significant geometric spaces [34, 35]. Mathematical researchers have published from many perspectives on geometric phase, which has become one of the most interesting topics of recent times [36–41]. Bozkurt and her colleagues used mathematical programs to visualize the motion of magnetic curves [42, 43]. This topic is still being published on and can be developed further in many other fields of study [44–48].

In this paper, we investigate Berry's phase concerning multiplicative differential geometry in multiplicative 3D space. Firstly, we gave some fundamental background about multiplicative calculi and electromagnetic theory. The definition of multiplicative quaternion algebra and the basic definitions and theorems of mathematics that we used to create this publication are provided in the second section. In the third section, we demonstrate the Berry phase model for multiplicative curves concerning three cases about constant multiplicative angles between multiplicative Frenet frame vector fields and E. In the same section, we gave valuable theorems for the aims and relationships of multiplicative quaternion algebras. We showed electromagnetic multiplicative curves and calculated the multiplicative magnetic field in the following section. After that, we organized physical interpretations by finding theorems and calculating equations. To make what we accomplished more apparent, we provided examples in the next section and used mathematical tools to display them.

2. Basic materials

$\mathbb{R}_*^3 = \{(u_1, u_2, u_3) : u_1, u_2, u_3 \in \mathbb{R}_*\}$ be a multiplicative 3-space which has following operations,

$$+_* : \mathbb{R}_*^3 x \mathbb{R}_*^3 \to \mathbb{R}_*^3$$

$$u +_* v = (u_1 v_1, u_2 v_2, u_3 v_3),$$

$$-_* : \mathbb{R}_*^3 x \mathbb{R}_*^3 \to \mathbb{R}_*^3$$

$$u -_* v = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)$$

and

$$\cdot_* : \mathbb{R}_* x \mathbb{R}_*^3 \to \mathbb{R}_*^3,$$

$$c \cdot_* u = (e^{logclogu_1}, e^{logclogu_2}, e^{logclogu_3}),$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}_*^3$ and $c \in \mathbb{R}_*$ that $\mathbb{R}_*$ is the set of all the positive real numbers.

Moreover, $\mathbb{R}_*^3$ has a metric defined as follows;

$$\langle u, v\rangle_* : \mathbb{R}_*^3 x \mathbb{R}_*^3 \to \mathbb{R}_*^3$$

$$\langle u, v\rangle_* = e^{\log u_1 \log v_1 + \log u_2 \log v_2 + \log u_3 \log v_3},$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{R}_*^3$.

In 3-D multiplicative Euclidean space E_*^3, the multiplicative product of two 3-D multiplicative vectors are defined as;

$$u \times_* v = (e^{\log u_2 \log v_3 - \log v_2 \log u_3}, e^{\log v_1 \log u_3 - \log u_1 \log v_3}, e^{\log u_1 \log v_2 - \log v_1 \log u_2}),$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in E_*^3$.

On the other hand, we take $\theta \in [0_*, e^\pi]$.Then, it can be written as follows,

$$\cos_* \theta = e^{\cos(\log \theta)},$$

$$\arccos_* \psi = e^{\arccos(\log \psi)},$$

for $\psi \in [e^{-1}, 1_*]$,for more details see [13, 17]. The multiplicative angle is represented by θ in this entire study.

Moreover, $s \to p(s) \in \mathbb{R}_*$ is the multiplicative derivative of p at s defined by rule of L'Hospital as follow;

$$p^*(s) = e^{s(\log(p(s))'},$$

where $'$ is the usual derivative.

Let (I, f_*), $f_* : \mathbb{R}_* \to \mathbb{R}_*^3$ is a multiplicative biregular curve, namely nowhere $f^*(s)$ and f^{**} are multiplicative collinear, (see more details [17]).

Then the multiplicative Frenet frame equations concerning f are calculated as follows;

$$\tau^*(s) = k(s) \cdot_* \nu(s),$$

$$\nu^*(s) = -_* k(s) \cdot_* \tau(s) +_* \kappa(s) \cdot_* \beta(s),$$

$$\beta^*(s) = -_* \kappa(s) \cdot_* \nu(s),$$

where $k(s)$ and $\kappa(s)$ are called the multiplicative curvature and multiplicative torsion of the multiplicative curve, respectively. Moreover, $\tau(s), \nu(s)$, and $\beta(s)$ are called the multiplicative unit tangent, unit principal normal, and binormal at the point (s_0), respectively. Then the binormal satisfies that;

$$\beta(s) = \tau(s) \times_* \nu(s).$$

Definition 1. *Let (I, f) be a multiplicative parameterized curve. We can say the curve is the multiplicative generalized helix if its multiplicative tangents make a constant angle with a fixed multiplicative vector in the pace $\mathbb{R}_*^3$* [13].

The definition of H_*, the set of multiplicative quaternions, is:

$$H_* = \{q_e = (q_1, q_2, q_3, q_4) : q_1, q_2, q_3, q_4) :\in \mathbb{R}_*\} \subset \mathbb{R}_*^4.$$

One way to express a multiplicative quaternion q_e is as

$$q_e=(q_1 \cdot_* e_*)+_*(q_2 \cdot_* i_*)+_*(q_3 \cdot_* j_*)+_*(q_4 \cdot_* k_*),$$

where $1_* = e$, and $0_* = 1$.

$$e_* = (1_*, 0_*, 0_*, 0_*),$$
$$i_* = (0_*, 1_*, 0_*, 0_*),$$
$$j_* = (0_*, 0_*, 1_*, 0_*),$$
$$k_* = (0_*, 0_*, 0_*, 1_*),$$

abide by the norms of multiplication

$$(i_*)^{2_*} = e^{-0_*}, \ \ (j_*)^{2_*} = e^{-0_*}, \quad (k_*)^{2_*} = e^{-0_*}.$$

Theorem 1. *Let* $x_e = (x_1 \cdot_* e_*) +_* (x_2 \cdot_* i_*) +_* (x_3 \cdot_* j_*) +_* (x_4 \cdot_* k_*)$ *is unit multiplicative quaternion and* $\overset{\Lambda}{v}$ *is a pure multiplicative quaternion. Then, the* Φ *linear mapping represented as a matrix is given as the matrix form* $M = [M_1, M_2, M_3]$ *has the following columns;*

$$\overset{\Lambda}{M}_1 = \begin{bmatrix} e^{\log x_1^2+\log x_2^2-\log x_3^2-\log x_4^2} \\ e^{\log 2x_1 \log x_4+\log 2x_2 \log x_3} \\ e^{-\log 2x_1 \log x_3+\log 2x_2 \log x_4} \end{bmatrix},$$

$$\overset{\Lambda}{M}_2 = \begin{bmatrix} e^{-\log 2x_1 \log x_4+\log 2x_2 \log x_3} \\ e^{\log x_1^2+\log x_3^2-\log x_2^2-\log x_4^2} \\ e^{\log 2x_1 \log x_2+\log 2x_4 \log x_3} \end{bmatrix},$$

$$\overset{\Lambda}{M}_3 = \begin{bmatrix} e^{\log 2x_1 \log x_3+\log 2x_2 \log x_4} \\ e^{-\log 2x_1 \log x_2+\log 2x_4 \log x_3} \\ e^{\log x_1^2+\log x_4^2-\log x_3^2-\log x_2^2} \end{bmatrix},$$

where $(\overset{\Lambda}{M})^t \overset{\Lambda}{M} = \overset{\Lambda}{I}$ and $\Phi(\overset{\Lambda}{v}) = \overset{\Lambda}{M}\overset{\Lambda}{v}$.

We give a multiplicative quaternion $x_e = (x_1 \cdot_* e_*) +_* (x_2 \cdot_* i_*) +_* (x_3 \cdot_* j_*) +_* (x_4 \cdot_* k_*)$ and then conjugate, norm, modulus, and inverse of multiplicative quaternion, respectively, given by

$$x_e^{-*} = (x_1) -_* (x_2 \cdot_* i_*) -_* (x_3 \cdot_* j_*) -_* (x_4 \cdot_* k_*),$$

$$N_*(x_e) = e^{(\log x_1)^2+(\log x_2)^2+(\log x_3)^2+(\log x_4)^2},$$

$$\mid x_e \mid_* = e^{\sqrt{(\log x_1)^2+(\log x_2)^2+(\log x_3)^2+(\log x_4)^2}},$$

and

$$x_e{}^{-1_*} = x_e{}^{-*} /_* N_*(x_e).$$

3. A geometric phase representation of the polarized light wave in the optic fiber through multiplicative Frenet frame

In multiplicative Euclidean 3-space, we can write an optic fiber as a multiplicative biregular curve by β. We can describe polarized light's state's direction as the direction of **E**. Consequently, **E**'s direction can be represented as the linear combination of $\{\tau, \nu, \beta\}$ multiplicative Frenet frame fields. After that, we get as follows:

$$E^* = \lambda_1 \cdot_* \tau(s) +_* \lambda_2 \cdot_* \nu(s) +_* \lambda_3 \cdot_* \beta(s),$$

where λ_i, $i = 1, 2, 3$ are differentiable functions according to multiplicative calculus.

Next, for three distinct scenarios, the direction of the polarized light's state was investigated, considering the planes on which **E** lies.

Case 1. Assuming the first scenario, there is a constant multiplicative angle between $E(s)$ and $\tau(s)$. We know that the expression for an electric field is as follows:

$$E^* = \lambda_1 \cdot_* \tau(s) +_* \lambda_2 \cdot_* \nu(s) +_* \lambda_3 \cdot_* \beta(s).$$

After that, if we consider this case's sufficiency, we get,

$$\langle E, \tau(s)\rangle_* = \cos_* \theta, \quad \theta = \textit{const.}$$

Generally, we know, examining the optic fiber allows us to write,

$$\langle E, E\rangle_* = \textit{const.}$$

On the other hand, ψ and θ are the multiplicative angles, so we get as follows,

$$E = \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* \sin_* \psi \cdot_* \nu(s)$$
$$+_* \sin_* \theta \cdot_* \cos_* \psi \cdot_* \beta(s).$$

If we take a derivative of last equation, we get;

$$E^* = (-_* k(s) \cdot_* \sin_* \theta \cdot_* \sin_* \psi) \cdot_* \tau(s))$$
$$+_* (k(s) \cdot_* \cos_* \theta -_* \kappa(s) \cdot_* \sin_* \theta \cdot_* \cos_* \psi$$
$$+_* \psi^* \cdot_* \cos_* \psi \cdot_* \sin_* \theta) \cdot_* \nu(s)$$
$$+_* (\kappa(s) \cdot_* \sin_* \theta \cdot_* \sin_* \psi$$
$$-_* \psi^* \cdot_* \sin_* \psi \cdot_* \sin_* \theta) \cdot_* \beta(s).$$

Generally, we can write electric field with the help of multiplicative Frenet frame elements as follows,

$$E = \langle E, \tau(s)\rangle_* \cdot_* \tau(s) +_* \langle E, \nu(s)\rangle_* \cdot_* \nu(s)$$
$$+_* \langle E, \beta(s)\rangle_* \cdot_* \beta(s).$$

We can get the following equation by matching and correcting the appropriate values.

$$E^* = (\langle E, \tau(s)\rangle_*^* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s)$$
$$+_* (\langle E, \nu(s)\rangle_*^* +_* k(s) \cdot_* \langle E, \tau(s)\rangle_*$$
$$-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*) \cdot_* \nu(s)$$
$$+_* (\langle E, \beta(s)\rangle_*^* +_* \kappa(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \beta(s).$$

After making the required modifications, we can write it like this:

$$\begin{aligned}E^* &= (-_*k(s)\cdot_*\langle E,\nu(s)\rangle_*)\cdot_*\tau(s)\\&+_*(k(s)\cdot_*\langle E,\tau(s)\rangle_*)\cdot_*\nu(s)\\&+_*\langle E,\beta(s)\rangle\cdot_*(-_*\kappa(s)+_*\psi^*)\cdot_*\nu(s)\\&+_*\langle E,\nu(s)\rangle\cdot_*(\kappa(s)-_*\psi^*)\cdot_*\beta(s).\end{aligned}$$

When we contrast the top equation with the general equation of the electric field, we can get as follows;

$$\psi^* = \kappa(s).$$

We can display all of these in matrix form as follows. The polarization plane's rotational motion is displayed in this matrix.

$$\begin{pmatrix}\langle E,\nu(s)\rangle_*^*\\ \langle E,\beta(s)\rangle_*^*\end{pmatrix} = \begin{pmatrix}0_* & \kappa(s)\\ -_*\kappa(s) & 0_*\end{pmatrix}\begin{pmatrix}\langle E,\nu(s)\rangle_*\\ \langle E,\beta(s)\rangle_*\end{pmatrix},$$

$$\psi(s) = \int^* \kappa(s)\cdot_* d^*(s).$$

Theorem 2. *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning the multiplicative Frenet frame $\{\tau(s),\nu(s),\beta(s)\}$ and $\tau(s)$ has a constant angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of the unit multiplicative quaternion $a(s,\psi)=\cos_*\psi+_*\sin_*\psi\cdot_*\tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)}=f(s)+_*\cos_*\theta\cdot_*\tau(s)+_*\sin_*\theta\cdot_* a\circledast\nu(s).$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)}=f(s)+_*\cos_*\theta\cdot_*\tau(s)+_*\sin_*\theta\cdot_* R^a_\psi\cdot_*\nu(s).$$

Proof. Assuming that the optic fiber has a definite definition via the multiplicative curve $f(s)$ concerning $\{\tau(s),\nu(s),\beta(s)\}$. So, the polarization vector $E_{\tau(s)}$ respect to multiplicative calculus can be described as follows:

$$\begin{aligned}E &= \cos_*\theta\cdot_*\tau(s)+_*\sin_*\theta\cdot_*\sin_*\psi\cdot_*\nu(s)\\&+_*\sin_*\theta\cdot_*\cos_*\psi\cdot_*\beta(s),\end{aligned}$$

where $\psi = const.$ If we make the necessary calculations, we compute as follows:

$$a(s,\psi)\circledast\nu(s) = \cos_*\psi\cdot_*\nu(s)+_*\sin_*\psi\cdot_*\beta(s).$$

Thus, we can write $E_{\tau(s)}$ multiplicative Rytov curve two forms:

$$E_{\tau(s)} = f(s)+_*\cos_*\theta\cdot_*\tau(s)+_*\sin_*\theta\cdot_* a\circledast\nu(s),$$

and

$$E_{\tau(s)} = f(s)+_*\cos_*\theta\cdot_*\tau(s)+_*\sin_*\theta\cdot_* R^a_\psi\cdot_*\nu(s),$$

that $\psi = const.$

Corollary 1. *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s),\nu(s),\beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi)=\cos_*\psi+_*\sin_*\psi\cdot_*\tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternion approach

$$E_{\tau(s)} = f(s)+_* a\circledast\nu(s),$$

ii) by the multiplicative quaternion approach

$$E_{\tau(s)} = f(s)+_* R^a_\psi.$$

where R^a_ψ is a rotation multiplicative matrix and

$$\psi(s) = \int^* \kappa(s)\cdot_* d^*(s).$$

Corollary 2. *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s),\nu(s),\beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi)=\cos_*\psi+_*\sin_*\psi\cdot_*\tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternion approach

$$E_{\tau(s)} = f(s)+_* a\circledast\nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s)+_* R^a_\psi.$$

Case 2. Assuming the second scenario, there is a constant multiplicative angle between $E(s)$ and $\nu(s)$. We know that the following can be written as an electric field:

$$E^* = \lambda_1\cdot_*\tau(s)+_*\lambda_2\cdot_*\nu(s)+_*\lambda_3\cdot_*\beta(s).$$

After that, if we consider this case's sufficiency, we get,

$$\langle E,\nu(s)\rangle_* = \cos_*\theta,\quad \theta = const.$$

Generally, we know that when inspecting the optic fiber, we can write,

$$\langle E, E\rangle_* = const.$$

On the other hand, ψ and θ are the multiplicative angles, so we get as follows,

$$E = \sin_* \theta \cdot_* \cos_* \psi \cdot_* \tau(s) +_* \cos_* \theta \cdot_* \nu(s) +_* \sin_* \theta \cdot_* \sin_* \psi \cdot_* \beta(s)$$

Next, by taking the last equation's multiplicative derivative, we may get the following result:

$$\begin{aligned} E^* =& (-_*\psi^* \cdot_* \sin_* \psi \cdot_* \sin_* \theta -_* k(s) \cdot_* \cos_* \theta) \cdot_* \tau(s) \\ &+_* (\sin_* \theta \cdot_* \cos_* \psi \cdot_* k(s) \\ &-_* \kappa(s) \cdot_* \sin_* \theta \cdot_* \sin_* \psi) \cdot_* \nu(s) \\ &+_* (\kappa(s) \cdot_* \cos_* \theta +_* \psi^* \cdot_* \cos_* \psi \cdot_* \sin_* \theta) \cdot_* \beta(s). \end{aligned}$$

Generally, we can write electric field with the help of multiplicative Frenet frame elements as follows,

$$\begin{aligned} E =& \langle E, \tau(s)\rangle_* \cdot_* \tau(s) \\ &+_* \langle E, \nu(s)\rangle_* \cdot_* \nu(s) +_* \langle E, \beta(s)\rangle_* \cdot_* \beta(s). \end{aligned}$$

When we take the last equation's multiplicative derivative, we obtain;

$$\begin{aligned} E^* =& (\langle E, \tau(s)\rangle_*^* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s) \\ &+_* (\langle E, \nu(s)\rangle_*^* +_* k(s) \cdot_* \langle E, \tau(s)\rangle_* \\ &-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*) \cdot_* \nu(s) \\ &+_* (\langle E, \beta(s)\rangle_*^* +_* \kappa(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \beta(s). \end{aligned}$$

We can get the following equation by matching and correcting the appropriate values

$$\begin{aligned} E^* =& (-_*\psi^* \cdot_* \langle E, \beta(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s) \\ &+_* (k(s) \cdot_* \langle E, \tau(s)\rangle_* -_* \langle E, \beta(s)\rangle \cdot_* \kappa(s)) \cdot_* \nu(s) \\ &+_* (\kappa(s) \cdot_* \langle E, \nu(s)\rangle_* +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_*) \cdot_* \beta(s). \end{aligned}$$

When we contrast the top equation with the general equation of the electric field, we can get as follows

$$\psi^* = 0_*.$$

We can display all of these in matrix form as follows:. The polarization plane's rotational motion is displayed in this matrix.

$$\begin{pmatrix} \langle E, \nu(s)\rangle_*^* \\ \langle E, \beta(s)\rangle_*^* \end{pmatrix} = \begin{pmatrix} 0_* & 0_* \\ 0_* & 0_* \end{pmatrix} \begin{pmatrix} \langle E, \nu(s)\rangle_* \\ \langle E, \beta(s)\rangle_* \end{pmatrix}.$$

Theorem 3. *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\nu(s)$ has a constant angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \sin_* \psi +_* \cos_* \psi \cdot_* \nu(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve is acquired two perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* R_\psi^a \cdot_* \nu(s).$$

Proof. We suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$. So, the polarization vector $E_{\nu(s)}$ respect to multiplicative calculus can be described as follows;

$$\begin{aligned} E =& \sin_* \theta \cdot_* \cos_* \psi \cdot_* \tau(s) +_* \cos_* \theta \cdot_* \nu(s) \\ &+_* \sin_* \theta \cdot_* \sin_* \psi \cdot_* \beta(s), \end{aligned}$$

where $\psi = const.$ If we make the necessary calculations, we compute as ;

$$a(s,\psi) \circledast \beta(s) = \sin_* \psi \cdot_* \beta(s) +_* \sin_* \psi \cdot_* \tau(s).$$

Thus, we can write $E_{\nu(s)}$ multiplicative Rytov curve two forms;

$$E_{\nu(s)} = f(s) +_* \cos_* \theta \cdot_* \nu(s) +_* \sin_* \theta \cdot_* a \circledast \beta(s),$$

and

$$E_{\nu(s)} = f(s) +_* \cos_* \theta \cdot_* \nu(s) +_* \sin_* \theta \cdot_* R_\psi^a \cdot_* \beta(s),$$

that $\psi = const.$

Corollary 3 *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \cos_* \psi +_* \sin_* \psi \cdot_* \tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two perspectives;*

i) by the multiplicative quaternionic representation

$$E_{\tau(s)} = f(s) +_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* R^a_\psi.$$

Corollary 4 *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \cos_* \psi +_* \sin_* \psi \cdot_* \tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)} = f(s) +_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* R^a_\psi.$$

Case 3. Assuming the third scenario, there is a constant multiplicative angle between $E(s)$ and $\beta(s)$. We know that the expression for an electric field is as follows:

$$E^* = \lambda_1 \cdot_* \tau(s) +_* \lambda_2 \cdot_* \nu(s) +_* \lambda_3 \cdot_* \beta(s)$$

After that, if we consider this case's sufficiency, we get,

$$\langle E, \beta(s)\rangle_* = \cos_* \theta, \quad \theta = const.$$

Generally, we know, that when inspecting the optic fiber, we can write,

$$\langle E, E\rangle_* = \text{const.}$$

On the other hand, ψ and θ are the multiplicative angles, so we get as follows,

$$E = \sin_* \theta \cdot_* \cos_* \psi \cdot_* \tau(s) +_* \sin_* \theta \cdot_* \sin_* \psi \cdot_* \nu(s) +_* \cos_* \theta \cdot_* \beta(s).$$

Generally, we can write electric field with the help of multiplicative Frenet frame elements as follows,

$$E = \langle E, \tau(s)\rangle_* \cdot_* \tau(s) +_* \langle E, \nu(s)\rangle_* \cdot_* \nu(s) +_* \langle E, \beta(s)\rangle_* \cdot_* \beta(s).$$

Next, by taking the last equation's multiplicative derivative, we may get the following result

$$\begin{aligned} E^* &= (\langle E, \tau(s)\rangle^*_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s) \\ &+_* (\langle E, \nu(s)\rangle^*_* +_* k(s) \cdot_* \langle E, \tau(s)\rangle_* \\ &-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*) \cdot_* \nu(s) \\ &+_* (\langle E, \beta(s)\rangle^*_* +_* \kappa(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \beta(s). \end{aligned}$$

We can get the following equation by matching and correcting the appropriate value

$$\begin{aligned} E^* &= (-_*\psi^* \cdot_* \langle E, \nu(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s) \\ &+_* (k(s) \cdot_* \langle E, \tau(s)\rangle_*) +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_* \\ &-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*) \cdot_* \nu(s) \\ &+_* (\langle E, \nu(s)\rangle \cdot_* \kappa(s)) \cdot_* \beta(s). \end{aligned}$$

When we contrast the top equation with the general equation of the electric field, we can get as follows

$$\psi^* = -_* k(s).$$

We can display all of these in matrix form as follows. The polarization plane's rotational motion is displayed in this matrix

$$\begin{pmatrix} \langle E, \tau(s)\rangle^*_* \\ \langle E, \nu(s)\rangle^*_* \end{pmatrix} = \begin{pmatrix} 0_* & -_* k(s) \\ k(s) & 0_* \end{pmatrix} \begin{pmatrix} \langle E, \tau(s)\rangle_* \\ \langle E, \nu(s)\rangle_* \end{pmatrix},$$

$$\psi(s) = -_* \int^* k(s) \cdot_* d^*(s).$$

Theorem 4. *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\tau(s)$ has a constant angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \cos_* \psi +_* \sin_* \psi \cdot_* \tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve is acquired from two parametric perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* R^a_\psi \cdot_* \nu(s).$$

Proof. Let us suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$. So, the polarization vector $E_{\tau(s)}$ respect to multiplicative calculus can be described as follows;

$$E = \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* \sin_* \psi \cdot_* \nu(s) +_* \sin_* \theta \cdot_* \cos_* \psi \cdot_* \beta(s),$$

where $\psi = const$. If we make the necessary calculations, we compute as;

$$a(s,\psi) \circledast \nu(s) = \cos_* \psi \cdot_* \nu(s) +_* \sin_* \psi \cdot_* \beta(s).$$

Thus, we can write $E_{\tau(s)}$ multiplicative Rytov curve two forms;

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* a \circledast \nu(s),$$

and

$$E_{\tau(s)} = f(s) +_* \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* R^a_\psi \cdot_* \nu(s),$$

that $\psi = const.$

Corollary 5 *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \cos_* \psi +_* \sin_* \psi \cdot_* \tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)} = f(s) +_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* R^a_\psi.$$

Corollary 6 *Suppose that $f(s)$ is a multiplicative curve defined as optic fiber concerning $\{\tau(s), \nu(s), \beta(s)\}$ and $\tau(s)$ has a right angle between polarization vector E. So, we can examine the motion of this curve in two ways with the help of the multiplicative quaternion. Firstly, through the agency of $a(s,\psi) = \cos_* \psi +_* \sin_* \psi \cdot_* \tau(s)$ the polarization vector's homothetic motion is calculated and the $E_{\tau(s)}$ multiplicative Rytov curve acquired two parametric perspectives;*

i) by the multiplicative quaternionic approach

$$E_{\tau(s)} = f(s) +_* a \circledast \nu(s),$$

ii) by the multiplicative homotetic motion

$$E_{\tau(s)} = f(s) +_* R^a_\psi.$$

4. EM-trajectories through multiplicative Frenet frame in an optic fiber in 3D Multiplicative space

4.1. Electromagnetic trajectories following a light wave's polarization plane as it passes through an optic fiber if $E_{\tau(s)}$

In section third, first case, the following is how we determined the electric field's multiplicative derivative:

$$\begin{aligned} E^* &= (-_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s) \\ &\quad +_* (k(s) \cdot_* \langle E, \tau(s)\rangle_*) \cdot_* \nu(s) +_* \langle E, \beta(s)\rangle \\ &\quad \cdot_* (-_* \kappa(s) +_* \psi^*) \cdot_* \nu(s) \\ &\quad +_* \langle E, \nu(s)\rangle \cdot_* (\kappa(s) -_* \psi^*) \cdot_* \beta(s). \end{aligned}$$

Then, we used the Lorentz force equation and matched obtaining the conclusion in the first case, we get as follows

$$\begin{aligned} \langle \phi(E), \tau(s)\rangle_* &= -_* k(s) \cdot_* \langle E, \nu(s)\rangle_* = -_* \langle \phi(\tau(s)), E\rangle_* \\ \langle \phi(E), \nu(s)\rangle_* &= k(s) \cdot_* \langle E, \tau(s)\rangle_* \\ &\quad +_* (-_* \kappa(s) +_* \psi^*) \cdot_* \langle E, \beta(s)\rangle \\ \langle \phi(E), \beta(s)\rangle_* &= \langle E, \nu(s)\rangle \cdot_* (\kappa(s) -_* \psi^*). \end{aligned}$$

We can write the results obtained from the last equation in the following matrix form

$$\begin{pmatrix} \phi(\tau(s)) \\ \phi(\nu(s)) \\ \phi(\beta(s)) \end{pmatrix} = \begin{pmatrix} 0_* & k(s) & 0_* \\ -_* k(s) & 0_* & \kappa(s) -_* \psi^* \\ 0_* & -_* \kappa(s) +_* \psi^* & 0_* \end{pmatrix} \begin{pmatrix} \tau(s) \\ \nu(s) \\ \beta(s) \end{pmatrix}.$$

On the other hand, we know that if we can make the necessary calculations on the light of the Lorentz force, we can get a magnetic field equation. So we assume that B is a magnetic field and then we get the following;

$$B(s) = (\kappa(s) -_* \psi^*) \cdot_* \tau(s) +_* k(s) \cdot_* \beta(s).$$

Theorem 5. *Assume E has a multiplicative constant angle with $\tau(s)$. Thus, the multiplicative magnetic field B of the $E_{\tau(s)}$ trajectories according to E implies;*

$$B(s) = (\kappa(s) -_* \psi^*) \cdot_* \tau(s) +_* k(s) \cdot_* \beta(s).$$

Proof. Generally, we can write the multiplicative magnetic vector as follows;

$$B = a_1 \cdot_* \tau(s) +_* a_2 \cdot_* \nu(s) +_* a_3 \cdot_* \beta(s).$$

With the help of the Lorentz force equation, we can obtain the following;

$$\begin{aligned} B \times_* \tau(s) &= -_* a_2 \cdot_* \beta(s) +_* a_3 \cdot_* \nu(s) = \phi(\tau(s)), \\ B \times_* \nu(s) &= a_1 \cdot_* \beta(s) -_* a_3 \cdot_* \tau(s) = \phi(\nu(s)), \\ B \times_* \beta(s) &= -_* a_1 \cdot_* \nu(s) +_* a_2 \cdot_* \tau(s) = \phi(\beta(s)). \end{aligned}$$

Then, we can find $a_1 = \kappa(s) -_* \psi^*$, $a_3 = k(s)$.

The proof is completed. □

4.2. Electromagnetic trajectories following a light wave's polarization plane as it passes through an optic fiber if $E_{\nu(s)}$

In section third, second case, the following is how we determined the electric field's multiplicative derivative:

$$E^* = (-_*\psi^* \cdot_* \langle E, \beta(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s)$$
$$+_* (k(s) \cdot_* \langle E, \tau(s)\rangle_* -_* \langle E, \beta(s)\rangle \cdot_* \kappa(s)) \cdot_* \nu(s)$$
$$+_* (\kappa(s) \cdot_* \langle E, \nu(s)\rangle_* +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_*) \cdot_* \beta(s).$$

Then, we used the Lorentz force equation and matched obtaining the conclusion in the second case, we get as follows

$$\langle \phi(E), \tau(s)\rangle_* = -_*\psi^* \cdot_* \langle E, \beta(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*$$
$$= -_*\langle \phi(\tau(s)), E\rangle_*$$
$$\langle \phi(E), \nu(s)\rangle_* = k(s) \cdot_* \langle E, \tau(s)\rangle_* -_* \langle E, \beta(s)\rangle \cdot_* \kappa(s)$$
$$\langle \phi(E), \beta(s)\rangle_* = \kappa(s) \cdot_* \langle E, \nu(s)\rangle_* +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_*.$$

We can write the results obtained from the last equation in the following matrix form

$$\begin{pmatrix} \phi(\tau(s)) \\ \phi(\nu(s)) \\ \phi(\beta(s)) \end{pmatrix} = \begin{pmatrix} 0_* & k(s) & \psi^* \\ -_*k(s) & 0_* & \kappa(s) \\ -_*\psi^* & -_*\kappa(s) & 0_* \end{pmatrix} \begin{pmatrix} \tau(s) \\ \nu(s) \\ \beta(s) \end{pmatrix}$$

On the other hand, we know that if we can make the necessary calculations on the light of the Lorentz force, we can get a magnetic field equation. So we assume that B is a magnetic field and then we get the following;

$$B(s) = \kappa(s) \cdot_* \tau(s) -_* \psi^* \cdot_* \nu(s) +_* k(s) \cdot_* \beta(s).$$

Theorem 6. *Assume E has a multiplicative constant angle with $\nu(s)$. Thus, the multiplicative magnetic field B of the $E_{\tau(s)}$ trajectories according to E implies;*

$$B(s) = \kappa(s) \cdot_* \tau(s) -_* \psi^* \cdot_* \nu(s) +_* k(s) \cdot_* \beta(s).$$

4.3. Electromagnetic trajectories following a light wave's polarization plane as it passes through an optic fiber if $E_{\beta(s)}$

In section third, third case, The following is how we determined the electric field's multiplicative derivative:

$$E^* = (-_*\psi^* \cdot_* \langle E, \nu(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*) \cdot_* \tau(s)$$
$$+_* (k(s) \cdot_* \langle E, \tau(s)\rangle_*) +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_*$$
$$-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*) \cdot_* \nu(s)$$
$$+_* (\langle E, \nu(s)\rangle \cdot_* \kappa(s)) \cdot_* \beta(s).$$

Then, we used the Lorentz force equation and matched obtaining the conclusion in the third case, we get as follows

$$\langle \phi(E), \tau(s)\rangle_* = -_*\psi^* \cdot_* \langle E, \nu(s)\rangle_* -_* k(s) \cdot_* \langle E, \nu(s)\rangle_*$$
$$\langle \phi(E), \nu(s)\rangle_* = k(s) \cdot_* \langle E, \tau(s)\rangle_*) +_* \psi^* \cdot_* \langle E, \tau(s)\rangle_*$$
$$-_* \kappa(s) \cdot_* \langle E, \beta(s)\rangle_*$$
$$\langle \phi(E), \beta(s)\rangle_* = \langle E, \nu(s)\rangle \cdot_* \kappa(s).$$

We can write the results obtained from the last equation in the following matrix form

$$\begin{pmatrix} \phi(\tau(s)) \\ \phi(\nu(s)) \\ \phi(\beta(s)) \end{pmatrix} =$$
$$\begin{pmatrix} 0_* & \psi^* +_* k(s) & 0_* \\ -_*\psi^* -_* k(s) & 0_* & \kappa(s) \\ 0_* & -_*\kappa(s) & 0_* \end{pmatrix} \begin{pmatrix} \tau(s) \\ \nu(s) \\ \beta(s) \end{pmatrix}.$$

On the other hand, we know that if we can make the necessary calculations on the light of the Lorentz force, we can get a magnetic field equation. So we assume that B is a magnetic field and then we get the following;

$$B(s) = (\kappa(s) -_* \psi^*) \cdot_* \tau(s) +_* k(s) \cdot_* \beta(s).$$

Theorem 7. *Assume E has a multiplicative constant angle with $\tau(s)$. Thus, the multiplicative magnetic field B of the $E_{\tau(s)}$ trajectories according to E implies;*

$$B(s) = (\kappa(s) -_* \psi^*) \cdot_* \tau(s) +_* k(s) \cdot_* \beta(s).$$

5. Physical interpretations

Case 1. Assume that the multiplicative unit tangent vector τ and the polarization vector E form a constant angle. Next, we calculate the magnetic field B and the polarization vector E in the following manner:

$$E = \cos_* \theta \cdot_* \tau(s) +_* \sin_* \theta \cdot_* \sin_* \left(\int^* \kappa(s) \cdot_* d^*(s) \right)$$
$$\cdot_* \nu(s) +_* \sin_* \theta \cdot_* \cos_* \left(\int^* \kappa(s) \cdot_* d^*(s) \right) \cdot_* \beta(s),$$
$$B(s) = (\kappa(s) -_* \psi^*) \cdot_* \tau(s) +_* k(s) \cdot_* \beta(s).$$

Therefore, in the following three situations, we can analyze the behavior of a charged particle in the Killing magnetic field B:

(I). The charged particle moves parallel to the magnetic field if E is parallel to the magnetic field B.

(II). If $\langle E, B\rangle_* = 0_*$, then

$$(\kappa(s) -_* \psi^*) \cdot_* \cos_* \theta$$
$$+_* \sin_* \theta \cdot_* \cos_* \left(\int^* \kappa(s) \cdot_* d^*(s) \right) \cdot_* k(s) = 0_*,$$

This gives $k(s) = 0_*$ and $\kappa(s) = 0_*$, the electromagnetic curve is a multiplicative line. As a result, we may state that the charged particle follows a multiplicative line trajectory in the Killing magnetic field B.

(III). If $\langle E, B\rangle_* = \mathit{const.}$, then

$$(\kappa(s) -_* \psi^*) \cdot_* \cos_* \theta +_* \sin_* \theta \cdot_* \cos_* \left(\int^* \kappa(s) \cdot_* d^*(s)\right) \cdot_* k(s) = \mathit{const.},$$

This gives $k(s) = const.$ and $\kappa(s) = \mathit{const.}$, namely the electromagnetic curve is a multiplicative helix (see, Fig. 2). Therefore, we can say that the charged particle follows a multiplicative helical trajectory in the Killing magnetic field B.

Case 2. Assume that the multiplicative unit normal vector ν and the polarization vector E form a constant angle. Next, we calculate the magnetic field B and the polarization vector E in the following manner:

$$E = \sin_* \theta \cdot_* \cos_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \tau(s) +_* \cos_* \theta \cdot_* \nu(s) +_* \sin_* \theta \cdot_* \sin_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \beta(s),$$

$$B(s) = \kappa(s) \cdot_* \tau(s) -_* \psi^* \cdot_* \nu(s) +_* k(s) \cdot_* \beta(s).$$

Therefore, in the following three situations, we can analyze the behavior of a charged particle in the Killing magnetic field B:

(i). The charged particle moves parallel to the magnetic field if E is parallel to the magnetic field B.

(ii). If $\langle E, B\rangle_* = 0_*$, then

$$\sin_* \theta \cdot_* \cos_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \kappa(s) -_* \psi^* \cdot_* \cos_* \theta +_* k(s) \cdot_* \sin_* \theta \cdot_* \sin_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* = 0_*.$$

Namely,

$$\sin_* \theta \cdot_* \left(\cos_*(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \kappa(s)) +_* \sin_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* k(s) = 0_*$$

So, we get $\kappa(s)/k(s)* = \mathit{const.}$ We can say that, the trajectory is a multiplicative general helix.

(iii). If $\langle E, B\rangle_* = \text{const.}$, then

$$\sin_* \theta \cdot_* \cos_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \kappa(s) -_* \psi^* \cdot_* \cos_* \theta +_* k(s) \cdot_* \sin_* \theta \cdot_* \sin_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* = \text{const.},$$

Namely,

$$\sin_* \theta \cdot_* \cos_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* \kappa(s)) +_* \sin_* \left(\int^* 0_* \cdot_* d^*(s)\right) \cdot_* k(s) = \mathit{const.}$$

So, we get $\kappa(s) = \sin_*(\int^* 0_* \cdot_* d^*(s))$ and $k(s) = \cos_*(\int^* 0_* \cdot_* d^*(s))$. We can say that, the trajectory is a multiplicative general helix.

Case 3. Assume that the multiplicative unit tangent vector β and the polarization vector E form a constant angle. Next, we calculate the magnetic field B and the polarization vector E in the following manner:

$$E = -_* \sin_* \theta \cdot_* \cos_* \left(\int^* k(s) \cdot_* d^*(s)\right) \cdot_* \tau(s) -_* \sin_* \theta \cdot_* \sin_* \left(\int^* k(s) \cdot_* d^*(s)\right) \cdot_* \nu(s) +_* \cos_* \theta \cdot_* \beta(s),$$

$$B(s) = \kappa(s) \cdot_* \tau(s) +_* (\psi^* +_* k(s)) \cdot_* \beta(s).$$

Therefore, in the following three situations, we can analyze the behavior of a charged particle in the Killing magnetic field B:

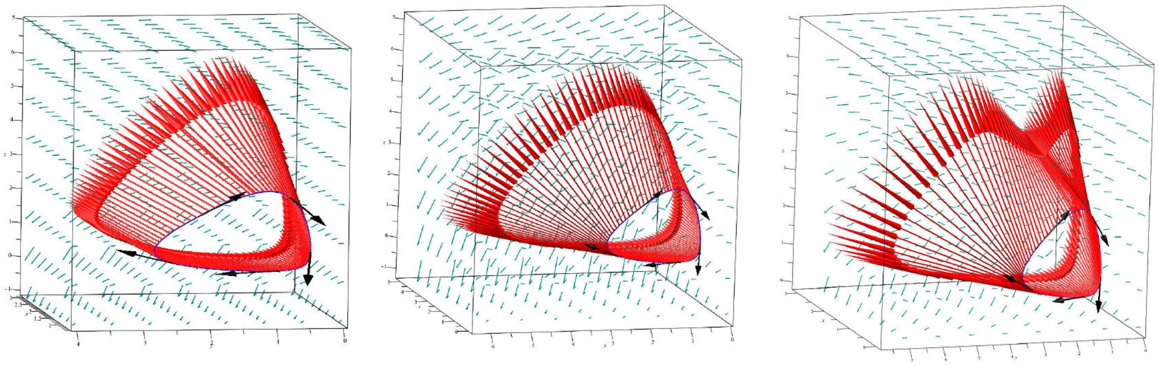
(i). The charged particle moves parallel to the magnetic field if E is parallel to the magnetic field B.

(ii). If $\langle E, B\rangle_* = 0_*$, then

$$-_* \sin_* \theta \cdot_* \cos_* \left(\int^* k(s) \cdot_* d^*(s)\right) \cdot_* \kappa(s)) +_* (\psi^* +_* k(s)) \cdot_* \cos_* \theta = 0_*.$$

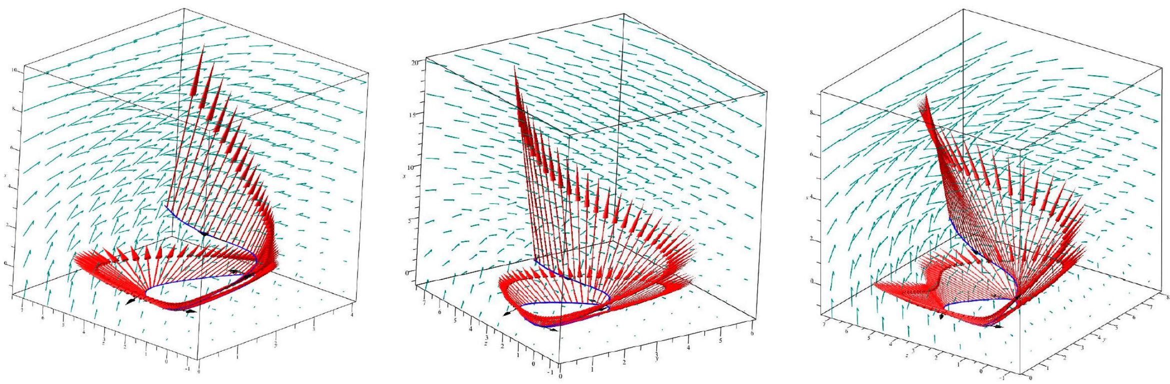
Namely, $\sin_* \theta \cdot_* \cos_*(\int^* k(s) \cdot_* d^*(s)) \cdot_* \kappa(s)) = 0_*$. Thus, we can say that $\kappa(s) = 0_*$ and the trajectory is multiplicative planar curve or $\cos_*(\int^* k(s) \cdot_* d^*(s)) = 0_*$, *i.e.* $k(s) = \pi/2*$ where the angle $\pi/2*$ is a multiplicative right angle, namely in Euclidean sense it is equal to $e^{\cos(\log[\pi/2])}$.

(iii). If $\langle E, B\rangle_* = \mathit{const.}$, then $-_* \sin_* \theta \cdot_* \cos_*(\int^* k(s) \cdot_* d^*(s))) \cdot_* \kappa(s)) +_* (\psi^* +_* k(s)) \cdot_* \cos_* \theta = \mathit{const.}$ Then, we obtain that $\theta_* = e^{\arcsin(logcos \int k(s) ds)}$.



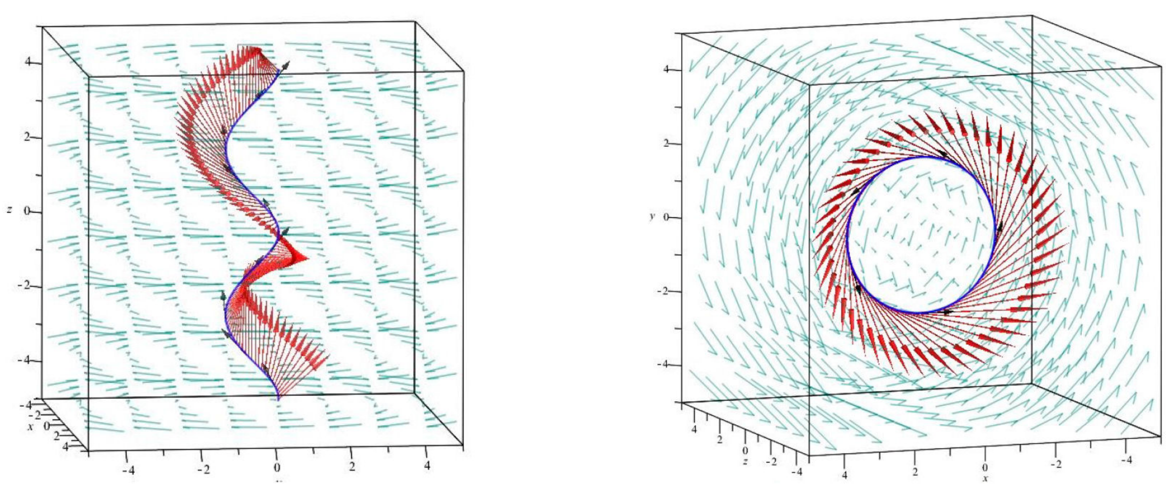
a) $\langle E^*, \tau^* \rangle_* = const.$ b) $\langle E^*, \nu^* \rangle_* = const.$ c) $\langle E^*, \beta^* \rangle_* = const.$

FIGURE 1. Three cases of propagation of the polarized light along the multiplicative circular fiber optic.



a) $\langle E^*, \tau^* \rangle_* = const.$ b) $\langle E^*, \nu^* \rangle_* = const.$ c) $\langle E^*, \beta^* \rangle_* = const.$

FIGURE 2. Three cases of propagation of the polarized light along the multiplicative helical fiber optic.



a) $\langle E, t \rangle = const.$ b) $\langle E, t \rangle = const.$

FIGURE 3. a) Propogation of the polarized light along the helical optic fiber and b) circular fiber in Euclidean space.

6. Example

Let us take the multiplicative circle $\beta_1(s) = (e^{\cos(\log s)}, e^{\sin(\log s)})$, we can say that β_1 is an electromagnetic curve. The charged particle motion along the curve β_1 in the related magnetic field and the three cases of propagation of the linearly polarized light in the optic fiber along the curve β_1 obtained as in Fig. 1.

Next, if we take the multiplicative helix $\beta_2(s) = (e^{\log s}, e^{\cos(\log s)}, e^{\sin(\log s)})$, we can say that β_2 is an electromagnetic curve. The charged particle motion along the curve β_2 in the related magnetic field and the three cases of propagation of the linearly polarized light in the optic fiber along the curve β_2 obtained as in Fig. 2.

7. Discussion and conclusion

Behavior of the linearly polarized light wave's geometric phase interactions throughout this effective optical fiber. In geometry, non-Newtonian calculus is advancing quite quickly. In this work, quaternion algebra has been used to simplify and make more comprehensible equations that were previously generated in a convoluted manner. The trajectories were obtained in a different way than the Euclidean since the work was done in a different space with a different metric, meaning that the mathematical tools were different. In order to understand the trajectories-which are not precisely circular-it is crucial to examine various space formations. With the new analysis approach and in this new environment, a different visualization may be obtained thanks to the trajectories we found when using the Newtonian approach [47,48]. The recognized helix curve and the equation that was computed as a helix curve are not the same [47,48]. This is because there is a difference in the circle picture defined in this space [45,47].

The case for Euclidean space when the tangent vector of the curve associated with the polarized light waves along the optical fiber is perpendicular to the electric field is shown visually in Fig. 3. It was obtained for the circular and helix curves visualized in Sec. 6. In this way, it is seen how the change in metric and space changes the propagation of the light wave. The Rytov curve, the path taken by the linearly polarized light wave as it moves along the optical fiber, can be described quaternionically and as homothetic motion thanks to recently defined multiplicative quaternions.

Acknowledgment

The authors would like to express their sincere gratitude to the referees for the valuable suggestions to improve the paper.

---

1. M. Grossman, R. Katz, Non-Newtonian Calculus. 1st. ed. (Lee Press. Pigeon Cove, Massachussets, 1972).
2. M. Grossman, An introduction to non-newtonian calculus. *Int. J. Math. Educ. Sci. Technol.* **10**(4) (1979) 525-528, https://doi.org/10.1080/0020739790100406.
3. M. Grossman, Bigeometric Calculus: A System with a Scale-Free Derivative. (Archimedes Foundation, Massachussets, 1983).
4. D. Stanley, A multiplicative calculus. *PRIMUS*, **9**(4) (1999) 310-326, https://doi.org/10.1080/10511979908965937.
5. D. Aniszewska, Multiplicative Runge-Kutta Methods. *Nonlinear Dynamics*, 50 (2007) 262-272 https://doi.org/10.1007/s11071-006-9156-3.
6. D. Aniszewska, M. Rybaczuk, Analysis of the multiplicative Lorentz system. *Chaos, Solitons and Fractals*, 25 (2005) 79-90 https://doi.org/10.1016/j.chaos.2004.09.060.
7. A. Bashirov, M. Riza, On complex multiplicative differentiation. *TWMS Journal of Applied and Engineering Mathematics*, **1**(1) (2011) 75-85.
8. A. E. Bashirov, E. M. Kurpınar, A. Ozyapıcı, Multiplicative calculus and its applications. *J. Math. Anal. Appl.* **337**(1) (2008) 36-48, https://doi.org/10.1016/j.jmaa.2007.03.081.
9. L. Florack, H. van Assen, Multiplicative calculus in biomedical image and analysis. *J. Math. Im. Vis.* 42 (2012) 64-75, https://doi.org/10.1007/s10851-011-0275-1.
10. M. Mora, F. Cordova-Lepe, R. Del-Valle R, 2012 A non-Newtonian gradient for contour detection in images with multiplicative noise. *Pattern Recognition Letters*, **33**(10) (2012) 1245-1256, https://doi.org/10.1016/j.patrec.2012.02.012.
11. K. Boruah, B. Hazarika, Application of geometric calculus in numerical analysis and difference sequence spaces. *J. Math. Anal. Appl.* **449**(2) (2017) 1265-1285, https://doi.org/10.1016/j.jmaa.2016.12.066.
12. K. Boruah, B. Hazarika, G-calculus. *TWMS J. Appl. Eng. Math.* **8**(1) (2018) 94-105.
13. S. G. Georgiev, Multiplicative Differential Geometry. 1st. ed., (Chapman and Hall/CRC., New York 2022).
14. S. G. Georgiev, K. Zennir, Multiplicative Differential Calculus. 1st. ed., (Chapman and Hall/CRC., New York, 2022).
15. S. G. Georgiev, K. Zenni, A. Boukarou, Multiplicative Analytic Geometry. 1st ed., (Chapman and Hall/CRC., New York, 2022).
16. S. Aslan, M. Bekar, Y. Yaylı, 2023 Geometric 3-space and multiplicative quaternions. *Int. J. of Geo. Methods in Modern Physics.* **20**(9) (2023) 2350151, https://doi.org/10.1142/S0219887823501517.
17. M. E. Aydin, A. Has, B. Yılmaz, A non-Newtonian approach in differential geometry of curves: multiplicative rectifying curves. *Bull. Korean Math. Soc.* **61**(3) (2024) 849-866, https://doi.org/10.4134/BKMS.b230491.
18. S. K. Nurkan, Ä°. GÃ$\frac{1}{4}$rgil, M. K. Karacan, Vector properties of geometric calculus. *Math. Meth. Appl. Sci.* (2023) 1-20, https://doi.org/10.1002/mma.9525.
19. H. Ceyhan, Z. Özdemir, İ Gök, Multiplicative generalized tube surfaces with multiplicative quaternions algebra. *Mathematical Methods in the Applied Sciences*, **47**(11) (2024) 9157-9168 http://doi.org/10.1002/mma.10065.
20. Z. Özdemir, H. Ceyhan, Multiplicative hyperbolic split quaternions and generating geometric hyperbolical rotation matrices, *Applied Mathematics and Computation*, **479** (2024) 128862, https://doi.org/10.1016/j.amc.2024.128862.

21. S. M. Rytov, 1938 Dokl. Akad. Nauk. SSSR 18, 263, reprinted in B. Markovski, S. I. Vinitsky, ((eds) Topological Phases in Quantum Theory, World Scientific, Singapore, 1989).
22. Y. A. Kravtsov, Y. I. Orlov, Geometrical Optics of Inhomogeneous Medium. (Nauka, Moscow, 1980; Springer-Verlag, Berlin, 1990).
23. J. N. Ross, 1984 The rotation of the polarization in low briefrigence monomode optical fibres due to geometric effects. *Opt. Quantum Electron*. **16**(5) (1984) 455, https://doi.org/10.1007/BF00619638.
24. M. V. Berry, Quantal phase factors accompanying adiabatic changes. (Proc. Roy. Soc. London. A 392, 45, 1984).
25. A. Comtet, On the Landau Hall levels on the hyperbolic plane. *Ann. Phys.* 173, (1987) 185, https://doi.org/10.1016/0003-4916(87)90098-4.
26. M. Kugler, S. Shtrikman, 1988 Berry's phase, locally inertial frames, and classical analogues. *Phys. Rev. D*. **37**(4), (1988) 934, https://link.aps.org/doi/10.1103/PhysRevD.37.934.
27. T. Sunada, Magnetic flows on a Riemann surface. (In Proceedings of the KAIST Mathematics Workshop: Analysis and Geometry, Taejeon, Korea 1993).
28. T. Adachi, Kahler magnetic on a complex projective space. *Proc. Jpn. Acad. Ser. A Math. Sci.* **70**(1), (1994) 12-13, DOI: 10.3792/pjaa.70.12.
29. T. Adachi, Kahler magnetic flow for a manifold of constant holomorphic sectional curvature. *Tokyo J. Math.* **18**(2) (1995) 473-483, DOI: 10.3836/tjm/1270043477.
30. M. Barros, General helices and a theorem of Lancret. *Proc. Amer. Math. Soc.* **125**(5) (1997) 1503-1509, https://api.semanticscholar.org/CorpusID:17451334.
31. E. M. Frins, W. Dultz, Rotation of the polarization plane in optical fibers. *J. Lightwave Technol.* **15**(1), (1997) 144, doi=10.1109/50.552122.
32. M. Barros, A. Romero, J. L. Cabrerizo, M. Fernández, The Gauss-Landau–Hall problem on Riemanniansurfaces. *J. Math. Phys*. 46 (2005) 112905, https://doi.org/10.1063/1.2136215.
33. M. Barros, J. L. Cabrerizo, M. Fernández, A. Romero, Magnetic vortex filament flows. *J. Math. Phys*. 48 (2007) 1-27, https://doi.org/10.1063/1.2767535.
34. J. L. Cabrerizo, M. Fernandez, J. S. Gomez, 2009 On the existence of almost contact structure and the contact magnetic field. *Acta Math. Hung.* **125** (2009) 191-199, https://doi.org/10.1007/s10474-009-9005-1.
35. J. L. Cabrerizo, M. Fernandez, J. S. Gomez, The contact magnetic flow in 3D Sasakian manifolds. *J. Phys. A: Math. Theor.* **42** (2009) 195201, https://dx.doi.org/10.1088/1751-8113/42/19/195201.
36. I. I. Satija, R. Balakrishnan, Geometric phases in twisted strips. *Phys. Lett. A.* **373**(39) (2009) 3582, https://doi.org/10.1016/j.physleta.2009.07.083.
37. S. L. Druta-Romaniuc, M. I. Munteanu, Magnetic Curves corresponding to Killing magnetic fields in E^3. *J. Math. Phys.* **52** (2011) 113506, https://doi.org/10.1063/1.3659498.
38. O. Yamashita, Effect of the geometrical phase shift on the spin and orbital angular momenta of light traveling in a coiled optical fiber with optical activity. *Opt. Commun.* **285**(18) (2012) 3740-3747 https://doi.org/10.1016/j.optcom.2012.05.013.
39. O. Yamashita, Geometrical phase shift of the extrinsic orbital angular momentum density of light propagating in a helically wound optical fiber. *Opt. Commun.* **285**(13-14) (2012) 3061-3065 https://doi.org/10.1016/j.optcom.2012.02.041.
40. J. L. Cabrerizo, Magnetic fields in 2D and 3D sphere. *J. Nonlinear Math. Phys.* **20**(3) (2013) 440-450 https://doi.org/10.1080/14029251.2013.855052.
41. S. L. Druta-Romaniuc, M. I. Munteanu, Killing magnetic curves in a Minkowski 3-space. *Nonlinear Anal. RealWorld Appl.* 14 (2013) 383-396 https://doi.org/10.1016/j.nonrwa.2012.07.002.
42. Z. Bozkurt, İ. Gök, Y. Yaylı F. N. Ekmekci, A new approach for magnetic curves in 3D Riemannian manifolds. *J. Math. Phys.* **55**(5) (2014) 053501 https://doi.org/10.1063/1.4870583.
43. Z. Özdemir, İ. Gök, Y. Yaylı F. N. Ekmekci, Notes on Magnetic Curves in 3D semi-Riemannian Manifolds. *Turk. J. Math.* **39**(3) (2015) 412-426 https://doi.org/10.3906/mat-1408-31.
44. T. Körpinar, R. C. Demırkol, Electromagnetic curves of the linearly polarized light wave along an optical fiber in a 3D semi-Riemannian manifold. *Journal of Modern Optics*, **66**(8) (2019) 857-867, https://doi.org/10.1080/09500340.2019.1579930.
45. H. Ceyhan, Z. Özdemir, İ Gök, F. N. Ekmekci, 2020 Electromagnetic Curves and Rotation of the Polarization Plane through Alternative Moving Frame. *European Physical Journal Plus* **135**, (2020) https://doi.org/10.1140/epjp/s13360-020-00881-z.
46. T. Körpinar, R. C. Demırkol, Electromagnetic curves of the linearly polarized light wave along an optical fiber in a 3D Riemannian manifold with Bishop equations. *Optik - International Journal for Light and Electron Optics.* **200**, (2020) 163334, https://doi.org/10.1016/j.ijleo.2019.163334.
47. Z. Özdemir, A New Calculus for the Treatment of Rytov's Law in the Optical Fiber. *Optik-International Journal for Light and Electron Optics.* **216** (2020) 164892, https://doi.org/10.1016/j.ijleo.2020.164892.
48. H. Ceyhan, Z. Özdemir, İ. Gök, F. N. Ekmekci, A Geometric Interpretation of Polarized Light and Electromagnetic Curves Along an Optical Fiber with Surface Kinematics. *Mediterr. J. Math*. **19** (2022) 265, https://doi.org/10.1007/s00009-022-02160-w.