Exact chirped solutions, stability analysis, chaotic behaviors and dynamical properties of the nonlinear Schrödinger equation with anti-cubic law nonlinearity

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In this paper, the dynamical properties of the nonlinear Schrödinger equation with anti-cubic law nonlinearity is studied. By using the trial equation method and the complete discrimination system for polynomial method, the exact chirped solutions of the equation are obtained, and the parametric stability of these modes is analyzed. Finally, we study the chaotic behaviors of the equation with perturbation terms.

Keywords: The nonlinear Schrödinger equation; dynamical properties; the trial equation method; chaotic behaviors; parameter stability.

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1. Introduction

The nonlinear Schrödinger equation (NLSE) is a very classical physical model, which plays an important role in physics and other related fields. It is also considered as the pulse propagation equation in optical communication [1–6]. Under certain conditions, optical solitons can be transmitted in optical fibers over long distances, which completely gets rid of the limitation of optical fiber dispersion on transmission rate and communication capacity [7–9]. And it can achieve all-optical communication, without the conversion between light and electricity, playing a huge role in the new generation of research communication technology [10,11]. With the development of research, a growing number of researchers begin to research optical solitons. In order to extract these optical solitons, the unified solver method [12], F-expansion method [13, 14], and extended hyperbolic function method [15] are applied to obtain dark, bright, periodic singular and periodic soliton solutions, which enrich the understanding of optical solitons.

In the NLSE, the chirped soliton is a special solitary wave solution. A soliton is a wave packet that propagates through a medium and maintains its shape and velocity without attenuation. Then the chirp describes the frequency modulation of the soliton in time [16–18]. Specifically, the frequency of solitons can vary with time. This frequency change can be linear or non-linear. Chirped solitons are formed when the frequency change is nonlinear and very fast [19,20]. Chirped solitons have important applications in nonlinear optics and ultrafast optics. For example, in the optical communication and optical storage, chirped solitons can enable information transmission and processing. At the same time, there are many other nonlinear effects and interactions in nonlinear media [21–23]. However, it is difficult to obtain the chirped solution in mathematics.

In this paper, we research the nonlinear Schrödinger equation with anti-cubic law nonlinearity [24]

$$i(-\delta\phi_x+\phi_t)+\alpha_1\phi_{xx}+\beta_1\phi_{xt} + \lambda F[(|\phi|)^2]\phi+\frac{\gamma|\phi|_{xt}\phi}{|\phi|}=0, \qquad (1)$$

where ϕ is a complex-valued function and represents the profile of a complex wave; λ and γ are the coefficients of nonlinear terms; δ is the coefficient of inter-modal dispersion; α_1 and β_1 denote the group velocity dispersion coefficient and the spatio-temporal dispersion coefficient; F is a real function. And $F(|\phi|)$ has a lot of laws, such as anti-cubic law nonlinearity, kerr law nonlinearity and parabolic law nonlinearitya. Here, we only discuss the anti-cubic law nonlinearity,

$$F(|\phi|)=\frac{b_1}{|\phi|}+b_2|\phi|+b_3|\phi|^2, \qquad (2)$$

where b_1 b_2 and b_3 are constants.

The nonlinear Schrödinger equation with anti-cubic law nonlinearity is a commonly used mathematical model in the field of nonlinear optics. In quantum optics, it can be used to study the dynamics of optical pulse propagation in nonlinear media such as optical fibers or waveguides [25,26]. The use of optical fiber as a nonlinear medium began in 1970, when Kapron *et al*. [27] found that the loss of optical fiber could be reduced to less than 20 dB/km. In 1972, Stolen *et al*. [28] observed stimulated Raman radiation of visible optical in glass fiber waveguides, and subsequently investigated other nonlinear effects. The anti-cubic nonlinearity in the equation explains the interaction between the optical pulse and its propagating medium, and this nonlinearity can lead to interesting phenomena such as soliton formation, self-focusing, and self-phase modulation [29–31]. In fiber and laser communications, optical signals can be disturbed by randomness and noise, and the nonlinear Schrodinger equation with inverse cubic law nonlinearity can help us build mathematical models to reduce the influence of interference [32–34].

The exact solution of this equation can simulate and analyze the behavior of quantum systems under the influence of randomness, and plays an important role in explaining complex dynamic phenomena [35–39]. In addition, there are a lot of useful methods and theories to solve exact solutions, including Riemann-Hilbert formulation [40], Exp-function method [41–43], semi-inverse variational principle [44] and Kudryashov's method [45] and so on.

Akram *et al*. [24] used traveling wave transforation and the proposed method [46, 47] to extract many soliton solutions of Eq. (1), including periodic soliton, bell shaped soliton, dark soliton, and singular soliton solutions. Compared with them, we use the more general complex envelope traveling wave transformation, namely the chirped wave transformation, and the trial equation method advanced by Liu [48–58] to study Eq. (1). Then, we obtain a variety of solutions including periodic solutions, elliptic function solutions, rational solutions, and trigonometric function solutions. In this paper, we use the trial equation method to analyze the equation qualitatively, give the dynamical properties of the NLSE, and prove the existence of soliton solutions and periodic solutions. We can use the complete discrimination system for polynomial method to judge the type of solution in advance according to the measured physical parameters. We also show the chaotic behaviors of the NLSE by adjusting its disturbance term [59–64].

In this paper, the general structure is as follows. In Sec. 2, we obtained the standard form of the equation by mathematical method. In Sec. 3, we predict the solution of the equation by qualitative analysis. In Sec. 4, we obtain the exact solutions and chirps of the NLSE, and analyze the stability of the parameters. In Sec. 5, we obtain the exact chirped solutions with specific parameters. In Sec. 6, we use three perturbation terms to represent chaotic motion. In Sec. 7, a summary of the article is given.

2. Mathematical analysis of Eq. (1)

We choose the chirp wave transformation [18] to solve Eq.(1)

$$\phi(x,t) = \rho(\xi) e^{i(\chi(\xi)-kx)}, \quad \xi = t - \mu x, \tag{3}$$

where $\rho(\xi)$ and $\chi(\xi)$ are real functions, and $\mu = 1/v$ is the inverse velocity. The chirp is as follows [23]

$$\delta\omega(t,x) = -\frac{\partial}{\partial t}(\chi(\xi) - kx) = -\chi'(\xi). \tag{4}$$

Substituting Eq. (2) and Eq. (3) into Eq. (1) to separate the real and imaginary part, we get

$$\begin{aligned}(-1 - \delta\mu - 2\alpha_1\mu k + \beta_1 k)\chi'\rho - (\delta k + \alpha_1 k^2)\rho \\ + (\alpha_1\mu^2 - \mu\beta_1 - \mu\lambda)\rho'' + (\mu\beta_1 - \alpha_1\mu^2)(\chi')^2\rho \\ + \lambda b_1\rho^{-1} + \lambda b_2\rho^3 + \lambda b_3\rho^5 = 0,\end{aligned} \tag{5}$$

and

$$\begin{aligned}(\mu\delta + 1 + 2\mu\alpha_1 k - k\beta_1)\rho' + (2\mu^2\alpha_1 - 2\mu\beta_1)\chi'\rho' \\ + (\mu^2\alpha_1 - \mu\beta_1)\chi''\rho = 0.\end{aligned} \tag{6}$$

Multiplying both sides of Eq.(6) by ρ and then integrate ρ to get

$$\chi' = \frac{s_0}{\mu^2\alpha_1 - \mu\beta_1}\rho^{-2} - \frac{\mu\delta + 1 + 2\mu\alpha_1 k - k\beta_1}{2\mu^2\alpha_1 - 2\mu\beta_1}. \tag{7}$$

Thus, the chirp takes the following form

$$\delta\omega = -\frac{s_0}{\mu^2\alpha_1 - \mu\beta_1}\rho^{-2} + \frac{\mu\delta + 1 + 2\mu\alpha_1 k - k\beta_1}{2\mu^2\alpha_1 - 2\mu\beta_1}. \tag{8}$$

Denote

$$A = \frac{s_0}{\mu^2\alpha_1 - \mu\beta_1}, B = \frac{\mu\delta + 1 + 2\mu\alpha_1 k - k\beta_1}{2\mu^2\alpha_1 - 2\mu\beta_1}. \tag{9}$$

Substituting Eq. (7) into Eq. (5), we get

$$\begin{aligned}(\alpha_1\mu^2 - \mu\beta_1 - \mu\lambda)\rho^3\rho'' + (\mu\beta_1 A^2 - \alpha_1\mu^2 A^2) \\ + (-A - \delta\mu A - 2\alpha_1\mu k A + \beta_1 k A + 2\alpha_1\mu^2 AB \\ - 2\mu\beta_1 AB + \lambda b_1)\rho^2 + (B + \delta\mu B + 2\alpha_1\mu k B \\ - \beta_1 k B + \delta k - \alpha_1 k^2 - \alpha_1\mu^2 B^2 + \mu\beta_1 B^2)\rho^4 \\ + \lambda b_2\rho^6 + \lambda b_3\rho^8 = 0.\end{aligned} \tag{10}$$

We adopt the trial equation method [55] to solve Eq.(10)

$$(\rho')^2 = a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho + a_0, \tag{11}$$

$$\begin{aligned}\rho'' = \frac{1}{2}[n a_n\rho^{n-1} + (n-1)a_{n-1}\rho^{n-2} \\ + \cdots + 2a_2\rho + a_1].\end{aligned} \tag{12}$$

Inserting Eq. (12) into Eq. (10), we notice $n = 6$ in the light of the balance algorithm. And Eq. (11) becomes

$$(\rho')^2 = a_6\rho^6 + a_4\rho^4 + a_2\rho^2 + a_0, \tag{13}$$

where a_0 is constant and

$$\begin{aligned}a_6 &= -\frac{\lambda b_3}{3\alpha_1\mu^2 - 3\beta_1\mu - 3\gamma\mu}, \\ a_4 &= -\frac{\lambda b_2}{2\alpha_1\mu^2 - 2\beta_1\mu - 2\gamma\mu}, \\ a_2 &= \frac{(\mu\delta + 1 + 2\alpha_1\mu k - k\beta_1)^2}{4(\alpha_1\mu^2 - \beta_1\mu)(\alpha_1\mu^2 - \beta_1\mu - \gamma\mu)} \\ &\quad - \frac{\alpha_1 k^2 + \delta k}{\alpha_1\mu^2 - \beta_1\mu - \gamma\mu}.\end{aligned} \tag{14}$$

Replacing ρ^2 with y, we get

$$(y')^2 = c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y, \tag{15}$$

where

$$c_4 = 4a_6, \quad c_3 = 4a_4, \quad c_2 = 4a_2, \quad c_1 = 4a_0. \tag{16}$$

Taking the transformation

$$z = y + \frac{c_3}{4c_4}, \tag{17}$$

we get

$$(z')^2 = F(z) = c_4 z^4 + h_2 z^2 + h_1 z + h_0, \tag{18}$$

where

$$h_2 = c_2 - \frac{3c_3^2}{8c_4}, \quad h_1 = c_1 + \frac{c_3^3}{8c_4^2} - \frac{c_2 c_3}{2c_4},$$
$$h_0 = -\frac{3c_3^4}{256c_4^3} + \frac{c_2 c_3^2}{16c_4^2} - \frac{c_1 c_3}{4c_4}. \tag{19}$$

We only consider the case of $c_4 > 0$ in the Eq. (18), and the case of $c_4 < 0$ is similar and not be discussed here. Eq. (18) can be written as the following dynamical system [62]

$$\begin{cases} z' = Y, \\ Y' = 2c_4 z^3 + h_2 z + \frac{1}{2} h_1, \end{cases} \tag{20}$$

the Hamiltonian is

$$H(z, Y) = \frac{Y^2}{2} - \frac{1}{2}(c_4 z^4 + h_2 z^2 + h_1 z + h_0). \tag{21}$$

Thus, the potential energy is

$$U(Y) = -\frac{1}{2}(c_4 z^4 + h_2 z^2 + h_1 z + h_0). \tag{22}$$

3. Qualitative analysis

We analyze the dynamical the system Eq.(20), and need to compute the zeros of $U'(Y)$

$$U'(Y) = -2c_4(z^3 + pz + q), \tag{23}$$

where $p = h_2/2c_4$, $q = h_1/4c_4$, we apply the complete discrimination system for the third degree polynomial [52]

$$\Delta = -\left(\frac{q^2}{4} + \frac{p^3}{27}\right). \tag{24}$$

Case 1. $\Delta = 0$, $p < 0$, we have

$$U'(Y) = -2c_4(z - \alpha)^2(z - \beta)(2\alpha + \beta = 0). \tag{25}$$

Thus, $(\alpha, 0)$ and $(\beta, 0)$ are the two equilibrium points of the dynamical system. For instance, if $c_4 = 1$, $p = -48$ and $q = -128$, then $\alpha = -4$, $\beta = 8$, we can know that $(-4, 0)$ is a cuspidal point and $(8, 0)$ is a saddle point. The global phase of system Eq. (20) as shown in Fig. 1.

From the figure above, we can see that Eq. (20) has twisted wave solutions and lone wave solutions.

Case 2. $\Delta = 0$, $p = 0$, we have

$$U'(Y) = -2c_4 z^3. \tag{26}$$

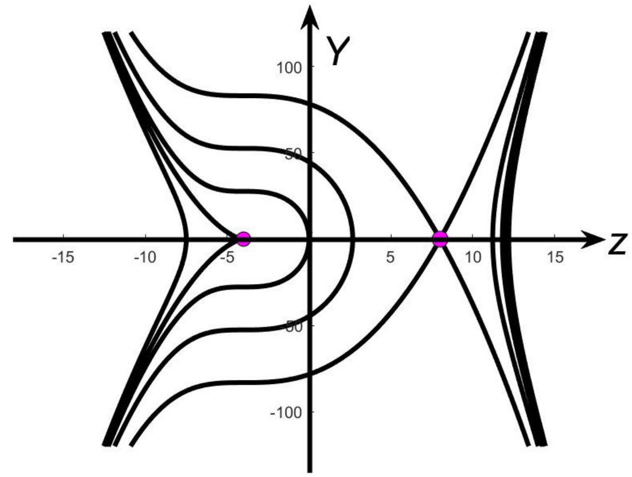

FIGURE 1. $c_4 = 1$, $p = -48$, $q = -128$.

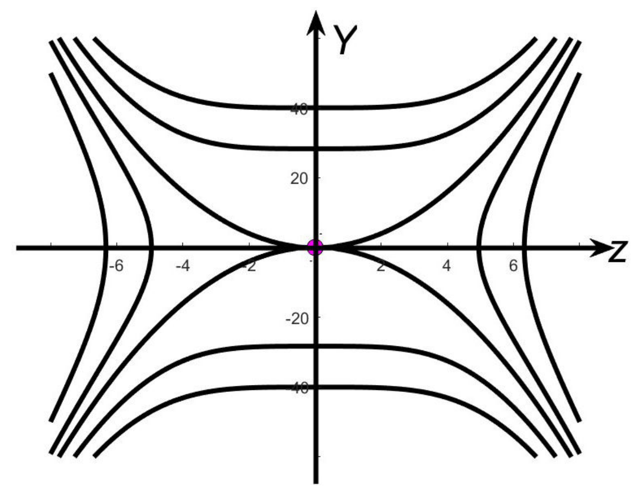

FIGURE 2. $c_4 = 1$, $p = 0$, $q = 0$.

We find that the dynamical system has only one cuspidal point $(0, 0)$. And when $c_4 = 1$, $p = 0$ and $q = 0$, we give the global phase of system Eq. (20) as shown in Fig. 2.

From the figure above, we can see that Eq. (20) has torsional wave solutions.

Case 3. $\Delta > 0$, $p < 0$, we have

$$U'(Y) = -2c_4(z - f_1)(z - f_2)(z - f_3),$$
$$(f_1 + f_2 + f_3 = 0), \tag{27}$$

where $f_1 > f_2 > f_3$, $(f_1, 0)$, $(f_2, 0)$ and $(f_3, 0)$ are the three equilibrium points of the dynamical system. For instance, if $c_4 = 1$, $p = -9$ and $q = 0$, then $f_1 = 3$, $f_2 = 0$ and $f_3 = -3$, we can know that $(f_1, 0)$ and $(f_3, 0)$ are two saddle points, and $(f_2, 0)$ is the center point. The global phase of system Eq. (20) as shown in Fig. 3.

From the figure above, we can see that Eq. (20) has kink, periodic and anti-kink solutions. We find that the figure above is symmetric, so let's discuss the asymmetric case. For instance, when $c_4 = 1$, $p = -28$ and $q = -48$, we have

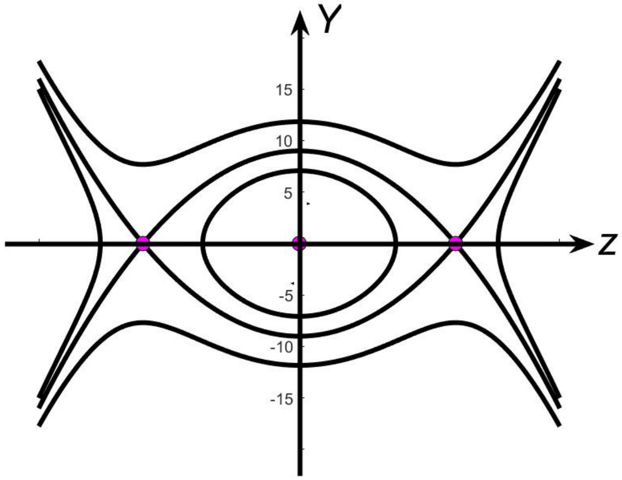

FIGURE 3. $c_4 = 1$, $p = -9$, $q = 0$.

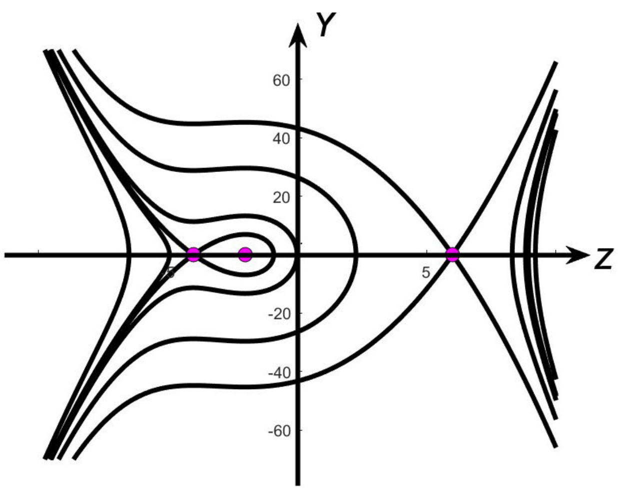

FIGURE 4. $c_4 = 1$, $p = -28$, $q = -48$.

$f_1 = 6$, $f_2 = -2$ and $f_3 = -4$, and the global phase of system Eq. (20) as shown in Fig. 4.

From the figure above, we can that the existence of solitary wave solutions.

Case 4. $\Delta < 0$, we have

$$U'(Y) = -2c_4(z - f_1)[(z - f_2)^2 + f_3^2] \quad .(2f_2 + f_1 = 0). \tag{28}$$

We find that the dynamical system has only one saddle point $(f_1, 0)$. This case is similar to case 2, which we not discuss here.

4. Classification and parameter stability analysis of exact chirped solutions

When we set $c_4 = 1$, Eq. (18) can be written as

$$\pm(\xi - \xi_0) = \int \frac{dz}{\sqrt{(z^4 + h_2 z^2 + h_1 z + h_0)}}. \tag{29}$$

We can obtain a classification of all roots conforming to the equality by the complete discrimination system for the fourth order polynomial [51]

$$\begin{aligned} &M_1 = 1, \quad M_2 = -h_2, \quad M_3 = -2{h_2}^3 + 8h_2h_0 - 9{h_1}^2, \\ &M_4 = -{h_2}^3{h_1}^2 + 4{h_2}^4 h_0 + 36h_2{h_1}^2 h_0 \\ &\quad - 32{h_2}^2{h_0}^2 - \frac{27}{4}{h_1}^4 + 64{h_0}^3, \\ &E_2 = -32h_2h_0 + 9{h_1}^2. \end{aligned} \tag{30}$$

Then, we discuss the topological stability in nine different modes of light waves. We according to the change of parameters to study the stability of topology. The form of the solution does not change when the parameters are disturbed, it is stable. Or else, it is unstable or semi-stable [58].

Case 1. $M_2 = M_3 = M_4 = 0$, then $F(z) = z^4$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{z^2}. \tag{31}$$

The exact solution and chirp are

$$\rho_1 = \left[\pm(\xi - \xi_0)^{-1} - \frac{3b_2}{8b_3}\right]^{\frac{1}{2}}, \tag{32}$$

$$\delta\omega_1 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left[\pm(\xi - \xi_0)^{-1} - \frac{3b_2}{8b_3}\right]^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}. \tag{33}$$

This singular rational solution has an unstable topology.

Case 2. $M_2 < 0$, $M_3 = M_4 = 0$, then $F(z) = ((z - \alpha)^2 + \beta^2)^2$, where $\alpha = 0$, $\beta > 0$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{\sqrt{(z - \alpha)^2 + \beta^2}}. \tag{34}$$

The exact solution and chirp are

$$\rho_2 = \left(\beta \tan[\beta(\xi - \xi_0)] + \alpha - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{35}$$

$$\delta\omega_2 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left[\beta \tan[\beta(\xi - \xi_0)] + \alpha - \frac{3b_2}{8b_3}\right]^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}. \tag{36}$$

This singular periodic solution has a semis-table topology.

Case 3. $M_2 > 0$, $M_3 = M_4 = 0$, $E_2 > 0$, then $F(z) = (z - f_1)^2(z - f_2)^2$, where $f_1 > f_2$, $f_1 + f_2 = 0$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{(z - f_1)(z - f_2)}. \tag{37}$$

If $z > f_1,\ z < f_2$, we have

$$\rho_3 = \left(\frac{f_2 - f_1}{2}\left[\coth\frac{(f_1 - f_2)(\xi - \xi_0)}{2} - 1\right] + f_2 - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{38}$$

$$\delta\omega_3 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left(\frac{f_2 - f_1}{2}\left[\coth\frac{(f_1 - f_2)(\xi - \xi_0)}{2} - 1\right] + f_2 - \frac{3b_2}{8b_3}\right)^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}, \tag{39}$$

If $f_2 < z < f_1$, the exact solution and chirp are

$$\rho_4 = \left(\frac{f_2 - f_1}{2}\left[\tanh\frac{(f_1 - f_2)(\xi - \xi_0)}{2} - 1\right] + f_2 - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{40}$$

$$\delta\omega_4 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left(\frac{f_2 - f_1}{2}\left[\tanh\frac{(f_1 - f_2)(\xi - \xi_0)}{2} - 1\right] + f_2 - \frac{3b_2}{8b_3}\right)^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}. \tag{41}$$

These two solitary wave solutions have the semi-stable topology.

Case 4. $M_2 > 0$, $M_3 = 0$, $M_4 = 0$, $E_2 = 0$, then $F(z) = (z - \alpha)^3(z - \beta)$, where $3\alpha + \beta = 0$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{(z - \alpha)\sqrt{(z - \alpha)(z - \beta)}}. \tag{42}$$

The exact solution and corresponding chirp are

$$\rho_5 = \left(\frac{4(\alpha - \beta)}{(\beta - \alpha)^2(\xi - \xi_0)^2 - 4} + \alpha - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{43}$$

$$\delta\omega_5 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left(\frac{4(\alpha - \beta)}{(\beta - \alpha)^2(\xi - \xi_0)^2 - 4} + \alpha - \frac{3b_2}{8b_3}\right)^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}. \tag{44}$$

This singular rational solution has a semis-table topology.

Case 5. $M_2 > 0$, $M_3 > 0$, $M_4 = 0$, then $F(z) = (z - f_1)^2(z - f_2)(z - f_3)$, where $f_2 > f_3$, $2f_1 + f_2 + f_3 = 0$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{(z - f_1)\sqrt{(z - f_2)(z - f_3)}}. \tag{45}$$

For $(f_2 - f_1)(f_3 - f_1) > 0$,

$$\rho_6 = \left(f_3 + \frac{f_2 - f_3}{1 + \frac{f_2 - f_1}{f_3 - f_1}\tan^2[\sqrt{(f_2 - f_1)(f_3 - f_1)}(\xi - \xi_0)]} - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{46}$$

$$\delta\omega_6 = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left(f_3 + \frac{f_2 - f_3}{1 + \frac{f_2 - f_1}{f_3 - f_1}\tan^2[\sqrt{(f_2 - f_1)(f_3 - f_1)}(\xi - \xi_0)]} - \frac{3b_2}{8b_3}\right)^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}. \tag{47}$$

This singular periodic solution has a semi-stable topology.

For $(f_2-f_1)(f_3-f_1)<0$,

$$\rho_7=\left(f_3+\frac{f_2-f_3}{1+\frac{f_2-f_1}{f_1-f_3}\tanh^2[\sqrt{(f_2-f_1)(f_1-f_3)}(\xi-\xi_0)]}-\frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \quad (48)$$

$$\delta\omega_7=\frac{s_0}{\alpha_1\mu^2-\beta_1\mu}\left(f_3+\frac{f_2-f_3}{1+\frac{f_2-f_1}{f_1-f_3}\tanh^2[\sqrt{(f_2-f_1)(f_1-f_3)}(\xi-\xi_0)]}-\frac{3b_2}{8b_3}\right)^{-1}-\frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}, \quad (49)$$

$$\rho_8=\left(f_3+\frac{f_2-f_3}{1+\frac{f_2-f_1}{f_1-f_3}\coth^2[\sqrt{(f_2-f_1)(f_1-f_3)}(\xi-\xi_0)]}-\frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \quad (50)$$

$$\delta\omega_8=\frac{s_0}{\alpha_1\mu^2-\beta_1\mu}\left(f_3+\frac{f_2-f_3}{1+\frac{f_2-f_1}{f_1-f_3}\coth^2[\sqrt{(f_2-f_1)(f_1-f_3)}(\xi-\xi_0)]}-\frac{3b_2}{8b_3}\right)^{-1}-\frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}. \quad (51)$$

These two solitary wave solutions have the semi-stable topology.

Case 6. $M_2M_3<0$, $M_4=0$, then $F(z)=(z-f_1)^2((z-f_2)^2+{f_3}^2)$, where $f_1+f_2=0$.

We obtain

$$\pm(\xi-\xi_0)=\int\frac{dz}{(z-f_1)\sqrt{(z-f_2)^2+{f_3}^2}}. \quad (52)$$

The exact solution and corresponding chirp are written as

$$\rho_9=\left(\frac{e^{\pm\sqrt{(f_1-f_2)^2+{f_3}^2}(\xi-\xi_0)}-N+\sqrt{(f_1-f_2)^2+{f_3}^2}(2-N)}{(e^{\pm\sqrt{(f_1-f_2)^2+{f_3}^2}(\xi-\xi_0)}-N)^2-1}-\frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \quad (53)$$

$$\delta\omega_9=\frac{s_0}{\alpha_1\mu^2-\beta_1\mu}\left(\frac{e^{\pm\sqrt{(f_1-f_2)^2+{f_3}^2}(\xi-\xi_0)}-N+\sqrt{(f_1-f_2)^2+{f_3}^2}(2-N)}{(e^{\pm\sqrt{(f_1-f_2)^2+{f_3}^2}(\xi-\xi_0)}-N)^2-1}-\frac{3b_2}{8b_3}\right)^{-1}-\frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}, \quad (54)$$

where

$$N=\frac{f_1-2f_2}{\sqrt{(f_1-f_2)^2+{f_3}^2}}. \quad (55)$$

This exponential solution has a semi-stable topology.

Case 7. $M_2>0$, $M_3>0$, $M_4>0$, then $F(z)=(z-f_1)(z-f_2)(z-f_3)(z-f_4)$, where $f_1+f_2+f_3+f_4=0$, $f_1>f_2>f_3>f_4$.

We obtain

$$\pm(\xi-\xi_0)=\int\frac{dz}{\sqrt{(z-f_1)(z-f_2)(z-f_3)(z-f_4)}}. \quad (56)$$

The exact solution and corresponding chirp are given by

$$\rho_{10} = \left(\frac{f_2(f_1-f_4)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - f_1(f_2-f_4)}{(f_1-f_4)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - (f_2-f_4)} - \frac{3b_2}{8b_3} \right)^{\frac{1}{2}}, \tag{57}$$

$$\delta\omega_{10} = \frac{s_0}{\alpha_1\mu^2-\beta_1\mu} \left(\frac{f_2(f_1-f_4)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - f_1(f_2-f_4)}{(f_1-f_4)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - (f_2-f_4)} - \frac{3b_2}{8b_3} \right)^{-1}$$
$$- \frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}, \tag{58}$$

$$\rho_{11} = \left(\frac{f_4(f_2-f_3)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - f_3(f_2-f_4)}{(f_2-f_3)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - (f_2-f_4)} - \frac{3b_2}{8b_3} \right)^{\frac{1}{2}}, \tag{59}$$

$$\delta\omega_{11} = \frac{s_0}{\alpha_1\mu^2-\beta_1\mu} \left(\frac{f_4(f_2-f_3)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - f_3(f_2-f_4)}{(f_2-f_3)\mathrm{sn}^2\left[\frac{\sqrt{(f_1-f_3)(f_2-f_4)}}{2}(\xi-\xi_0), N_1\right] - (f_2-f_4)} - \frac{3b_2}{8b_3} \right)^{-1}$$
$$- \frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}, \tag{60}$$

where

$$N_1{}^2 = \frac{(f_2-f_3)(f_1-f_4)}{(f_1-f_3)(f_2-f_4)}. \tag{61}$$

These two elliptic function double periodic solutions have the stable topology.

Case 8. There are three different situations, namely **(8.1)** $M_2 > 0$, $M_4 < 0$ **(8.2)** $M_2 < 0$, $M_3 < 0$, $M_4 < 0$, and **(8.3)** $M_2 = 0$, $M_3 \leq 0$, $M_4 < 0$ then $F(z) = (z-f_1)(z-f_2)((z-f_3)^2+f_4{}^2)$, where $f_1 > f_2$, $f_4 > 0$ $f_1+f_2+2f_3 = 0$.

We obtain

$$\pm(\xi-\xi_0) = \int \frac{dz}{\sqrt{(z-f_1)(z-f_2)((z-f_3)^2+f_4{}^2)}}. \tag{62}$$

The exact solution and corresponding chirp are given by

$$\rho_{12} = \left(\frac{g_1\mathrm{cn}\left[\frac{\sqrt{\pm 2f_4d_1(f_1-f_2)}}{2dd_1}(\xi-\xi_0), d\right] + g_2}{g_3\mathrm{cn}\left[\frac{\sqrt{\pm 2f_4d_1(f_1-f_2)}}{2dd_1}(\xi-\xi_0), d\right] + g_4} - \frac{3b_2}{8b_3} \right)^{\frac{1}{2}}, \tag{63}$$

$$\delta\omega_{12} = \frac{s_0}{\alpha_1\mu^2-\beta_1\mu} \left(\frac{g_1\mathrm{cn}\left[\frac{\sqrt{\pm 2f_4d_1(f_1-f_2)}}{2dd_1}(\xi-\xi_0), d\right] + g_2}{g_3\mathrm{cn}\left[\frac{\sqrt{\pm 2f_4d_1(f_1-f_2)}}{2dd_1}(\xi-\xi_0), d\right] + g_4} - \frac{3b_2}{8b_3} \right)^{-1} - \frac{\mu\delta+1+2\alpha_1\mu k-\beta_1 k}{2\alpha_1\mu^2-2\beta_1\mu}, \tag{64}$$

where

$$g_1 = \frac{1}{2}(f_1+f_2)g_3 - \frac{1}{2}(f_1-f_2)g_4, \quad g_2 = \frac{1}{2}(f_1+f_2)g_4 - \frac{1}{2}(f_1-f_2)g_3, \quad g_3 = f_1-f_3-\frac{f_4}{d-1},$$
$$g_4 = f_1-f_3-f_4d_1, \quad D = \frac{f_4^2+(f_1-f_3)(f_2-f_3)}{f_4(f_1-f_2)}, \quad d_1 = D\pm\sqrt{D^2+1}, \quad d^2 = \frac{1}{1+d_1^2}. \tag{65}$$

This elliptic function double periodic solution has the stable topology at case 8.1 and 8.2, and the semi-stable at case 8.3.

Case 9. There are two situations, namely **(9.1)** $M_2 \leq 0$, $M_4 > 0$, and **(9.2)** $M_2 > 0$, $M_3 \leq 0$, $M_4 > 0$, then $F(z) = ((z-f_1)^2 + {f_2}^2)((z-f_3)^2 + {f_4}^2)$, where $f_2 \geq f_4 > 0$, $f_1 + f_3 = 0$.

We obtain

$$\pm(\xi - \xi_0) = \int \frac{dz}{\sqrt{((z-f_1)^2 + {f_2}^2)((z-f_3)^2 + {f_4}^2)}}. \tag{66}$$

The exact solution and corresponding chirp are presented by

$$\rho_{13} = \left(\frac{g_1 \mathrm{sn}(d_2(\xi-\xi_0), d) + g_2 \mathrm{cn}(d_2(\xi-\xi_0), d)}{g_3 \mathrm{sn}(d_2(\xi-\xi_0), d) + g_4 \mathrm{cn}(d_2(\xi-\xi_0), d)} - \frac{3b_2}{8b_3}\right)^{\frac{1}{2}}, \tag{67}$$

$$\delta\omega_{13} = \frac{s_0}{\alpha_1\mu^2 - \beta_1\mu}\left(\frac{g_1 \mathrm{sn}(d_2(\xi-\xi_0), d) + g_2 \mathrm{cn}(d_2(\xi-\xi_0), d)}{g_3 \mathrm{sn}(d_2(\xi-\xi_0), d) + g_4 \mathrm{cn}(d_2(\xi-\xi_0), d)} - \frac{3b_2}{8b_3}\right)^{-1} - \frac{\mu\delta + 1 + 2\alpha_1\mu k - \beta_1 k}{2\alpha_1\mu^2 - 2\beta_1\mu}, \tag{68}$$

where

$$g_1 = f_1 g_3 + f_2 g_4, \quad g_2 = f_1 g_4 - f_2 g_3, \quad g_3 = -f_2 - \frac{f_4}{d}, \quad g_4 = f_1 - f_3, \quad D = \frac{(f_1 - f_3)^2 + f_2^2 + f_4^2}{2 f_2 f_4},$$
$$d_1 = D + \sqrt{D^2 - 1}, \quad d = \frac{d_1^2 - 1}{d_1^2}, \quad d_2 = \frac{f_4\sqrt{(g_3^2 + g_4^2)(d_1^2 g_3^2 + g_4^2)}}{g_3^2 + g_4^2}. \tag{69}$$

This elliptic function double periodic solution has the semi-stable topology.

Analyzing the above nine cases, we get thirteen different optical wave patterns. Case 7, Case 8.1 and Case 8.2 have stable topologies, Case 1 has unstable topology and all others have semi-stable topologies. In fact, when parameters are perturbed, the equation $M_2 = M_3 = M_4 = 0$ becomes an inequality, and the instability becomes stable. Such as, we compare Case 1 with Case 7, when some parameters change, Case 1 can become Case 7, so there is no stability becomes stable.

5. Typical solutions and their graphs

Example 1. Triangular function solutions.

Taking $s_0 = f_2 = 1$, $f_1 = 2$, $b_2 = -\frac{8}{3}$, $\delta = k = 0$, $\xi_0 = -4$, $x = 0$ we get

$$\rho_3 = \left(-\frac{1}{2}\coth\left[\frac{1}{2}\xi + 2\right] + \frac{5}{2}\right)^{\frac{1}{2}}, \tag{70}$$

$$\delta\omega_3 = \frac{2}{5 - \coth(\frac{1}{2}t + 2)} - \frac{1}{2}. \tag{71}$$

The 2D graphics of ρ_3 and $\delta\omega_3$ are presented in Fig. 5 and 6.

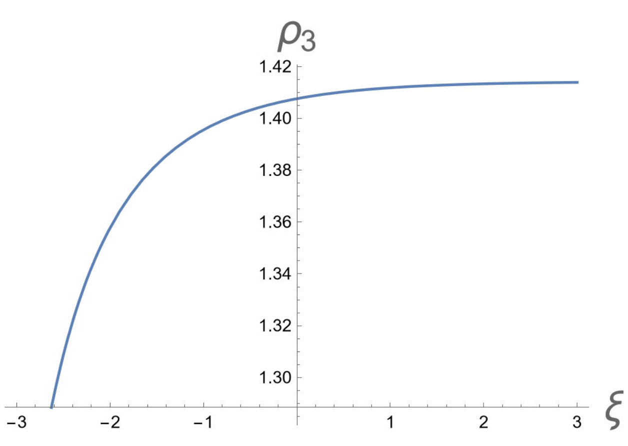

FIGURE 5. ρ_3.

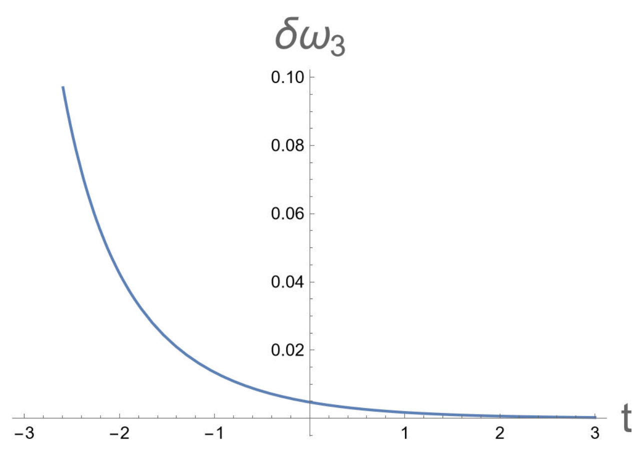

FIGURE 6. $\delta\omega_3$.

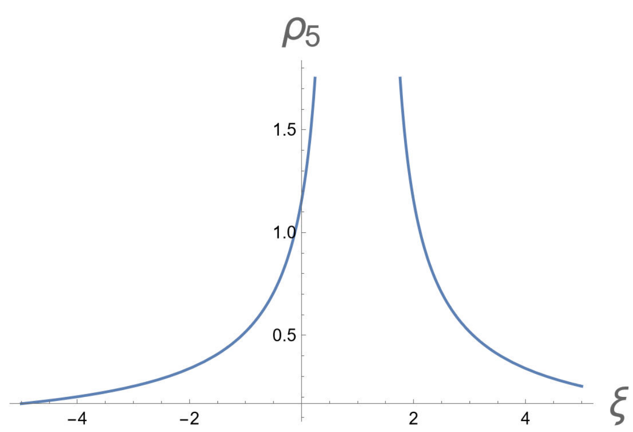

FIGURE 7. ρ_5.

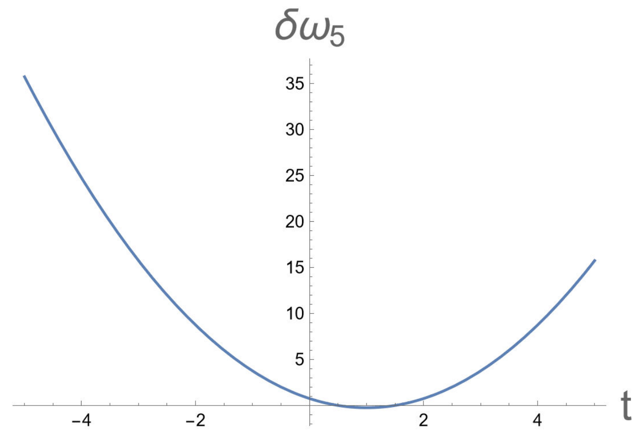

FIGURE 8. $\delta\omega_5$.

Example 2. Rational solutions.

Taking $\alpha = s_0 = \xi_0 = 1$, $\beta = -3$, $b_2 = \frac{8}{3}$, $\delta = 2$, $k = -1$, $x = 0$ we have

$$\rho_5 = \left(\frac{16}{16(\xi - 1)^2 - 4} \right)^{\frac{1}{2}}, \tag{72}$$

$$\delta\omega_5 = (t - 1)^2 - \frac{1}{4}. \tag{73}$$

The 2D illustrations of ρ_5 and $\delta\omega_5$ are shown in Fig. 7 and 8.

Example 3. Solitary wave solutions:

Taking $f_3 = s_0 = 1$, $f_1 = -2$, $f_2 = 3$, $b_2 = -\frac{16}{3}$, $\delta = 2$, $k = -4$, $\xi_0 = -\sqrt{15}/15$, $x = 0$ we obtain

$$\rho_6 = \left(\frac{2}{1 + \frac{5}{3}\tan^2\left[\sqrt{15}\xi - \frac{\sqrt{15}}{15}\right]} + 3 \right)^{\frac{1}{2}}, \tag{74}$$

$$\delta\omega_6 = \left(\frac{2}{1 + \frac{5}{3}\tan^2\left[\sqrt{15}t - \frac{\sqrt{15}}{15}\right]} + 3 \right)^{-1} + \frac{9}{2}. \tag{75}$$

The 2D diagrams of ρ_6 and $\delta\omega_6$ are separately portrayed in Fig. 9 and Fig. 10.

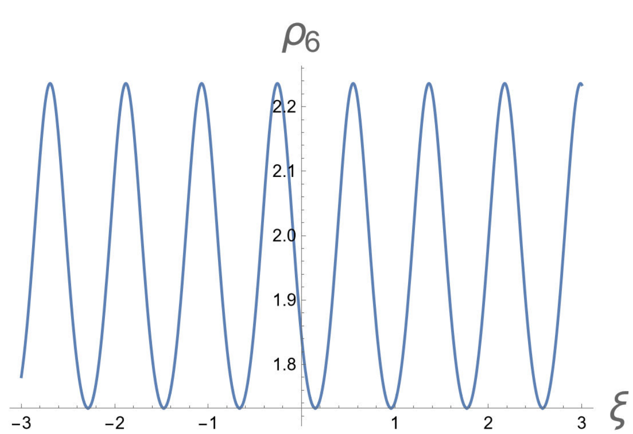

FIGURE 9. ρ_6.

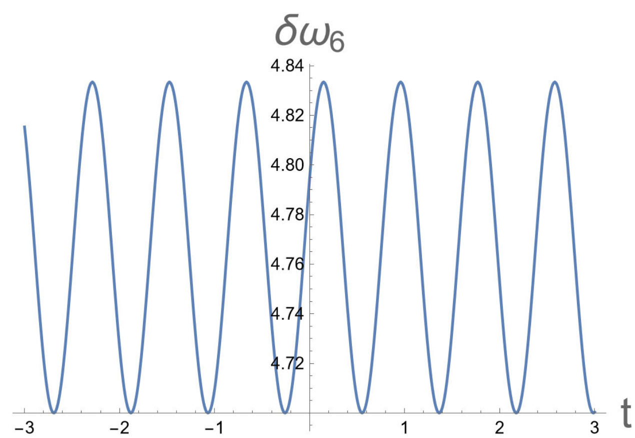

FIGURE 10. $\delta\omega_6$.

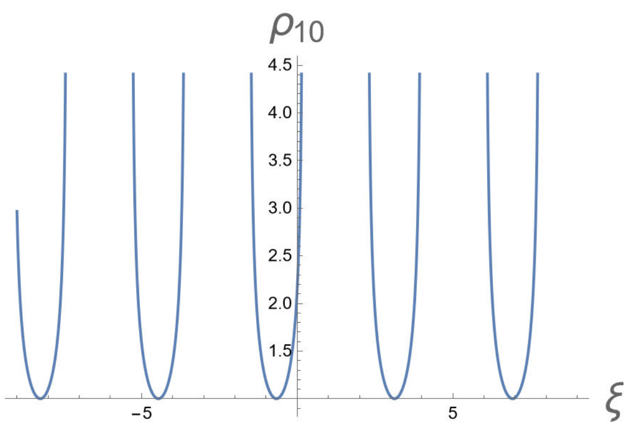

FIGURE 11. ρ_{10}.

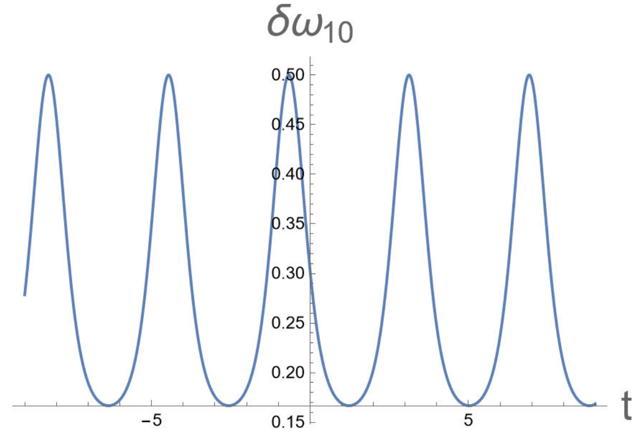

FIGURE 12. $\delta\omega_{10}$.

Example 4. Jacobi elliptic function double periodic solutions.

Taking $f_2 = 1$, $s_0 = -1/4$, $f_1 = 2$, $f_3 = -1$, $f_4 = -2$, $b_2 = 8/3$, $\delta = 1/2$, $k = -1/2$, $N_1 = 2\sqrt{2}/3$, $\xi_0 = -2/3$, $x = 0$ we have

$$\rho_{10} = \left(\frac{4\mathrm{sn}^2\left[\frac{3}{2}\xi + 1, \frac{2\sqrt{2}}{3}\right] - 6}{4\mathrm{sn}^2\left[\frac{3}{2}\xi + 1, \frac{2\sqrt{2}}{3}\right] - 3} - 1 \right)^{\frac{1}{2}}, \tag{76}$$

$$\delta\omega_{10} = \frac{-2\mathrm{sn}^2\left(\frac{3}{2}t + 1, \frac{2\sqrt{2}}{3}\right) - 3}{6}. \tag{77}$$

The 2D drawings of ρ_{10} and $\delta\omega_{10}$ are displayed in Fig.11 and 12.

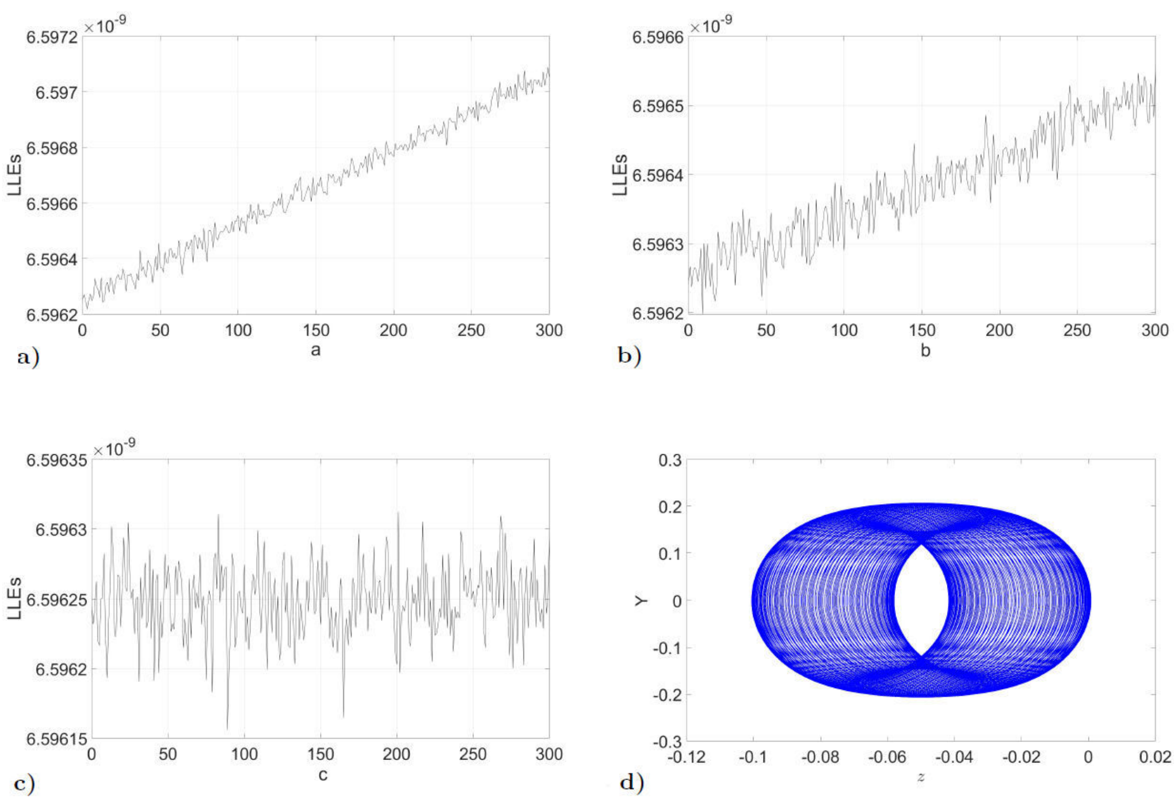

FIGURE 13. System (78): a)LLEs for a; b) LLEs for b; c) LLEs for c; Two-dimensional phase portrait, when $a = -1$, $b = -2$ and $c = -0.1$, $G(\xi) = \cos(6\xi)$.

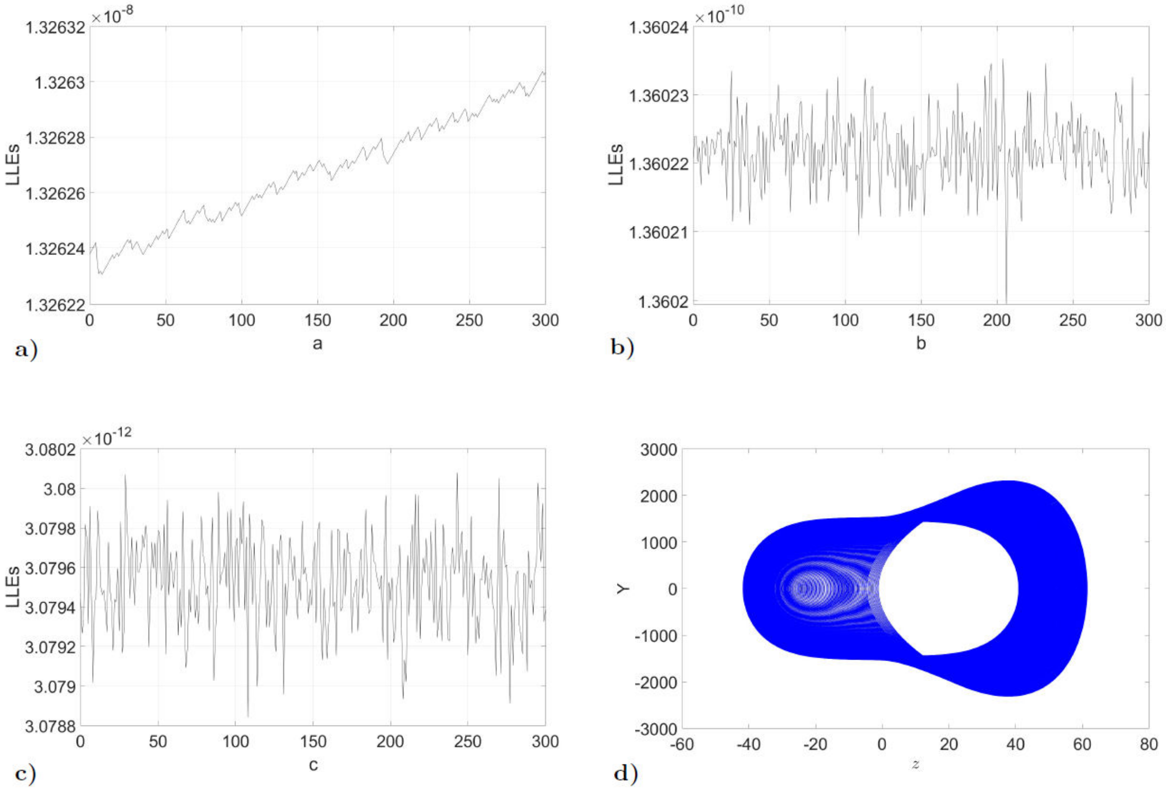

FIGURE 14. System(78): a)LLEs for a; b)LLEs for b; c)LLEs for c; Two-dimensional phase portrait, when $a = -1.5$, $b = -1.0$ and $c = -0.1$, $G(\xi) = \cosh(0.12\xi)$.

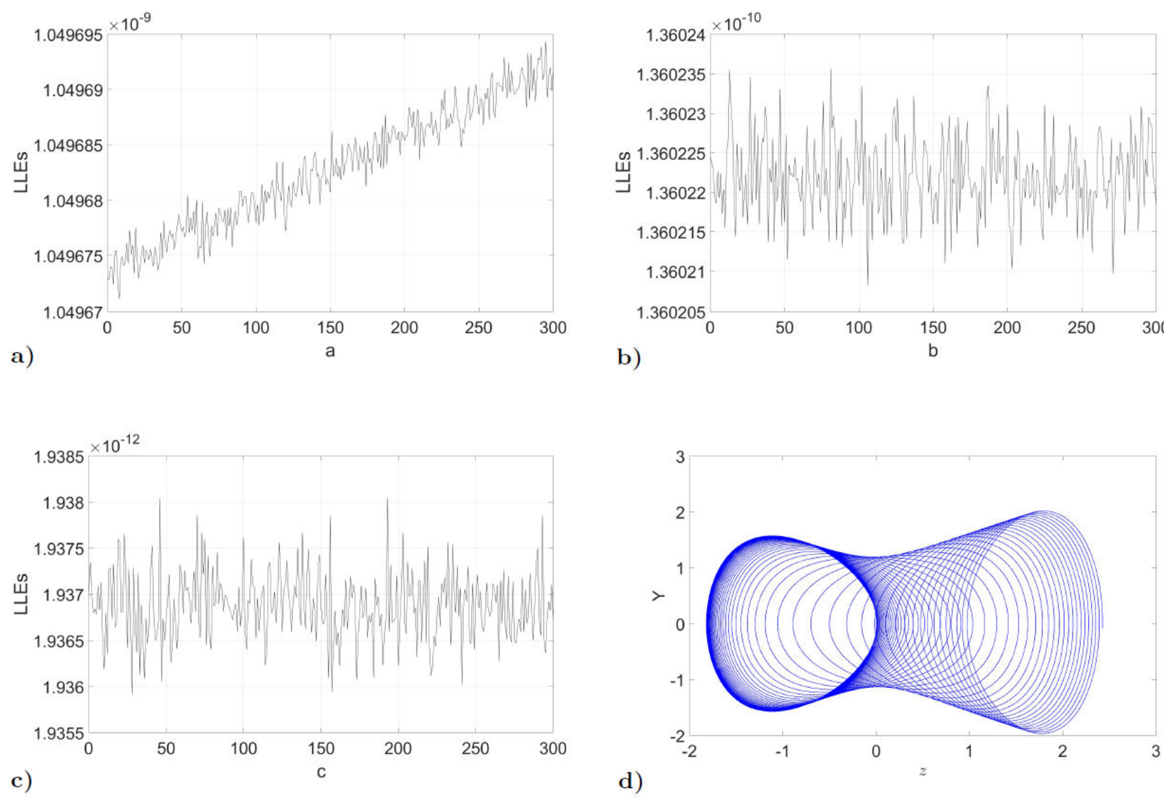

FIGURE 15. System(78): a)LLEs for a; b)LLEs for b; c)LLEs for c; Two-dimensional phase portrait, when $a = -0.8$, $b = -0.5$ and $c = -1.8$, $G(\xi) = e^{0.02\xi}$.

6. Chaotic behaviors

According to the previous analysis, power system does not exist the chaotic behaviors. However, if we increase the perturbation term $G(\xi)$, we obtain the following perturbation system [62–64]

$$\begin{cases} z' = Y, \\ Y' = 2c_4 z^3 + h_2 z + \dfrac{1}{2}h_1 + G(\xi), \end{cases} \tag{78}$$

where $a = 2c_4$, $b = h_2$ and $c = (1/2)h_1$.

We choose three types of perturbed terms here, namely $G(\xi) = \cos(6\xi)$, $\cosh(0.12\xi)$ and $e^{0.02\xi}$. In addition, we give the largest Lyapunov exponents for each case. Specific examples are shown in Fig. 13-15.

From Fig. 13-15, We can find that Eq. (78) has the chaotic behaviors. The largest Lyapunov exponent of parameter a is largest. We conclude that parameter a is more influential than parameter b and parameter c. Moreover, parameter a, parameter b and parameter c are all greater than zero, so they all affect the chaotic behaviors of Eq. (78). By changing the perturbation terms, we obtain different the largest Lyapunov exponents and phase diagrams, which prove the existence of different chaotic behaviors.

7. Conclusion

In this paper, we obtain the phase diagram and equilibrium point of the equation by qualitative analysis. Therefore, we prove the existence of soliton solutions and periodic solutions. We use the complete discrimination system for the fourth order polynomial to generate a series of chirped solutions, and analyze the parameter stability of various modes. We find that NLSE has the chaotic behaviors for somes disturbance terms. We choose three types of perturbation terms and draw the largest Lyapunov exponents and the corresponding global phase diagrams. Compared with the existing research, our method is more comprehensive and concise. The results of this study are helpful to complement the relevant physical systems and provide a new direction for the study of the equation.

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1. I. Onder *et al.*, Stochastic optical solitons of the perturbed nonlinear Schrödinger equation with Kerr law via Ito calculus, *Eur. Phys. J. Plus.* **138** (2023) 872.
2. H. E. Ibarra-Villalon, O. Pottiez, A. Gómez-Vieyra, J.P. Lauterio-Cruz, Comparative study of finite difference methods and pseudo-spectral methods for solving the nonlinear Schrödinger equation in optical fiber, *Phys. Scr.* **98** (2023) 065514.
3. Y. Zhang *et al.*, Interactions of vector anti-dark solitons for the coupled nonlinear Schrödinger equation in inhomogeneous fibers, *Nonlinear Dyn.* **94** (2018) 1351-1360.
4. M.A.S. Murad *et al*., Higher-order time-fractional Sasa-Satsuma equation: Various optical soliton solutions in optical fiber, *Results Phys*. **55** (2023) 107162.
5. S. Arshed *et al*., Optical soliton solutions of perturbed nonlinear Schrödinger equation with parabolic law nonlinearity, *Opt. Quantum Electron.* **56** (2024) 50.
6. T. Mathanaranjan *et al*., Chirped optical solitons and stability analysis of the nonlinear Schrödinger equation with nonlinear chromatic dispersion, *Commun. Theor. Phys.* **75** (2023) 085005.
7. Z. Islam *et al*., Optical solitons to the fractional order nonlinear complex model for wave packet envelope, *Results Phys.* **43** (2022) 106095.
8. S. Wang *et al*., Dynamic behavior of optical soliton interactions in optical communication systems, *Chin. Phys. Lett.* **39** (2022) 114202.
9. S. Wang *et al*., Study on the regulation of amplitude for the optical soliton through nonlinear effects in the optical communication system, *Optik* **269** (2022) 169839.
10. A.M. Elsherbeny *et al*., Optical solitons and another solutions for Radhakrishnan-Kundu-Laksmannan equation by using improved modified extended tanh-function method, *Opt. Quantum Electron.* **53** (2021) 1-15.
11. T. Shahzad *et al*., Extraction of optical solitons for nonlinear Biswas-Milovic equation in magneto-optic waveguide, *Opt. Quantum Electron.* **56** (2024) 64.
12. S.M. Boulaaras *et al*., Unveiling optical solitons: Solving two forms of nonlinear Schrödinger equations with unified solver method, *Optik* **295** (2023) 171535.
13. N.Z. Petrović, M. Belić, and W.P. Zhong, Exact traveling-wave and spatiotemporal soliton solutions to the generalized (3+1)-dimensional Schrödinger equation with polynomial nonlinearity of arbitrary order, *Phys. Rev. E.* **83** (2011) 02660.
14. M. Ozisik, A. Secer, and M. Bayram, On solitary wave solutions for the extended nonlinear Schrödinger equation via the modified F-expansion method, *Opt. Quantum Electron.* **55** (2023) 215.
15. H.U. Rehman *et al*., Analysis of cubic-quartic-nonlinear Schrödinger's equation with cubic-quintic-septic-nonic form of self-phase modulation through different techniques, *Optik* (2023) 171028.
16. A.I. Konyukhov and P.A. Mavrin, Chirped soliton fission and fusion in dispersion oscillating fibers, *Laser Phys.* **33** (2022) 015401.

17. S.T.R. Rizvi, A.R. Seadawy and U. Raza, Chirped optical wave solutions for a nonlinear model with parabolic law and competing weakly nonlocal nonlinearities, *Opt. Quantum Electron.* **54** (2022) 756.

18. A. Goyal *et al.*, Chirped femtosecond solitons and double-kink solitons in the cubic-quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift, *Phys. Rev. A.* **84** (2011) 063830.

19. L.Q. Feng *et al*., Nonlinear chirp combination effect on the improvement of high harmonic spectra and attosecond pulses, *Int. J. Mod. Phys. B.* **35** (2021) 2150225.

20. H. Triki *et al*., Chirped optical solitons having polynomial law of nonlinear refractive index with self-steepening and nonlinear dispersion, *Phys. Lett. A.* **417** (2021) 127698.

21. V.K. Sharma, Chirped soliton-like solutions of generalized nonlinear Schrödinger equation for pulse propagation in negative index material embedded into a Kerr medium, Indian *J. Phys.* **90** (2016) 1271-1276.

22. H. Triki *et al*., Chirped femtosecond pulses in the higher-order nonlinear Schrödinger equation with non-Kerr nonlinear terms and cubic-quintic-septic nonlinearities, *Opt. Commun.* **366** (2016) 362-369.

23. H. Triki, K. Porsezian, and P. Grelu, Chirped soliton solutions for the generalized nonlinear Schrödinger equation with polynomial nonlinearity and non-Kerr terms of arbitrary order, *J. Opt*. **18** (2016) 075504.

24. G. Akram, M. Sadaf, and M.A.U. Khan, Soliton solutions of the resonant nonlinear Schrödinger equation using modified auxiliary equation method with three different nonlinearities, *Math. Comput. Simul.* **206** (2023) 1-20.

25. P. Franken, A. E. Hill, C. W. Peters, and G. Weinrich, Generation of optical harmonics, *Phys. Rev. Lett*. **7** (1961) 118.

26. E. P. Ippen and R. H. Stolen, Stimulated Brillouin scattering in optical fibers, *Appl. Phys. Lett*. **21** (1972) 539.

27. F. P. Kapron, D. B. Keck, and R. D. Maurer, Radiation losses in glass optical waveguides, *Appl. Phys. Lett.* **17** (1970) 423.

28. R. H. Stolen, E. P. Ippen, and A. R. Tynes, Raman oscillation in glass optical waveguide, *Appl. Phys. Lett.* **20** (1972) 62.

29. R. H. Stolen, Nonlinearity in fiber transmission, *Proc. IEEE.* **68** (1980) 1232.

30. M.D. Perry, T. Ditmire, and B.C. Stuart, Self-phase modulation in chirped-pulse amplification, *Opt. Lett.* **24** (1994) 2149-2151.

31. H.U. Rehman *et al*., Analysis of cubic-quartic-nonlinear Schrödinger's equation with cubic-quintic-septic-nonic form of self-phase modulation through different techniques, *Optik* **287** (2023) 171028.

32. X.Z. Xu, and M.Y. Wang, Exact solutions of perturbed stochastic NLSE with generalized anti-cubic law nonlinearity and multiplicative white noise, *Results Phys.* **56** (2024) 10720.

33. E.M.E. Zayed *et al*., Optical solitons for Biswas-Arshed equation with multiplicative noise via Itô calculus using three integration algorithms, *Optik* **258** (2022) 168847.

34. E.M.E. Zayed *et al.*, Dispersive optical solitons in magneto-optic waveguides for perturbed stochastic NLSE with generalized anti-cubic law nonlinearity and spatio-temporal dispersion having multiplicative white noise, *Optik* **271** (2022) 170131.

35. N.A. Kudryashov, Q. Zhou, C. Dai, Solitary waves of the complex Ginzburg-Landau equation with anti-cubic nonlinearity, *Phys. Lett. A*. **490** (2023) 129172.

36. W.W. Mohammed *et al*., The exact solutions of the stochastic Ginzburg-Landau equation, *Results Phys*. **23** (2021) 103988.

37. W.W. Mohammed *et al*., Exact solutions of the stochastic new coupled Konno-Oono equation, *Results Phys.* **21** (2021) 103830.

38. E.H.M. Zahran, A. Bekir, R.A. Ibrahim, New impressive analytical optical soliton solutions to the Schrödinger-Poisson dynamical system against its numerical solutions, *Opt. Quantum Electron.* **55** (2023) 212.

39. S Kumar, A Kumar, Dynamical behaviors and abundant optical soliton solutions of the cold bosonic atoms in a zig-zag optical lattice model using two integral schemes, *Math. Comput. Simul.* **201** (2022) 254-274.

40. D.S. Wang *et al*., Integrability and bright soliton solutions to the coupled nonlinear Schrödinger equation with higher-order effects, *Appl. Math. Comput.* **229** (2014) 296-309.

41. W.X. Ma, Z. Zhu, Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm, *Appl. Math. Comput*. **218** (2021) 11871-11879.

42. R. Ellahi, S.T. Mohyud-Din, U. Khan, Exact traveling wave solutions of fractional order Boussinesq-like equations by applying Exp-function method, *Results Phys.* **8** (2018) 114-120.

43. W.X. Ma, T. Huang, Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, *Phys Scr.* **82** (2010) 065003.

44. A.K. Alzahrani, M.R. Belic, Cubic-quartic optical soliton perturbation with Lakshmanan-Porsezian-Daniel model by semi-inverse variational principle, *Ukr J Phys Opt.* **22** (2021) 123.

45. E.M.E. Zayed, K.A. Gepreel, M.E.M. Alngar, Addendum to Kudryashov's method for finding solitons in magneto-optics waveguides to cubic-quartic NLSE with Kudryashov's sextic power law of refractive index, *Optik* **230** (2011) 166311.

46. M.M.A. Khater, R.A.M. Attia, D. Lu, Modified auxiliary equation method versus three nonlinear fractional biological models in present explicit wave solutions, *Math Comput Appl*. **24** (2018) 1.

47. M. Khater, D. Lu, R.A.M. Attia, Dispersive long wave of nonlinear fractional Wu-Zhang system via a modified auxiliary equation method, *AIP Advances.* **9** (2019) 025003.

48. C.S. Liu, Exact travelling wave solutions for (1+1)-dimensional dispersive long wave equation, *Chin Phys.* **14** (2005) 1710.

49. C.S. Liu, Using trial equation method to solve the exact solutions for two kinds of KdV equations with variable coefficients. (2005) 4506-4510.

50. C.S. Liu, Trial equation method based on symmetry and applications to nonlinear equations arising in mathematical physics, *Found Phys*. **41** (2011) 793-804.

51. C.S. Liu, Exact travelling wave solutions for (1+1)-dimensional dispersive long wave equation, *Chin Phys.* **14** (2005) 1710-06.

52. C.S. Liu, Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations, *Comput Phys Comm.* **181** (2010) 317-24.
53. C.S. Liu, Classification of all single travelling wave solutions to Calogero Degasperis-Focas equation, *Commun Theor Phys.* **48** (2007) 601.
54. C.S. Liu, All single traveling wave solutions to (3+1)-dimensional Nizhnok Novikov-Veselov equation, *Commun Theor Phys*. **45** (2006) 991-2.
55. C.S. Liu, The classification of travelling wave solutions and superposition of multi-solutions to Camassa-Holm equation with dispersion, *Chin Phys.* **16** (2007) 1832.
56. C.S. Liu, Two model equations with a second degree logarithmic nonlinearity and their Gaussian solutions, *Commun Theor Phys*. **73** (2021) 045007.
57. C.S. Liu, The Gaussian soliton in the Fermi-Pasta-Ulam chain, *Nonlinear Dynam.* **106** (2021) 899-905.
58. X. Wang, Topological stability and patterns of traveling wave for a micro-polar non-Newtonian fluid model, *Mod Phys Lett B.* **35** (2021) 2150237.
59. Y. Kai *et al*., Qualitative and quantitative analysis of nonlinear dynamicals by the complete discrimination system for polynomial method, *Chaos Solitons Fractals*. **141** (2020) 110314.
60. Y. Kai, J. Ji, Z. Yin, Exact solutions and dynamic properties of Ito-Type coupled nonlinear wave equations, *Phys Lett A.* **421** (2022) 127780.
61. Y. Kai, Z. Yin, Asymptotic analysis to domain walls between traveling waves modeled by real coupled Ginzburg-Landau equations, *Chaos Solitons Fractals*. **152** (2021) 111266.
62. Y. Kai *et al*., A study of the shallow water waves with some Boussinesq-type equations, *Waves Random Complex Media*. (2021) 1-18.
63. Y. Li, Y. Kai, Wave structures and the chaotic behaviors of the cubic-quartic nonlinear Schrödinger equation for parabolic law in birefringent fibers, *Nonlinear Dynam.* **111** (2023) 8701-8712.
64. C.S. Liu, The renormalization method for singular perturbation of solitons, *Chaos Solitons Fractals*. (2022) 112074.