New exact solutions of the nonlinear space-time conformable symmetric regularized long wave equation using the extended fan sub-equation method

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The non-linearity in numerous problems occurs due to the complexity of the given physical phenomena. This work aims to present the extended Fan sub-equation method, which is successfully applied to get the analytical solutions to the space-time conformable symmetric regularized long wave (SRLW) equation. The results may be useful for analyzing the depth and spacing of parallel subsurface drains and long waves with small amplitudes on the water's surface in channels. The results that are obtained of the proposed approach are highly accurate and give beneficial information on the actual dynamics of every problem. The recommended method may be extended to solve more significant fractional order problems due to its simple implementation.

Keywords: Nonlinear space-time conformable equations; conformable derivative; symmetric regularized long wave; exact solutions.

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1. Introduction

Nonlinear Partial Differential Equations (NLPDEs) are important equations used to understand and solve various real-world problems in fields like oceanography [1], meteorology [2], fluid mechanics [3], and more [4,5]. They have been crucial in studying phenomena in a wide range of areas including solid state physics, water waves, chemical physics, and bio-genetics. Finding exact solutions to these equations has been a fascinating and significant advancement in nonlinear science and theoretical physics. Different analytical methods have been employed to discover these solutions to these complex equations.

Among nonlinear evolution equations, the SRLW model is considered to be an important model that explains the propagation of long waves with small amplitudes in dispersive media. It has extensive applications in shallow water wave theory, ion-acoustic waves, and other subsurface flow problems, including the analysis of depth and spacing of parallel subsurface drains. Combining the SRLW equation with fractional derivatives will surely introduce memory and hereditary properties into this equation, making it more realistic for modeling physical phenomena. Seyler and Fenstermacher developed the SRLW model [6], motivated by weak nonlinear ionic acoustic and spatially charged wave models. This problem may also be used to describe other physical phenomena such as solitary waves, ion-acoustic waves in plasma, and shallow water waves. Due to the significance of these equations in the understanding of physics, several effective and advanced methods have been proposed to get exact SRLW solutions such as functional variable method [7], (G'/G^2)-expansion [8], solitary wave ansatz method [9], the improved F-expansion method [10], the new auxiliary method [11], the Lie group analysis method [12], the Jacobi elliptic function [13], the extended complex method [14], the finite difference method [15], the $\exp(-\phi(\xi))$-expansion method [16] and ϕ^6 method [17].

Fractional Partial Differential Equations (FPDEs) are an expansion of the usual PDEs, involving fractional derivatives of the dependent variable in a specific order. FPDEs have caught people's attention because they offer insights into the peculiar movement and intricate occurrences within diverse physical systems.

Conformable fractional derivative

The conformable fractional derivative of order β with respect to the variable t is defined as follows:

Let $j : (0,\infty) \longrightarrow \mathbb{R}$ be a function. Then, the conformable derivative of a function of order β is

$$D_t^{\beta} j(t) = \lim_{\varepsilon \to 0} \frac{j(t + \varepsilon t^{1-\beta}) - j(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$ and $\beta \in (0, 1]$ signifies the order of derivative.

The following theorem lists some significant characteristics of conformable derivatives.

Theorem 1. *Assume that* $\beta \in (0, 1]$*, and* j *and* k *are* β*-differentiable at* $t > 0$*. Then*

$$D_t^{\beta}(cj + dk) = cD_t^{\beta}(j) + dD_t^{\beta}(k), \quad \forall\, c, d \in \mathbb{R}, \tag{2}$$

$$D_t^{\beta}(jk) = jD_t^{\beta}(k) + kD_t^{\beta}(j), \tag{3}$$

$$D_t^{\beta}(t^{\alpha}) = \alpha t^{\alpha-\beta}, \quad \forall \alpha \in \mathbb{R}, \tag{4}$$

$$D_t^{\beta}\left(\frac{j}{k}\right) = \frac{kD_t^{\beta}(j) - jD_t^{\beta}(k)}{k^2}. \tag{5}$$

Additionally, if k *is a differentiable function, then*

$$D_t^{\beta}(j)(k) = t^{1-\beta} \frac{dk}{dt}(t). \tag{6}$$

A novel and well-structured idea of the fractional derivative, known as the conformable fractional derivative, was introduced by [18]. In comparison to ordinary fractional derivatives, this concept is technically simpler and easier to apply. Furthermore, it retains certain essential aspects of classical calculus that other fractional derivatives often do not, such as the chain rule [19]. However, one recognized drawback of the conformable fractional derivative is that the fractional derivative of any differentiable function evaluated at 0 disappears. To address this problem, alternate formulations and appropriate adjustments were presented in Refs. [19,20], thus overcoming this limitation.

In recent years, a number of analytical approaches have been successfully used to determine accurate solutions for conformable fractional differential equations (CFPDE). These methods include the modified Kudryashov method [21, 22], the sardar-sub equation method [23–25], the auxiliary equation method [26–28], the extended Fan sub-equation method [29, 30], the (G'/G)-expansion method [31], Hirota bilinear technique [32], Lie symmetry method [33, 34], exp-function method [35, 36], the Adomian decomposition method [37,38] and lots of others.

In this article, we investigate new and more general closed form traveling wave solutions to the nonlinear conformable space-time fractional SRLW equation

$$D_t^{2\alpha} u + D_x^{2\alpha} u + uD_t^{\alpha}(D_x^{\alpha} u) + D_x^{\alpha} u D_t^{\alpha} u + D_t^{2\alpha}(D_t^{2\alpha} u) = 0, \quad 0 < \alpha \leqslant 1, \tag{7}$$

which arises in several physical applications including ion sound waves in plasma. This equation describes weakly nonlinear ion acoustic and space-charge waves and the real-valued $u(x, t)$ corresponds to the dimensionless fluid velocity with a decay condition.

Utilizing fractional derivatives into the SRLW equation enables the incorporation of memory and hereditary features, resulting in a more realistic depiction of physical processes. The novelty of current study stems from the investigation of the SRLW problem using the extended Fan-sub equation method, which has never been addressed in a previous work. The expanded Fan sub-equation method, in particular, is a powerful and a systematic manner to developing precise analytical results for nonlinear equations. It is distinguished by its simple implementation and capacity to produce several exact solutions in closed form. However, this method has certain limitations. It is extremely dependent on the suitable choice of an auxiliary equation and a balancing process, and it may be less effective when applied to evolution equations with very intricate nonlinearities or changing coefficients.

In contrast to previous investigations, which focused on the conventional integer order form, this research explores the SRLW equation using the conformable fractional calculus framework. As a result, it generates numerous new exact analytic solutions, demonstrating that the given approach is powerful and dependable for nonlinear conformable fractional models. The motivation for the present work is to further understand nonlinear wave propagation emanating from real-world applications, such as shallow water waves, the motion of long waves in dispersive media, and certain problems of subsurface flow, including the analysis of depth and spacing of parallel subsurface drains. The space-time conformable SRLW equation is a more realistic model, taking into consideration fractional-order effects due to memory and hereditary properties, which are not expressed by classical integer-order models.

This problem is considered in order to further our insight into wave dynamics and develop an efficient analytical framework that can be extended to other nonlinear fractional evolution equations. The discussion in the article proceeds as follows: The approach for the analytical procedure is presented in Sec. 2. In Sec. 3, we derive the analytical solutions for the model. The next Sec. 4 contains a survey of our results. Finally, Sec. 5 includes concluding remarks.

2. The strategy of extended fan sub-equation method

Consider the following nonlinear fractional differential equation of the type

$$S(u, D_t^{\alpha} u, D_x^{\beta} u, D_t^{\alpha} D_t^{\alpha} u, D_t^{\alpha} D_x^{\beta} u, D_x^{\beta} D_x^{\beta} u, \ldots) = 0, \tag{8}$$

where u is an unknown function and S is a polynomial in $u = u(x, t)$.

Consider the nonlinear fractional complex transformation:

$$u(x, t) = u(\eta), \qquad \eta = \frac{kx^{\alpha}}{\alpha} + \frac{ct^{\alpha}}{\alpha}, \tag{9}$$

where k and c are non-zero arbitrary constants. Now putting Eq. (9) into Eq. (8), the PDE is reduced to an ODE

$$T(U, U', U'', \ldots) = 0. \tag{10}$$

Let the solution of Eq. (10) be assumed as:

$$u(\eta) = \sum_{i=0}^{n} \varpi_i \, \theta^i(\eta), \tag{11}$$

where $\theta(\eta)$ is a function satisfying the following Riccati differential equation:

$$\theta'(\eta) = b_0 + b_1\theta(\eta) + b_2\theta^2(\eta) + b_3\theta^3(\eta) + b_4\theta^4(\eta), \tag{12}$$

where b_i $(i = 0, 1, 2, 3, 4)$ are constants, and n is a balancing term to be determined by balancing the highest order linear term with the nonlinear terms.

Putting Eq. (11) and Eq. (12) into Eq. (10) and gathering the identical terms of θ^j and θ^k and setting them to zero gives a system of algebraic equations that can be solved further to obtain the required solutions.

Thus, the exact solutions of Eq. (9) are in the form of θ_ι^{I}, θ_ι^{II}, θ_ι^{III}, θ_ι^{IV}, θ_ι^{V}. The superscripts represent the group of solutions, whereas the subscripts show the rank of the solutions.

3. Implementation

Let us consider Eq. (7) for constructing exact solutions. Substituting the following travelling wave transformation:

$$u(x,t) = u(\eta), \qquad \eta = \frac{kx^\alpha}{\alpha} + \frac{ct^\alpha}{\alpha}, \tag{13}$$

which yields a NODE:

$$c^2u'' + k^2u'' + ckuu'' + ck(u')^2 + c^2k^2u^{iv} = 0. \tag{14}$$

Integration of Eq. (14) twice yields,

$$(c^2 + k^2)u + \frac{1}{2}cku^2 + c^2k^2u'' + ru + s = 0. \tag{15}$$

Using the homogeneous balancing principle, we deduce that $n = 2$. Thus, the solutions of Eq. (11) can be assumed in the form

$$u(\eta) = \varpi_0 + \varpi_1\theta(\eta) + \varpi_2\theta(\eta)^2, \tag{16}$$

where

$$\begin{aligned} \theta'(\eta)^2 = b_0 + b_1\theta(\eta) + b_2\theta(\eta)^2 \\ + b_3\theta(\eta)^3 + b_4\theta(\eta)^4. \end{aligned} \tag{17}$$

By setting Eq. (16) and its derivatives into Eq. (15) and comparing coefficients, we derive the following system of algebraic equations:

$$b_1c^2k^2\varpi_1 + 4b_0c^2k^2\varpi_2 + 2c^2\varpi_0 + ck\varpi_0^2 + 2k^2\varpi_0 + 2r\varpi_0 + 2s = 0, \tag{18}$$

$$2\varpi_1(b_2c^2k^2 + c^2 + k^2 + r) + 6b_1c^2k^2\varpi_2 + 2ck\varpi_0\varpi_1 = 0, \tag{19}$$

$$2\varpi_2(4b_2c^2k^2 + c^2 + k^2 + r) + ck\varpi_1(3b_3ck + \varpi_1) + 2ck\varpi_0\varpi_2 = 0, \tag{20}$$

$$2ck[2b_4ck\varpi_1 + \varpi_2(5b_3ck + \varpi_1)] = 0, \tag{21}$$

$$ck\varpi_2(12b_4ck + \varpi_2) = 0. \tag{22}$$

We select variable suitability which gives

$$\begin{aligned} b_0 = {} & \frac{3b_3^4c^3k^3 + 12b_4b_3^2c^3k + 12b_4b_3^2c^2k^2\varpi_0}{768b_4^3c^3k^3} + \frac{32b_4^2c^2\varpi_0 + 12b_4b_3^2ck^3 + 12b_4b_3^2ckr}{768b_4^3c^3k^3} \\ & + \frac{16b_4^2ck\varpi_0^2 + 32b_4^2k^2\varpi_0 + 32b_4^2r\varpi_0 + 32b_4^2s}{768b_4^3c^3k^3}, \end{aligned} \tag{23}$$

$$b_1 = -\frac{b_3(b_3^2c^2k^2 + 4b_4c^2 + 4b_4ck\varpi_0)}{32b_4^2c^2k^2} - \frac{b_3(4b_4k^2 + 4b_4r)}{32b_4^2c^2k^2}, \tag{24}$$

$$b_2 = \frac{3b_3^2c^2k^2 - 4b_4c^2 - 4b_4ck\varpi_0}{16b_4c^2k^2} - \frac{4b_4k^2 + 4b_4r}{16b_4c^2k^2}, \tag{25}$$

$$\varpi_1 = -6b_3ck, \tag{26}$$

$$\varpi_2 = -12b_4ck. \tag{27}$$

Case I

If b_0, b_1, b_2, b_3, b_4 are all non-zero constants then there exists parameters, such that $b_0 = d^2$, $b_1 = 2ed$, $b_2 = 2ad + e^2$, $b_3 = 2ae$, $b_4 = a^2$. Let us define

$$\xi = \frac{ct^\alpha}{\alpha} + \frac{kx^\alpha}{\alpha}, \qquad \Delta = \sqrt{e^2 - 4ad}. \tag{28}$$

Type I: $e^2 - 4ad > 0$, $ea \neq 0$, $ad \neq 0$

$$u_1^I(x,t) = \varpi_0 + \frac{\varpi_2}{4a^2}\left(\Delta \tanh\left(\frac{\Delta}{2}\xi\right) + e\right)^2 - \frac{\varpi_1}{2a}\left(\Delta \tanh\left(\frac{\Delta}{2}\xi\right) + e\right), \tag{29}$$

$$u_2^I(x,t) = \varpi_0 + \frac{\varpi_2}{4a^2}\left(\Delta \coth\left(\frac{\Delta}{2}\xi\right) + e\right)^2 - \frac{\varpi_1}{2a}\left(\Delta \coth\left(\frac{\Delta}{2}\xi\right) + e\right). \tag{30}$$

The kink soliton arises in the form as follows:

$$u_3^I(x,t) = \varpi_0 - \frac{\varpi_1(\Delta + e)}{2a}\left[\tanh(\Delta\xi) + i\,\mathrm{sech}(\Delta\xi)\right] + \frac{\varpi_2(\Delta + e)^2}{4a^2}\left[\tanh(\Delta\xi) + i\,\mathrm{sech}(\Delta\xi)\right]^2. \tag{31}$$

$$u_6^I(x,t) = \varpi_0 + \frac{\varpi_2}{4a^2}\left(\Phi - e\right)^2 + \frac{\varpi_1}{2a}\left(\Phi - e\right), \tag{32}$$

where $\Phi = (\sqrt{(A^2 + B^2)\Delta^2} - A\Delta\cosh(\Delta\xi))/(A\sinh(\Delta\xi) + B)$,

$$u_7^I(x,t) = \varpi_0 + \frac{\varpi_2}{4a^2}\left(\Psi - e\right)^2 + \frac{\varpi_1}{2a}\left(\Psi - e\right), \tag{33}$$

where $\Psi = -(\sqrt{(B^2 - A^2)\Delta^2} + A\Delta\sinh(\Delta\xi))/(A\cosh(\Delta\xi) + B)$.

Type II: $e^2 - 4ad < 0$, $ea \neq 0$, $ad \neq 0$

Let $\delta = \sqrt{4ad - e^2}$. Then:

$$u_{13}^I(x,t) = \varpi_0 + \frac{\varpi_1}{2a}\left(\delta \tan\left(\frac{\delta}{2}\xi\right) - e\right) + \frac{\varpi_2}{4a^2}\left(\delta \tan\left(\frac{\delta}{2}\xi\right) - e\right)^2, \tag{34}$$

$$u_{17}^I(x,t) = \varpi_0 + \frac{\varpi_1(\delta - 2e)}{4a}\left[-\cot\left(\frac{\delta}{4}\xi\right) + \tan\left(\frac{\delta}{4}\xi\right)\right] + \frac{\varpi_2(\delta - 2e)^2}{16a^2}\tan^2\left(\frac{\delta}{4}\xi\right), \tag{35}$$

$$u_{21}^I(x,t) = \varpi_0 + \frac{2r\varpi_1 \sin\left(\frac{\delta}{2}\xi\right)}{\delta\cos\left(\frac{\delta}{2}\xi\right)} - \frac{2r\varpi_1}{e} + \frac{4r^2\varpi_2 \sin^2\left(\frac{\delta}{2}\xi\right)}{\delta^2\cos^2\left(\frac{\delta}{2}\xi\right)} - \frac{4r^2\varpi_2}{e^2}, \tag{36}$$

$$u_{24}^I(x,t) = \varpi_0 + \frac{4r\varpi_1 \sin\left(\frac{\delta}{4}\xi\right)\cos\left(\frac{\delta}{4}\xi\right)}{\delta\left[2\cos^2\left(\frac{\delta}{2}\xi\right) - 1\right]} - \frac{2r\varpi_1}{e} + \frac{16r^2\varpi_2 \sin^2\left(\frac{\delta}{4}\xi\right)\cos^2\left(\frac{\delta}{4}\xi\right)}{\delta^2\left[2\cos^2\left(\frac{\delta}{2}\xi\right) - 1\right]^2} - \frac{4r^2\varpi_2}{e^2}. \tag{37}$$

Case II

If $b_0 = d^2$, $b_1 = 2da$, $b_2 = 2ae$, $b_4 = e^2$, then Eq. (17) reduces to

$$\theta'(\eta)^2 = b_0 + b_1\theta(\eta) + b_2\theta(\eta)^2 + b_4\theta(\eta)^4. \tag{38}$$

The corresponding limitations are $e^2 = -2da$ and $da < 0$. Let $\mu = \sqrt{-ad}$.

Type I: $ad < 0$, $ad \neq 0$

$$u_1^{II}(x,t) = \varpi_0 + \frac{\varpi_2}{4a^2}\left(\sqrt{6}\mu \tanh\left(\sqrt{\frac{3}{2}}\mu\xi\right) + \sqrt{2}\mu\right)^2 - \frac{\varpi_1}{2a}\left(\sqrt{6}\mu \tanh\left(\sqrt{\frac{3}{2}}\mu\xi\right) + \sqrt{2}\mu\right), \tag{39}$$

$$\begin{aligned} u_3^{II}(x,t) = \varpi_0 - \varpi_1 &\left[\sqrt{2}\mu + \frac{\sqrt{6}\mu}{2a}\left(i\,\mathrm{sech}(\sqrt{6}\mu\xi)\right) + \frac{\tanh(\sqrt{6}\mu\xi)}{2a}\right] \\ &+ \varpi_2\left[\sqrt{2}\mu + \frac{\sqrt{6}\mu}{4a^2}\left(i\,\mathrm{sech}(\sqrt{6}\mu\xi)\right)^2 + \frac{\tanh^2(\sqrt{6}\mu\xi)}{4a^2}\right]. \end{aligned} \tag{40}$$

Case III

If $b_0 = b_1 = 0$, then

$$\theta'(\eta)^2 = b_2\theta(\eta)^2 + b_3\theta(\eta)^3 + b_4\theta(\eta)^4. \tag{41}$$

Type I: $b_2 = 1,\ \ b_3 = -(2r/p),\ \ b_4 = ([r^2 - q^2]/p^2)$

$$u_1^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\,\mathrm{sech}(\xi)}{r\,\mathrm{sech}(\xi) + q} + \frac{p^2\varpi_2\,\mathrm{sech}^2(\xi)}{(r\,\mathrm{sech}(\xi) + q)^2}. \tag{42}$$

Type II: $b_2 = 1$, $b_3 = -2r/p$, $b_4 = ([r^2 + q^2]/p^2)$

$$u_2^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\,\mathrm{csch}(\xi)}{r\,\mathrm{csch}(\xi) + q} + \frac{p^2\varpi_2\,\mathrm{csch}^2(\xi)}{(r\,\mathrm{csch}(\xi) + q)^2}. \tag{43}$$

Type III: $b_2 = 4$, $b_3 = -(c4[2q + s]/p)$, $b_4 = ([r^2 + 4q^2 + 4qs]/p^2)$

$$u_3^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\,\mathrm{sech}^2(\xi)}{r\tanh(\xi) + q\,\mathrm{sech}^2(\xi) + s} + \frac{p^2\varpi_2\,\mathrm{sech}^4(\xi)}{(r\tanh(\xi) + q\,\mathrm{sech}^2(\xi) + s)^2}. \tag{44}$$

Type IV: $b_2 = 4$, $b_3 = -[4(s - 2q)]/p$, $b_4 = ([r^2 + 4q^2 - 4qs]/p^2)$

$$u_4^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\,\mathrm{csch}^2(\xi)}{r\coth(\xi) + q\,\mathrm{csch}^2(\xi) + s} + \frac{p^2\varpi_2\,\mathrm{csch}^4(\xi)}{(r\coth(\xi) + q\,\mathrm{csch}^2(\xi) + s)^2}. \tag{45}$$

Type V: $b_2 = p^2$, $b_3 = 2pq$, $b_4 = q^2$
Let $\zeta = \sinh(p\xi) + \cosh(p\xi)$. Then:

$$u_5^{III}(x,t) = \varpi_0 - \frac{pr\varpi_1}{q(\zeta + r)} + \frac{p^2r^2\varpi_2}{q^2(\zeta + r)^2}, \tag{46}$$

$$u_6^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\zeta}{q(\zeta + r)} + \frac{p^2\varpi_2\zeta^2}{q^2(\zeta + r)^2}. \tag{47}$$

Type VI: $b_2 = -1$, $b_3 = 2r/p$, $b_4 = ([-r^2 - q^2]/p^2)$

$$u_7^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\sec(\xi)}{r\sec(\xi) + q} + \frac{p^2\varpi_2\sec^2(\xi)}{(r\sec(\xi) + q)^2}, \tag{48}$$

$$u_8^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\csc(\xi)}{r\csc(\xi) + q} + \frac{p^2\varpi_2\csc^2(\xi)}{(r\csc(\xi) + q)^2}. \tag{49}$$

Type VIII: $b_2 = -4$, $b_3 = 4(2q + s)/p$, $\ \ b_4 = ([-r^2 + 4q^2 + 4qs]/p^2)$

$$u_9^{III}(x,t) = \varpi_0 + \frac{p\varpi_1\sec^2(\xi)}{q\sec^2(\xi) + r\tan(\xi) + s} + \frac{p^2\varpi_2\sec^4(\xi)}{(q\sec^2(\xi) + r\tan(\xi) + s)^2}. \tag{50}$$

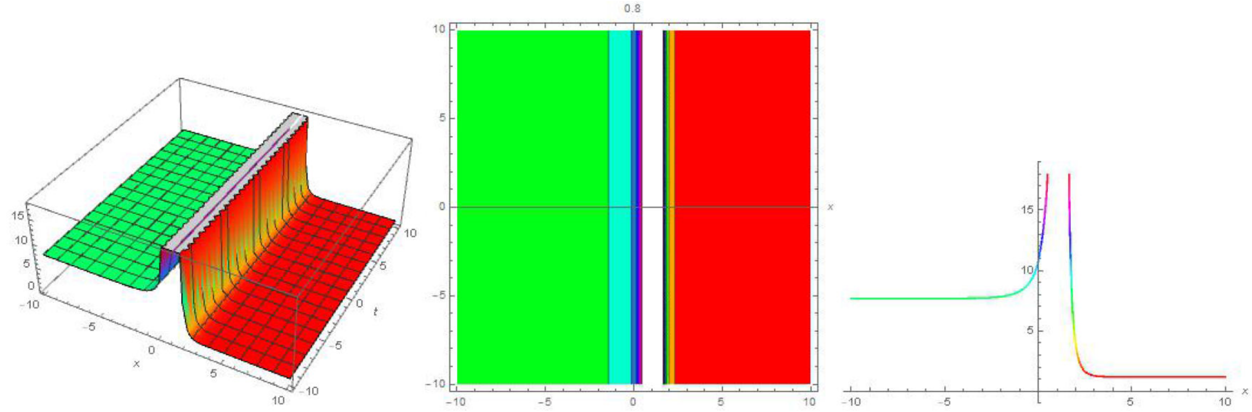

FIGURE 1. The singular solution of Eq. (30) for $\varpi_0 = -1.2$, $\varpi_1 = 0.3$, $\varpi_2 = -1.4$, $a = 1$, $d = -0.1$, $e = 2$, $c = 0.9$, $\alpha = 0.9$, $k = -0.6$.

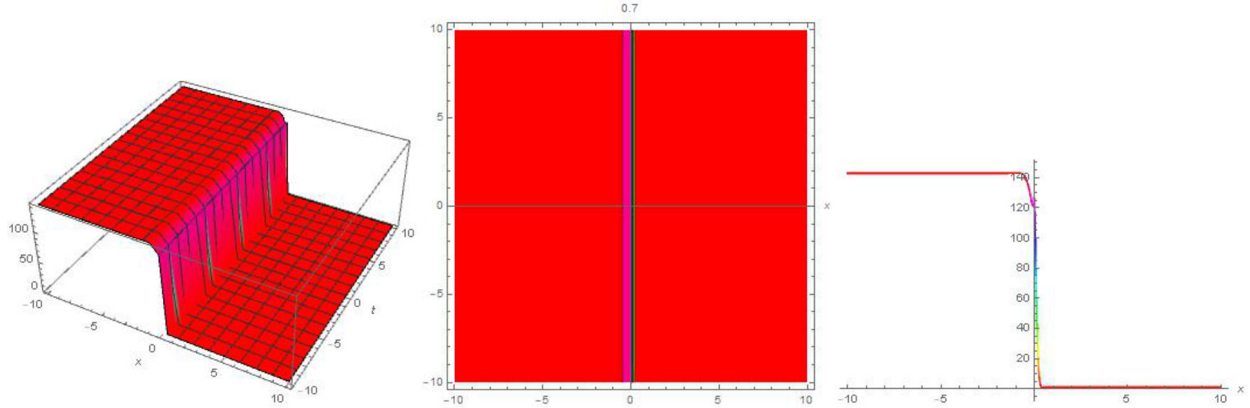

FIGURE 2. The kink-shaped solution of Eq. (31) for $\varpi_0 = -1.2$, $\varpi_1 = 0.3$, $\varpi_2 = -1.4$, $a = 0.2$, $d = 0.1$, $e = 2$, $c = 0.6$, $\alpha = 0.8$, $k = -2.9$.

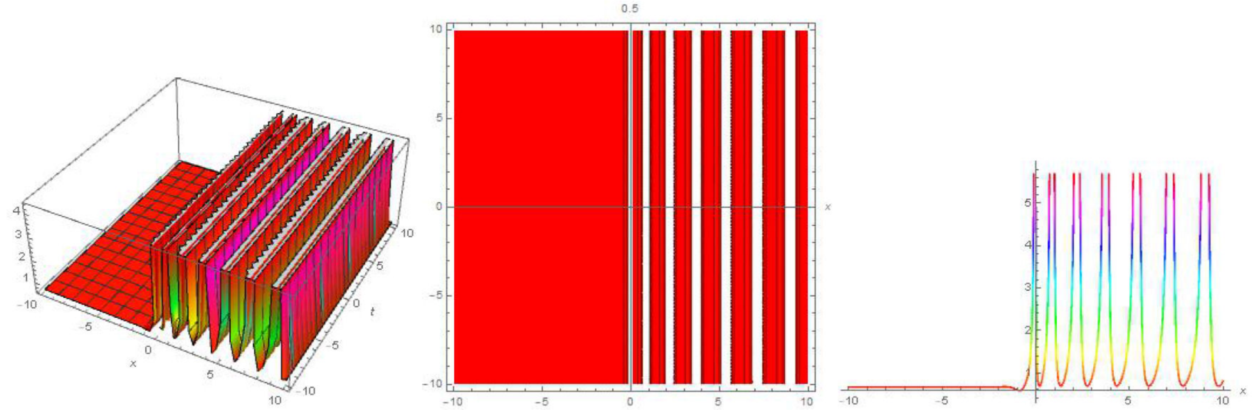

FIGURE 3. The periodic solution of Eq. (34) for $\varpi_0 = 0.7$, $\varpi_1 = -0.3$, $\varpi_2 = 1.4$, $a = 2$, $d = 1$, $e = 0.9$, $c = -1.9$, $\alpha = 0.8$, $k = -1.9$.

4. Results and discussion

In this section, we illustrate the space-time conformable symmetric regularized long wave equation graphically. The extended Fan sub-equation method was used to obtain the exact traveling wave solutions. The obtained results are visualized in 3D, contour, and 2D graphs to illustrate the physical behavior of the solutions.

Figures 1-5 show the graphical solutions of the space-time conformable symmetric regularized long wave equation for various values of the parameters a, d, e, p, q, r, s, c, k, and α. Figures 1 and 4 display singular soliton solutions. A kink-shaped solution is shown in Fig. 2. Figure 3 presents a periodic solution. For the constant values indicated below, a singular-periodic solution is depicted in Fig. 5. The contour

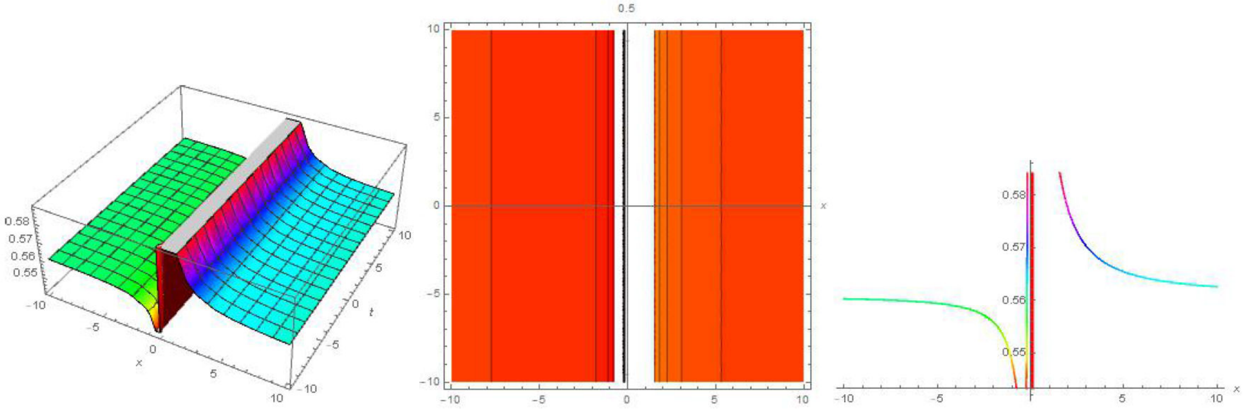

FIGURE 4. The singular solution of Eq. (37) for $\varpi_0 = -0.3$, $\varpi_1 = 1.7$, $\varpi_2 = -1.4$, $a = 0.3$, $d = 0.5$, $e = 0.4$, $c = 0.9$, $\alpha = -1.4$, $k = 0.4$.

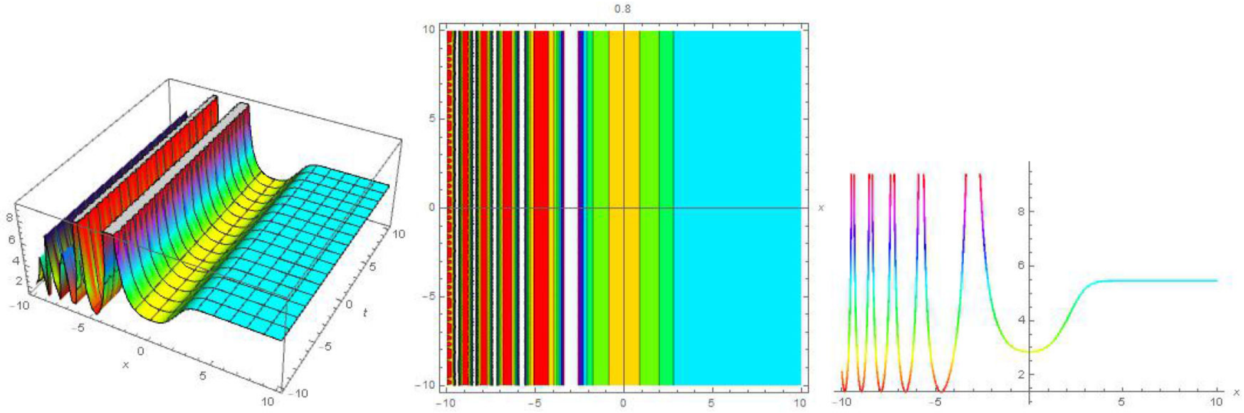

FIGURE 5. The singular-periodic solution of Eq. (40) for $\varpi_0 = 0.1$, $\varpi_1 = 2$, $\varpi_2 = 0.4$, $a = -1$, $d = 1$, $e = 0.4$, $c = -0.4$, $\alpha = 2.5$, $k = 0.1$.

tour and 2D charts reveal the nature of the nonlinear waves produced by Eq. (7).

5. Conclusion

In this paper, the space-time fractional symmetric regularized long wave (SRLW) equation has been effectively solved using the extended Fan sub-equation method within the conformable fractional framework. The obtained solutions are new and take the form of singular, kink, singular periodic, and periodic solitons. Three-dimensional, contour, and two-dimensional plots have been used to demonstrate the behavior of the nonlinear model. All required mathematical calculations and graphical representations were obtained using Mathematica 11.0.

The discovered solutions contribute to a better understanding of the motion and dynamics of long waves with small amplitudes in dispersive media, and they could serve as useful tools for simulating physical phenomena such as shallow water waves and subsurface flow problems. Furthermore, the findings demonstrate that the extended Fan sub-equation approach is a reliable analytical tool for solving nonlinear conformable fractional partial differential equations.

Future research could extend the current technique to higher-dimensional and variable-coefficient fractional models, as well as other nonlinear evolution equations arising in mathematical physics and engineering. Additionally, exploring the stability and physical interpretation of the obtained solutions, along with numerical simulations to validate the analytical results, will provide further insight into the presented technique and its effectiveness.

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