Relating the free particle with the harmonic oscillator

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We show that by means of a coordinate transformation in the extended configuration space the problem of a free particle can be related to that of a harmonic oscillator in classical mechanics and in quantum mechanics.

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1. Introduction

The quantum harmonic oscillator is one of the basic examples studied in the textbooks on quantum mechanics and modern physics. The Schrödinger equation for the harmonic oscillator can be solved exactly by various methods. The aim of this paper is to add another method to the ones already known. We show that, essentially, by means of a coordinate transformation mixing the space and time variables it is possible to relate the solutions of the Schrödinger equation for a free particle with the solutions of the Schrödinger equation for a harmonic oscillator.

In Sec. 2 we start by showing that in the framework of classical mechanics, certain coordinate transformation in the extended configuration space relates the motion a free particle with that of a harmonic oscillator. In Sec. 2.1 we show that this coordinate transformation leads from the standard Lagrangian for a free particle to one *equivalent* to that of a harmonic oscillator. In Sec. 2.2 we show in an elementary manner that the solutions of the Hamilton–Jacobi (HJ) equation for a free particle are related to the solutions of the HJ equation for a harmonic oscillator. This result has been obtained previously in Ref. [1] as an example of the effect on the HJ equation of a coordinate transformation in the extended configuration space.

In Sec. 3 we show that the solutions of the (time-dependent) Schrödinger equation for a harmonic oscillator are equal to the solutions of the Schrödinger equation for a free particle multiplied by a fixed factor and, making use of this relation, we express the propagator for the harmonic oscillator in terms of that of a free particle.

2. The relation in the framework of classical mechanics

Throughout this paper we make use of the coordinate transformation in the extended configuration space given by

$$q = q' \sec \omega t', \qquad t = \frac{\tan \omega t'}{\omega}, \tag{1}$$

where ω is a constant. This coordinate transformation has been employed in Ref. [1] relating the HJ equation for a one-dimensional harmonic oscillator with that of the free particle. In fact, a straightforward computation shows that the coordinate transformation (1) relates the solutions of the equations of motion of the two systems. Indeed, the position of a free particle as a function of the time is given by

$$q = At + B, \tag{2}$$

where A and B are constants. Substituting Eqs. (1) into Eq. (2) one obtains $q' \sec \omega t' = (A/\omega) \tan \omega t' + B$, which is equivalent to

$$q' = (A/\omega) \sin \omega t' + B \cos \omega t'. \tag{3}$$

The last equation can be recognized as the general solution of the equations of motion for a harmonic oscillator with frequency ω. As we shall see in the following two subsections this result can be obtained in more elaborated ways making use of the Lagrangian formalism and the HJ equation.

It may be noticed that the relation between t and t' is not one-to-one; all the real values of t are mapped to an interval of t' of length π/ω. However, this behavior does not seem to affect the results of the following sections; in this sense, we can say that (1) is just a transformation that *locally* relates the two problems.

2.1. Relation via the Lagrangian formalism

The standard Lagrangian for a free particle is

$$L = \frac{m}{2} \dot{q}^2. \tag{4}$$

In the case of a coordinate transformation in the extended configuration space ($q' = q'(q,t)$, $t' = t'(q,t)$) it is not enough to make the substitutions in a given Lagrangian. Instead, in order to maintain the form of the equations of motion, the Lagrangian must be replaced by $L' = L\,(\mathrm{d}t/\mathrm{d}t')$ (see, *e.g.*, Ref. [2]). In this case we have

$$
\begin{aligned}
L\,\frac{\mathrm{d}t}{\mathrm{d}t'} &= \frac{1}{2}m\left(\frac{\mathrm{d}(q'\sec\omega t')}{\mathrm{d}(\tan\omega t'/\omega)}\right)^2\frac{\mathrm{d}(\tan\omega t'/\omega)}{\mathrm{d}t'} \\
&= \frac{1}{2}m\left(\frac{\sec\omega t'\mathrm{d}q' + q'\sec\omega t'\tan\omega t'\omega\,\mathrm{d}t'}{\sec^2\omega t'\mathrm{d}t'}\right)^2\sec^2\omega t' \\
&= \frac{1}{2}m\left(\frac{\mathrm{d}q'}{\mathrm{d}t'} + \omega q'\tan\omega t'\right)^2 \\
&= \frac{1}{2}m\left[\left(\frac{\mathrm{d}q'}{\mathrm{d}t'}\right)^2 + 2\omega q'\frac{\mathrm{d}q'}{\mathrm{d}t'}\tan\omega t' + \omega^2 q'^2\sec^2\omega t' - \omega^2 q'^2\right] \\
&= \frac{1}{2}m\left[\left(\frac{\mathrm{d}q'}{\mathrm{d}t'}\right)^2 - \omega^2 q'^2\right] + \frac{\partial F}{\partial q'}\frac{\mathrm{d}q'}{\mathrm{d}t'} + \frac{\partial F}{\partial t'},
\end{aligned} \tag{5}
$$

where $F = \frac{1}{2}m\omega q'^2\tan\omega t'$. Owing to the form of the last two terms, they do not contribute in the Lagrange equations and therefore, $L\,\mathrm{d}t/\mathrm{d}t'$ yields the same equations of motion as the Lagrangian

$$
\frac{1}{2}m\left[\left(\frac{\mathrm{d}q'}{\mathrm{d}t'}\right)^2 - \omega^2 q'^2\right],
$$

which is the standard Lagrangian for a harmonic oscillator.

2.2. Relation via the Hamilton–Jacobi formalism

The HJ equation corresponding to the standard Hamiltonian of a free particle is given by

$$
0 = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t}. \tag{6}
$$

With the aid of the chain rule and Eqs. (1) we have

$$
\frac{\partial}{\partial q} = \cos\omega t'\frac{\partial}{\partial q'} \tag{7}
$$

$$
\frac{\partial}{\partial t} = \cos^2\omega t'\frac{\partial}{\partial t'} - \omega q'\sin\omega t'\cos\omega t'\frac{\partial}{\partial q'} \tag{8}
$$

and therefore Eq. (6) amounts to

$$
\begin{aligned}
0 &= \frac{1}{2m}\left(\cos\omega t'\frac{\partial S}{\partial q'}\right)^2 + \cos^2\omega t'\frac{\partial S}{\partial t'} - \omega q'\sin\omega t'\cos\omega t'\frac{\partial S}{\partial q'} \\
&= \cos^2\omega t'\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q'}\right)^2 + \frac{\partial S}{\partial t'} - \omega q'\tan\omega t'\frac{\partial S}{\partial q'}\right] \\
&= \cos^2\omega t'\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q'} - m\omega q'\tan\omega t'\right)^2 + \frac{m}{2}\omega^2 q'^2 - \frac{m}{2}\omega^2 q'^2\sec^2\omega t' + \frac{\partial S}{\partial t'}\right] \\
&= \cos^2\omega t'\left\{\frac{1}{2m}\left[\frac{\partial}{\partial q'}\left(S - \frac{m}{2}\omega q'^2\tan\omega t'\right)\right]^2 + \frac{m}{2}\omega^2 q'^2 + \frac{\partial}{\partial t'}\left(S - \frac{m}{2}\omega q'^2\tan\omega t'\right)\right\},
\end{aligned}
$$

which shows that S satisfies the HJ equation corresponding to the Hamiltonian for a free particle if and only if

$$
S' \equiv S - \tfrac{1}{2}m\omega q'^2\tan\omega t' \tag{9}
$$

satisfies the HJ equation for the harmonic oscillator [*cf.* Ref. [1], Eq. (6.98)]. Note that the last term in Eq. (9) is the function F appearing in Eq. (5).

3. The relation in the framework of quantum mechanics

In this section we shall show that the solutions of the Schrödinger equation for a free particle can be expressed in terms of the solutions of the Schrödinger equation for a harmonic oscillator making use of the coordinate transformation (1).

We proceed essentially as in the preceding section starting from the Schrödinger equation for a free particle,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial q^2} = \mathrm{i}\hbar\frac{\partial\Psi}{\partial t}. \tag{10}$$

Making use of (7) and (8) we find that Eq. (10) is equivalent to

$$-\frac{\hbar^2}{2m}\cos^2\omega t'\frac{\partial^2\Psi}{\partial q'^2} = \mathrm{i}\hbar\cos^2\omega t'\frac{\partial\Psi}{\partial t'} - \mathrm{i}\hbar\omega q'\sin\omega t'\cos\omega t'\frac{\partial\Psi}{\partial q'}$$

and eliminating the common factor $\cos^2\omega t'$, we have

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial q'^2} = \mathrm{i}\hbar\frac{\partial\Psi}{\partial t'} - \mathrm{i}\hbar\omega q'\tan\omega t'\frac{\partial\Psi}{\partial q'}. \tag{11}$$

Guided by Eq. (9), we look for a relation of the form

$$\Psi = f\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m}{2}\omega q'^2\tan\omega t'\right)\Psi', \tag{12}$$

where Ψ' is a solution of the Schrödinger equation for a harmonic oscillator,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial q'^2} + \frac{m}{2}\omega^2 q'^2\Psi' = \mathrm{i}\hbar\frac{\partial\Psi'}{\partial t'}, \tag{13}$$

and f is a function to be determined. Substituting (12) into Eq. (11), making use of (13), one finds that f is a function of t' only, such that

$$\frac{\mathrm{d}f}{\mathrm{d}t'} = -\frac{\omega}{2}\tan\omega t'\, f$$

and, therefore, we can take $f = \cos^{1/2}\omega t'$.

Thus, we conclude that Ψ' is a solution of the Schrödinger equation for a harmonic oscillator [Eq. (13)] if and only if

$$\Psi = \cos^{1/2}\omega t'\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m}{2}\omega q'^2\tan\omega t'\right)\Psi', \tag{14}$$

is a solution of the Schrödinger equation for a free particle [Eq. (10)]. Then, according to Eq. (1),

$$\int_{-\infty}^{\infty}|\Psi|^2\mathrm{d}q = \int_{-\infty}^{\infty}\cos\omega t'\,|\Psi'|^2\mathrm{d}q = \int_{-\infty}^{\infty}|\Psi'|^2\mathrm{d}q',$$

which means that $\Psi(q,t)$ is normalized if and only if $\Psi'(q',t')$ is normalized.

Since the solutions of the Schrödinger equation for a free particle can be obtained in a simple way, one would think that the interesting applications of Eq. (14) correspond to expressing Ψ' in terms of Ψ; however, a nice example of the application of Eq. (14) as it stands is the following: the ground state solution of the Schrödinger equation for a harmonic oscillator is given by

$$\Psi'(q',t') = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}q'^2 - \frac{\mathrm{i}}{2}\omega t'\right).$$

Substituting this expression into the right-hand side of Eq. (14), the result, written in terms of q and t, is

$$\Psi(q,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{1+\mathrm{i}\omega t}}\exp\left[-\frac{m\omega q^2}{2\hbar(1+\mathrm{i}\omega t)}\right], \tag{15}$$

which must be a solution of the Schrödinger equation for a free particle. Evaluating the right-hand side of Eq. (15) at $t=0$ we see that

$$\Psi(q,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}q^2\right). \tag{16}$$

This means that Eq. (15) is the wave function of the free particle with the initial condition (16). (The standard procedure to find (15) involves the calculation of two Fourier transforms (see, *e.g.*, Ref. [3]).)

3.1. Relation between propagators

As is well known, the solution to the time-dependent Schrödinger equation can be expressed in the form

$$\Psi(q_{\rm f}, t_{\rm f}) = \int_{-\infty}^{\infty} K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i}) \Psi(q_{\rm i}, t_{\rm i}) \, {\rm d}q_{\rm i}, \tag{17}$$

where $K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i})$ is the so-called propagator, and in the case of a free particle one readily finds that

$$K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i}) = \sqrt{\frac{m}{2\pi {\rm i}\hbar (t_{\rm f} - t_{\rm i})}} \, \exp\left[\frac{\rm i}{\hbar} \frac{m(q_{\rm f} - q_{\rm i})^2}{2(t_{\rm f} - t_{\rm i})}\right]. \tag{18}$$

Hence, with the aid of Eq. (14) we can obtain the propagator for a harmonic oscillator. In fact, making use of (1), (14) and (17) we have (with $q'_{\rm f}, t'_{\rm f}$ related to $q_{\rm f}, t_{\rm f}$ by means of (1) and, similarly, $q'_{\rm i}, t'_{\rm i}$ related to $q_{\rm i}, t_{\rm i}$)

$$\begin{aligned}
\Psi'(q'_{\rm f}, t'_{\rm f}) &= \frac{1}{\cos^{1/2} \omega t'_{\rm f}} \exp\left(-\frac{\rm i}{\hbar} \frac{m}{2} \omega q'^2_{\rm f} \tan \omega t'_{\rm f}\right) \Psi(q_{\rm f}, t_{\rm f}) \\
&= \frac{1}{\cos^{1/2} \omega t'_{\rm f}} \exp\left(-\frac{\rm i}{\hbar} \frac{m}{2} \omega q'^2_{\rm f} \tan \omega t'_{\rm f}\right) \int_{-\infty}^{\infty} K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i}) \Psi(q_{\rm i}, t_{\rm i}) \, {\rm d}q_{\rm i} \\
&= \frac{1}{\cos^{1/2} \omega t'_{\rm f}} \exp\left(-\frac{\rm i}{\hbar} \frac{m}{2} \omega q'^2_{\rm f} \tan \omega t'_{\rm f}\right) \\
&\quad \times \int_{-\infty}^{\infty} K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i}) \cos^{1/2} \omega t'_{\rm i} \exp\left(\frac{\rm i}{\hbar} \frac{m}{2} \omega q'^2_{\rm i} \tan \omega t'_{\rm i}\right) \Psi'(q'_{\rm i}, t'_{\rm i}) \, \frac{1}{\cos \omega t'_{\rm i}} {\rm d}q'_{\rm i}.
\end{aligned}$$

Comparison with Eq. (17) shows that the propagator for the harmonic oscillator must be given by

$$K'(q'_{\rm f}, t'_{\rm f}; q'_{\rm i}, t'_{\rm i}) = \frac{K(q_{\rm f}, t_{\rm f}; q_{\rm i}, t_{\rm i})}{\cos^{1/2} \omega t'_{\rm f} \cos^{1/2} \omega t'_{\rm i}} \exp\left[\frac{\rm i}{\hbar} \frac{m}{2} \omega (q'^2_{\rm i} \tan \omega t'_{\rm i} - q'^2_{\rm f} \tan \omega t'_{\rm f})\right].$$

Substituting (18), expressing the result in terms of the primed variables making use of (1) one finds

$$K'(q'_{\rm f}, t'_{\rm f}; q'_{\rm i}, t'_{\rm i}) = \sqrt{\frac{m\omega}{2\pi {\rm i}\hbar \sin \omega (t'_{\rm f} - t'_{\rm i})}} \, \exp\left\{\frac{\rm i}{\hbar} \frac{m\omega \left[(q'^2_{\rm i} + q'^2_{\rm f}) \cos \omega (t'_{\rm f} - t'_{\rm i}) - 2 q'_{\rm i} q'_{\rm f}\right]}{2 \sin \omega (t'_{\rm f} - t'_{\rm i})}\right\}. \tag{19}$$

As is well known, one can find the energy levels of the quantum harmonic oscillator and the corresponding wavefunctions from the propagator (19) (see, *e.g.* Ref. [4]). It might seem strange that the solutions of the Schrödinger equation for the free particle, whose Hamiltonian has a continuous spectrum, can be put in a one-to-one correspondence with the solutions of the Schrödinger equation for the harmonic oscillator, whose Hamiltonian has a discrete spectrum. The reason is that, among the solutions of the Schrödinger equation for the harmonic oscillator that can be obtained by means of (14) are the non-normalizable wavefunctions. One should obtain the stationary states of the harmonic oscillator looking for those solutions with a time-dependence of the form $\exp(-{\rm i}Et'/\hbar)$, but that seems as complicated as solving the Schrödinger equation directly.

An entirely different approach and different results from those given above are presented in Ref. [5], where the solutions of the time-independent Schrödinger equation for a free particle are obtained as *limiting cases* of the time-independent Schrödinger equation for a harmonic oscillator.

4. Concluding remarks

It is natural to ask if a coordinate transformation similar to the one considered here can be useful in other cases or in connection with other equations (*i.e.*, not only the HJ equation or the Schrödinger equation).

Apart from the transformations that relate two different problems, there exist transformations relating a problem with itself (that is, symmetry transformations); if these transformations contain arbitrary parameters, such parameters will be incorporated in the new solution.

It may be remarked that the coordinate transformation (1) is not (part of) a canonical transformation since the time t is substituted by a new variable t', and that the relation (14) does not involve integral transforms. Even though the free particle can be regarded as a limiting case of the harmonic oscillator, the relations established here are not based in this fact, the connection between the two problems works in both ways.

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1. G.F. Torres del Castillo, *An Introduction to Hamiltonian Mechanics* (Springer, Cham, 2018). Sec. 6.4. https://doi.org/10.1007/978-3-319-95225-3.
2. G.F. Torres del Castillo, *op. cit.* Sec. 2.4.
3. D.J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2005), Prob. 2.22.
4. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965, reprinted by Dover, New York, 2010). Sec. 8-1.
5. G.S. Pogosyan, L.E. Vicent y K.B. Wolf, El límite del oscilador cuántico a la partícula libre, *Rev. Mex. Fís. E* **51** (2005) 18.