

Behavior of the Feshbach-Villars oscillator in Gürses space-time under Coulomb-type potential

A. Bouzenada* and A. Boumali†

*Laboratory of theoretical and applied Physics,
Echahid Cheikh Larbi Tebessi University, Algeria.*

* *e-mail: abdelmalek.bouzenada@univ-tebessa.dz; abdelmalekbouzenada@gmail.com*

† *e-mail: boumali.abdelmalek@gmail.com*

O. Mustafa

*Department of Physics, Eastern Mediterranean University,
G. Magusa, north Cyprus, Mersin 10 - Turkiye.
e-mail: omar.mustafa@emu.edu.tr*

H. Hassanabadi

*Department of Physics, Faculty of Science, University of Hradec Králové,
Rokitanského 62, 500 03 Hradec Králové, Czechia,
e-mail: h.hasanabadi@shahroodut.ac.ir*

Received 23 June 2024; accepted 18 March 2025

This study investigates the impact of gravitational fields on the spectroscopic structure of the Feshbach-Villars oscillator (FVO) within Gürses space-time. Utilizing the first-order Feshbach-Villars formulation of the Klein-Gordon equation, which describes relativistic wave dynamics for spinless particles, we analyze the quantum mechanical properties of the oscillator under a Coulomb-type potential. The corresponding wave functions and energy levels are derived for both free and interacting cases. Furthermore, we explore the effects of the interaction between the Coulomb-type potential and Gürses space-time on the behavior of the Feshbach-Villars oscillator, particularly in relation to its spectroscopic characteristics. This research provides valuable insights into the intricate relationship between quantum mechanics, relativity, and gravitational fields at the microscopic level.

Keywords: Klein-Gordon equation; feshbach-villars oscillator; topological defects; Gürses space-time; coulomb-type potential; biconfluent heun function.

DOI: <https://doi.org/10.31349/RevMexFis.72.020801>

1. Introduction

The investigation the examination of the impact of gravitational fields on the dynamics of quantum mechanical systems is a topic of significant interest. General Relativity (GR) posits that gravity is a geometric characteristic of space-time. The classical gravitational field arises from the curvature of space-time and accounts for phenomena such as black holes and gravitational waves. Conversely, quantum mechanics (QM) offers a framework for comprehending particle behaviour at the microscopic level and elucidating interactions among fundamental forces. However, the integration of General Relativity and Quantum Mechanics into a unified theory of quantum gravity remains challenging due to numerous unresolved technical and philosophical issues. To investigate the influence of gravitational fields on relativistic quantum systems, it is customary to transition the particles' relativistic dynamics from a flat Minkowski space to a curved background. This approach is applicable to various curvature-based models and facilitates theoretical predictions for large-scale observables that may be empirically evaluated in cosmology and astrophysics. We can enhance our understand-

ing of the quantum dimensions of gravity by examining the thermodynamic behaviour of relativistic particles in gravitational contexts and the associated statistical properties. This aids in comprehending the peculiar phenomena occurring in space [1–12].

Topological defects such as domain walls, cosmic strings, monopoles, and textures have been a significant focus of research in cosmology, particle physics, condensed matter physics, and astrophysics for many years. These structures are thought to arise from the Kibble process during the symmetry-breaking phase transitions of the early universe. Cosmic strings have garnered significant interest for their potential contributions to galaxy formation and gravitational lensing. We can enhance our understanding of high-energy particle physics by examining their characteristics. The proposition that cosmic strings could function as superconducting wires has significant implications for contemporary physics [13–18].

The quantum harmonic oscillator (HO) is an essential instrument in theoretical physics, offering understanding of intricate quantum mechanical systems. Its relativistic equivalent is crucial for simulating the interactions of molecules,

atoms, and nuclei. It provides precise analytical solutions that facilitate the comprehension of many mathematical and physical events from diverse perspectives [19, 20].

The Dirac oscillator (DO) is crucial for comprehending the behaviours of diverse relativistic quantum systems. This system affects the dynamics of spin-1/2 particles exhibiting linear paths, as proven by Itô *et al* [20]. In its non-relativistic limit, the Dirac operator resembles a conventional harmonic oscillator with significant spin-orbit coupling. Moshinsky and Szczepaniak identified the Dirac operator by incorporating an external linear potential into the free Dirac equation [21]. This altered the radial momentum operator into a configuration that is not minimally linked. This transformation can be extended to a position-dependent mass (PDM) formulation by utilising the PDM-Schrödinger [23–25] and PDM-Klein-Gordon equations [26–30].

The Klein-Gordon oscillator (KGO) was constructed using a framework analogous to that of Moshinsky and Szczepaniak's work on Dirac oscillators, specifically for bosonic particles [31, 32]. Recent research have investigated its covariant formulation in several curved spacetime settings. Research has examined the KGO's interaction with topological defects in Kaluza-Klein theory, its behaviour in spinning cosmic string spacetime under Coulomb-type potentials, and the influences of rotation, non-commutativity, and magnetic quantum flux. Numerous research have examined the temporal evolution and wave functions of spin-0 bosons, spin-1 bosons, and spin-1/2 fermions within a relativistic framework utilising Hamiltonian equations, including the Schrödinger equation and the Feshbach-Villars (FV) equation. The FV formalism was developed to provide a relativistic interpretation of the second-order Klein-Gordon equation for an individual particle. It decomposes the wave function into two components, resulting in an equation featuring a first-order temporal derivative. Recent investigations have examined the dynamical characteristics of individual particles in relativistic contexts utilising the FV technique [33–58].

This article is organised as follows: Initially, we examine the FV representation of the Feshbach-Villars formalism for spin-0 particles in Minkowski spacetime. Subsequently, we discuss the FV formulation inside Gürses space-time and examine the position-dependent mass (PDM) FVO in this framework. Subsequently, we obtain solutions for the free equation and the oscillator in Gürses space-time [59]. We further examine the interaction with a Coulomb-type potential for both the free and oscillator scenarios within this context. We now offer concluding observations. In this article, we utilise natural units ($\hbar = c = 1$) and adopt the metric convention (+, −, −, −).

2. The FV Representation of Feshbach-Villars (Spin-0) in Minkowski Space-time

2.1. An Overview of the Feshbach-Villars Approximation

This section explores the relativistic quantum formulation of a spin-0 particle propagating in Minkowski space-time, employing the metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The covariant Klein-Gordon equation governing a massive scalar particle Φ with mass $m > 0$ is expressed as [60–62]:

$$(\eta^{\mu\nu} D_\mu D_\nu + m^2) \Phi(x, t) = 0, \quad (1)$$

where the minimally coupled covariant derivative is given by $D_\mu = \partial_\mu - ieA_\mu$. In this context, the classical four-momentum is defined as $p_\mu = (E, -p_i)$, while the electromagnetic four-potential takes the form $A_\mu = (A_0, -A_i)$, with e denoting the absolute value of the particle's charge.

This equation can be recast into Hamiltonian form, resembling a Schrödinger-type equation:

$$\mathcal{H}\Phi(x, t) = i \frac{\partial}{\partial t} \Phi(x, t). \quad (2)$$

The Hamiltonian \mathcal{H} is formulated using the Feshbach-Villars (FV) linearization procedure, which converts the equation into a first-order time differential equation. To achieve this, a two-component wave function is introduced [40–48, 63]:

$$\Phi(x, t) = \begin{pmatrix} \phi_1(x, t) \\ \phi_2(x, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{i}{m} \mathcal{D} \\ 1 - \frac{i}{m} \mathcal{D} \end{pmatrix} \psi(x, t), \quad (3)$$

where $\psi(x, t)$ satisfies the Klein-Gordon wave equation, and \mathcal{D} is given by:

$$\mathcal{D} = \frac{\partial}{\partial t} + ieA_0(x). \quad (4)$$

This transformation results in wave functions satisfying the relations:

$$\psi = \phi_1 + \phi_2, \quad i\mathcal{D}\psi = m(\phi_1 - \phi_2). \quad (5)$$

For convenience, the components ϕ_1 and ϕ_2 can be expressed as:

$$\phi_1 = \frac{1}{2m} \left[m + i \frac{\partial}{\partial t} - eA_0 \right] \psi, \quad (6)$$

$$\phi_2 = \frac{1}{2m} \left[m - i \frac{\partial}{\partial t} + eA_0 \right] \psi. \quad (7)$$

Substituting these expressions into the governing equation yields:

$$\left[i \frac{\partial}{\partial t} - eA_0 \right] (\phi_1 + \phi_2) = m(\phi_1 - \phi_2), \quad (8)$$

$$\left[i \frac{\partial}{\partial t} - eA_0 \right] (\phi_1 - \phi_2) = \left[\frac{(p_i - eA_i)^2}{m} + m \right] \times (\phi_1 + \phi_2). \quad (9)$$

By adding and subtracting these equations, we obtain a system of first-order coupled differential equations:

$$\frac{(p_i - eA_i)^2}{2m}(\phi_1 + \phi_2) + (m + eA_0)\phi_1 = i\frac{\partial\phi_1}{\partial t}, \quad (10)$$

$$\frac{-(p_i - eA_i)^2}{2m}(\phi_1 + \phi_2) - (m - eA_0)\phi_2 = i\frac{\partial\phi_2}{\partial t}. \quad (11)$$

The Feshbach-Villars Hamiltonian describing a scalar particle interacting with an electromagnetic field is given by:

$$\mathcal{H}_{K-G} = (\tau_3 + i\tau_2)\frac{(p_i - eA_i)^2}{2m} + m\tau_3 + eA_0(x), \quad (12)$$

where τ_i ($i = 1, 2, 3$) are the standard 2×2 Pauli matrices:

$$\begin{aligned} \tau_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \tau_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \tau_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (13)$$

The Hamiltonian \mathcal{H}_{K-G} satisfies the generalized hermiticity condition. Specifically, if there exists a Hermitian linear operator β such that:

$$\mathcal{H}^\dagger = \beta\mathcal{H}\beta^{-1}, \quad (14)$$

then the Hamiltonian is pseudo-Hermitian [70–74], which can be rewritten as:

$$\mathcal{H}_{K-G} = \tau_3\mathcal{H}_{K-G}^\dagger\tau_3, \quad \mathcal{H}_{K-G}^\dagger = \tau_3\mathcal{H}_{K-G}\tau_3. \quad (15)$$

For the case of free particle propagation, assuming no external interaction ($A_\mu = 0$), the one-dimensional FV Hamiltonian simplifies to:

$$\mathcal{H}_0 = (\tau_3 + i\tau_2)\frac{p_x^2}{2m} + m\tau_3. \quad (16)$$

The solutions to the time-independent free Hamiltonian correspond to stationary states.

Assuming a solution of the form:

$$\Phi(x, t) = \Phi(x)e^{-iEt} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} e^{-iEt}, \quad (17)$$

where E represents the system's energy, the equation can be rewritten as:

$$\mathcal{H}_0\Phi(x) = E\Phi(x). \quad (18)$$

This formulation represents the one-dimensional FV equation for a free relativistic spin-0 particle, providing an alternative Schrödinger-type equation to the Klein-Gordon equation. This approach will be utilized to derive the Gürses solutions for wave equations in curved space-time.

3. The FV representation of spin-0 particle in Gürses space-time

The objective of this section is to analyze the Klein-Gordon oscillator (KGO) within the Gürses space-time framework using the Feshbach-Villars (FV) approach. It is well established that the general ly covariant form of the relativistic wave equation for a scalar particle in a Riemannian space-time, defined by the metric tensor $g_{\mu\nu}$, can be obtained by reformulating the Klein-Gordon equation as [5, 6, 8, 9]:

$$(\square + m^2 - \xi R)\Phi(x, t) = 0, \quad (19)$$

where \square represents the Laplace-Beltrami operator, given by:

$$\square = g^{\mu\nu}D_\mu D_\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu). \quad (20)$$

Here, ξ is a real, dimensionless coupling constant that governs the non-minimal interaction between the scalar field and spacetime curvature. It plays a pivotal role in mediating interactions between the scalar field and spacetime curvature within general relativity. When $\xi = 0$, the coupling is minimal, meaning the field interacts solely through the geometry of the metric. The specific case $\xi = 1/6$ corresponds to conformal coupling in a four-dimensional spacetime, ensuring the equation remains invariant under conformal transformations. In Gürses spacetime, where $R = -2\Omega^2$, this coupling introduces an effective mass term given by $m_{\text{eff}}^2 = m^2 - 2\xi\Omega^2$. The second parameter, R , represents the Ricci scalar curvature, defined as $R = g^{\mu\nu}R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci curvature tensor. Here, $g^{\mu\nu}$ denotes the inverse metric tensor, and $g = \det(g_{\mu\nu})$ represents the determinant of the metric tensor.

In the subsequent sections, we focus on investigating the quantum dynamics of spin-0 particles within a (1+2)-dimensional space-time and formulating the corresponding Feshbach-Villars (FV) framework for this context.

3.1. Feshbach-Villars oscillators in Gürses space-time

This section closely follows the work of Silenko [69], adopting the metric signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$. The determinant-related term $\sqrt{-g}$ is expressed as:

$$\mathcal{Y} = \frac{1}{2} \left\{ \partial_i, \sqrt{-g} \frac{g^{0i}}{g^{00}} \right\}. \quad (21)$$

The Hamiltonian governing the transformation is given by [69]:

$$\begin{aligned} \mathcal{H}_{\text{GFVT}} &= \tau_z \left(\frac{\mathcal{N}^2 + \mathcal{T}'}{2\mathcal{N}} \right) \\ &+ i\tau_y \left(\frac{-\mathcal{N}^2 + \mathcal{T}'}{2\mathcal{N}} \right) - i\mathcal{Y}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \mathcal{T}' = & \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \frac{m^2 - \xi R}{g^{00}} + \frac{1}{\mathcal{F}} \nabla_i (\sqrt{-g} G^{ij}) \nabla_j \left(\frac{1}{\mathcal{F}} \right) \\ & + \sqrt{\frac{\sqrt{-g}}{g^{00}}} G^{ij} \nabla_i \nabla_j \left(\frac{1}{\mathcal{F}} \right) + \frac{1}{4\mathcal{F}^4} [\nabla_i (\mathcal{U}^i)]^2 - \frac{1}{2\mathcal{F}^2} \nabla_i \left(\frac{g^{0i}}{g^{00}} \right) \nabla_j (\mathcal{U}^i) - \frac{g^{0i}}{2g^{00}\mathcal{F}^2} \nabla_i \nabla_j (\mathcal{U}^j), \end{aligned} \quad (23)$$

where the relevant quantities are defined as:

$$G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \quad \mathcal{F} = \sqrt{g^{00}\sqrt{-g}}, \quad \mathcal{U}^i = \sqrt{-g}g^{0i}. \quad (24)$$

For $\mathcal{N} = m$, the initial Feshbach-Villars transformations hold.

Following the approach in Refs. [69–74], we introduce a nonunitary transformation:

$$\Psi' = \mathcal{F}\Psi, \quad (25)$$

to obtain a pseudo-Hermitian Hamiltonian, specifically a ρ_3 -pseudo-Hermitian one, given by:

$$\mathcal{H}' = \mathcal{F}\mathcal{H}\mathcal{F}^{-1}, \quad (26)$$

which satisfies the relation:

$$\mathcal{H}' = \rho_3 \mathcal{H}'^\dagger \rho_3. \quad (27)$$

It is noteworthy that pseudo-Hermiticity [70] and/or non-Hermitian PT-symmetric Hamiltonians [70–72] (and related references cited therein) generalizes the concept of Hermiticity in quantum mechanics, allowing certain non-Hermitian Hamiltonians to retain real eigenvalues under specific conditions. This property is particularly relevant in scenarios where conventional Hermiticity is relaxed while ensuring the physical consistency of observables.

The Gürses metric in (1+2)-dimensions with circular stationary and rotational symmetry is [59]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dr^2 + 2\Omega r^2 dt d\varphi - r^2 (1 - \Omega^2 r^2) d\varphi^2, \quad (28)$$

with $a_0 = b_0 = e_0 = 1$, $b_1 = c_0 = \lambda_0 = 0$, and the vorticity $\Omega = -\mu/3$, in the Gürses metric

$$ds^2 = -\phi dt^2 + 2q dt d\varphi + \frac{h^2 \psi - q^2}{a_0} d\varphi^2 + \frac{1}{\psi} dr^2, \quad (29)$$

(i.e., as in Eq.(5) of [59]), where

$$\phi = a_0, \quad \psi = b_0 + \frac{b_1}{r^2} + \frac{3\lambda_0}{4} r^2, \quad q = c_0 + \frac{e_0 \mu}{3} r^2, \quad h = e_0 r, \quad \lambda_0 = \lambda + \frac{\mu^2}{27}. \quad (30)$$

Here, $\Omega = \pm|\Omega|$ represents the vorticities of the rotating space-time. The metric tensor and its inverse are:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \Omega r^2 \\ 0 & -1 & 0 \\ \Omega r^2 & 0 & -r^2 (1 - \Omega^2 r^2) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} (1 - \Omega^2 r^2) & 0 & \Omega \\ 0 & -1 & 0 \\ \Omega & 0 & -1/r^2 \end{pmatrix}, \quad (31)$$

with $\det(g_{\mu\nu}) = -r^2$.

Following the procedure in Ref. [52], we obtain

$$\mathcal{T}' = \frac{1}{g^{00}\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j) + \frac{m^2 - \xi R}{g^{00}} - \mathcal{Y}^2, \quad (32)$$

with $i = 1, 2$

$$\mathcal{Y} = \frac{1}{g^{00}\sqrt{-g}} \{ \partial_i, \sqrt{-g} g^{0i} \} = \frac{g^{02}}{g^{00}} \partial_2. \quad (33)$$

Thus,

$$\mathcal{T}' = \frac{1}{g^{00}\sqrt{-g}} \left[\partial_1 (\sqrt{-g}g^{11}\partial_1 + \partial_2 (\sqrt{-g}g^{22}\partial_2)) \right] + \frac{m^2 - \xi R}{g^{00}} - \mathcal{Y}^2, \quad (34)$$

leading to

$$\mathcal{T}' = \frac{1}{g^{00}r} \left[-\partial_r r \partial_r - \partial_\varphi \left(\frac{r}{r^2} \partial_\varphi \right) \right] + \frac{m^2 - \xi R}{g^{00}} - \mathcal{Y}^2. \quad (35)$$

The Hamiltonian $\mathcal{H}_{\text{GFVT}}$ is:

$$\mathcal{H}_{\text{GFVT}} = \tau_3 \left(\frac{\mathcal{N}^2 + \mathcal{T}'}{2\mathcal{N}} \right) + i\tau_2 \left(\frac{-\mathcal{N} + \mathcal{T}'}{2\mathcal{N}} \right) - i\mathcal{Y}, \quad (36)$$

where

$$\mathcal{H}_{\text{GFVT}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{-iEt} = i\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{-iEt}. \quad (37)$$

This yields

$$(\mathcal{T}' + \mathcal{N}^2) \phi_1 + (\mathcal{T}' - \mathcal{N}^2) \phi_2 = (2i\mathcal{N}\mathcal{Y} + 2\mathcal{N}E)\phi_1, \quad (38)$$

and

$$-(\mathcal{T}' + \mathcal{N}^2) \phi_2 - (\mathcal{T}' - \mathcal{N}^2) \phi_1 = (2i\mathcal{N}\mathcal{Y} + 2\mathcal{N}E)\phi_2. \quad (39)$$

These equations lead to:

$$\mathcal{N}(\phi_1 - \phi_2) = (E + i\mathcal{Y})(\phi_1 + \phi_2), \quad (40)$$

$$\mathcal{T}'(\phi_1 + \phi_2) = \mathcal{N}(E + i\mathcal{Y})(\phi_1 - \phi_2). \quad (41)$$

Equation (40) suggests:

$$\tilde{\psi} = \phi_1 - \phi_2 = \frac{1}{\mathcal{N}}(E + i\mathcal{Y})\psi,$$

where

$$\psi = \phi_1 + \phi_2,$$

leads to

$$\mathcal{T}'\psi = (E + i\mathcal{Y})^2\psi = (E^2 + 2iE\mathcal{Y} - \mathcal{Y}^2)\psi. \quad (42)$$

Substituting \mathcal{T}' from (35) into (42) gives:

$$\left[-\frac{1}{r}\partial_r r \partial_r - \frac{1}{r^2}\partial_\varphi^2 + \tilde{\Omega}^2 r^2 \right] R(r) = \lambda R(r), \quad (43)$$

where

$$\lambda = E^2 - 2\ell\Omega E - m^2 - 2\xi\Omega^2, \tilde{\Omega}^2 = \Omega^2 E^2,$$

and R is the Ricci scalar curvature for the Gürses space-time, $R = g^{\mu\nu} R_{\mu\nu} = -2\Omega^2$.

3.2. PDM FV-oscillators in Gürses space-time

In this subsection, we generalize FV-oscillators to include PDM FV-oscillators. Using the nonminimal coupling form of the radial momentum operator ($\hat{p}_r \rightarrow \hat{p}_r - i\mathcal{M}_r$), Eq. (43) accommodates such generalization as:

$$\left[-\frac{1}{r}(\partial_r + \mathcal{M}_r)r(\partial_r - \mathcal{M}_r) - \frac{1}{r^2}\partial_\varphi^2 + (m^2 - \xi R) \right] \psi = [g^{00}E^2 + 2iEg^{02}\partial_\varphi] \psi. \quad (44)$$

This allows us to write:

$$\left[-\partial_r^2 + \frac{(\ell^2 - 1/4)}{r^2} + \tilde{\Omega}^2 r^2 + V_{PDM}(r) \right] R(r) = \lambda R(r), \quad (45)$$

where

$$V_{PDM}(r) = \frac{\mathcal{M}_r}{r} + \mathcal{M}'_r + \mathcal{M}_r^2; \mathcal{M}_r = \frac{f(r)'}{4f(r)}. \quad (46)$$

For $f(r) = 1$, our PDM FV-oscillators retrieve FV-oscillators in [52]. With $f(r) = e^{2\eta r^2} \Rightarrow \mathcal{M}_r = \eta r, V_{PDM} = \eta^2 r^2 + 2\eta$, and hence:

$$\left[-\partial_r^2 + \frac{(\ell^2 - 1/4)}{r^2} + \tilde{\gamma}^2 r^2 \right] R(r) = \tilde{\lambda} R(r), \quad (47)$$

where $\tilde{\gamma}^2 = \Omega^2 E^2 + \eta^2$ and

$$\tilde{\lambda} = E^2 - 2\ell\Omega E - m^2 - 2\xi\Omega^2 - 2\eta. \quad (48)$$

This is in the form of the two-dimensional radial Schrödinger oscillator. The solutions are:

$$\tilde{\lambda} = 2|\tilde{\gamma}|(2n_r + |\ell| + 1) = 2|\Omega E| \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} (2n_r + |\ell| + 1), \quad (49)$$

and

$$R(r) \sim r^{|\ell|+1/2} \exp\left(-\frac{|\tilde{\gamma}|r^2}{2}\right) L_{n_r}^{|\ell|}(|\tilde{\gamma}|r^2). \quad (50)$$

Consequently,

$$\psi(r) \sim r^{|\ell|} \exp\left(-\frac{|\tilde{\gamma}|r^2}{2}\right) L_{n_r}^{|\ell|}(|\tilde{\gamma}|r^2). \quad (51)$$

Comparing (48) with (49), we obtain:

$$E^2 - 2\ell\Omega E - 2|\Omega E| \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} (2n_r + |\ell| + 1) - \tilde{m} = 0, \quad (52)$$

where $\tilde{m} = m^2 + 2\xi\Omega^2 + 2\eta$.

This must be solved with care since $|\Omega E| = +\Omega_{\pm} E_{\pm}$ or $|\Omega E| = -\Omega_{\mp} E_{\pm}$, where $\Omega_{\pm} = \pm|\Omega|$ and $E_{\pm} = \pm|E|$. This yields:

$$E_{\pm}^2 - 2\Omega_{\pm} E_{\pm} \left(\sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} [2n_r + |\ell| + 1] + \ell \right) - \tilde{m} = 0, \quad (53)$$

for $|\Omega E| = +\Omega_{\pm} E_{\pm}$, and

$$E_{\pm}^2 + 2\Omega_{\mp} E_{\pm} \left(\sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} [2n_r + |\ell| + 1] - \ell \right) - \tilde{m} = 0. \quad (54)$$

Equation (53) results in:

$$E_{\pm} = \Omega_{\pm} k_1 \pm \sqrt{\Omega^2 k_1^2 + \tilde{m}}, k_1 = \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} [2n_r + |\ell| + 1] + \ell, \quad (55)$$

and Eq. (54) gives:

$$E_{\pm} = -\Omega_{\mp} k_2 \pm \sqrt{\Omega^2 k_2^2 + \tilde{m}}, k_2 = \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} [2n_r + |\ell| + 1] - \ell. \quad (56)$$

It is more convenient to report the energies in terms of positive $E_{\pm}^{(+)}$ and negative $E_{\pm}^{(-)}$ velocities so that for $\Omega = +|\Omega|$, we have:

$$E_{\pm}^{(+)} = \pm|\Omega|k_{\pm} \pm \sqrt{\Omega^2 k_{\pm}^2 + \tilde{m}}, \quad (57)$$

and

$$E_{\pm}^{(-)} = \pm|\Omega|k_{\mp} \pm \sqrt{\Omega^2 k_{\mp}^2 + \tilde{m}}, \quad (58)$$

where

$$k_{\pm} = \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}} [2n_r + |\ell| + 1] \pm \ell. \quad (59)$$

To derive the FV formulation of the Klein-Gordon wave equation in curved manifolds, we adopt the approach outlined in Refs. [63, 69], utilizing the Generalized Feshbach-Villars Transformation (GFVT).

In the GFVT framework, the components of the wave function Φ are expressed as:

$$\psi = \phi_1 + \phi_2, \quad i\tilde{\mathcal{D}}\psi = \mathcal{N}(\phi_1 - \phi_2), \quad (60)$$

where \mathcal{N} is an arbitrary nonzero real parameter and

$$\tilde{\mathcal{D}} = \frac{\partial}{\partial t} + \mathcal{Y}, \quad (61)$$

with

$$\mathcal{Y} = \frac{1}{2} \left\{ \partial_i, \sqrt{-g} \frac{g^{0i}}{g^{00}} \right\}. \quad (62)$$

A simple computation yields:

$$\mathcal{Y} = \left(\frac{\sqrt{-g}}{g^{00}} \frac{\partial}{\partial \varphi} \right). \quad (63)$$

Hence, the operator \mathcal{T}' becomes

$$\mathcal{T}' = \mathcal{F}^{-1} \left(\frac{\partial}{\partial r} (-\sqrt{-g}) \frac{\partial}{\partial r} \right) \mathcal{F}^{-1} + \left(\left(\frac{\ell^2}{r^2} \sqrt{-g} \right) + \frac{m^2 - 2\xi\Omega^2}{g^{00}} - \left(\frac{\Omega\ell}{g^{00}} \sqrt{-g} \right)^2 - E^2 \right) \mathcal{F}^{-2}. \quad (64)$$

Using these techniques to obtain the Hamiltonian (22), we suppose a solution of the type:

$$\Phi(t, r, \varphi) = \Phi(r) e^{-i(Et - \ell\varphi)}, \quad (65)$$

where $\ell = 0, \pm 1, \pm 2, \dots$

Now, the KG equation can be written equivalently as two coupled equations:

$$\begin{aligned} (\mathcal{N}^2 + \mathcal{T}) \phi_1 + (-\mathcal{N}^2 + \mathcal{T}') \phi_2 &= 2\mathcal{N}E\phi_1, \\ -(\mathcal{N}^2 + \mathcal{T}) \phi_2 - (-\mathcal{N}^2 + \mathcal{T}') \phi_1 &= 2\mathcal{N}E\phi_2. \end{aligned} \quad (66)$$

Adding and subtracting these equations yields a second-order differential equation for the field $\psi' = \mathcal{F}\psi$:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\zeta^2}{r^2} + \kappa \right] \psi(r) = 0, \quad (67)$$

where

$$\zeta = \ell, \quad \kappa = \sqrt{E^2 - m^2 + 2\xi\Omega^2}. \quad (68)$$

Equation (68) is a Bessel equation, and its general solution is [75, 78]:

$$\psi(r) = AJ_{|\zeta|}(\kappa r) + BY_{|\zeta|}(\kappa r), \quad (69)$$

where $J_{|\zeta|}(\kappa r)$ and $Y_{|\zeta|}(\kappa r)$ are Bessel functions of order ζ of the first and second kind, respectively. Here A and B are arbitrary constants. $J_{|\zeta|}(\kappa r) \neq 0$ at the origin when $\zeta = 0$. However, $Y_{|\zeta|}(\kappa r)$ is divergent at the origin. Thus, we consider only $J_{|\zeta|}(\kappa r)$ when $\zeta \neq 0$.

The solution to Eq. (67) is:

$$\psi(r) = \mathcal{A} J_{|\ell|} \left(\sqrt{E^2 - m^2 + 2\xi\Omega^2} r \right). \quad (70)$$

Using this, the two-component wave function of the spinless heavy KG particle in Gürses space-time is:

$$\Psi(\mathbf{r}) = |\mathcal{A}_1| \begin{pmatrix} 1 + \frac{E}{\mathcal{N}} \\ 1 - \frac{E}{\mathcal{N}} \end{pmatrix} e^{-i(Et - \ell\varphi)} J_{|\ell|} \left(\sqrt{E^2 - m^2 + 2\xi\Omega^2} r \right). \quad (71)$$

The constant $|\mathcal{A}_1|$ can be obtained by applying the normalization condition to the KG equation, but the failure to determine normalization constants throughout this manuscript does not affect the final results.

For the case of KGO, we replace the momentum operator in Eq. (15). Thus, Eq. (64) can be rewritten as:

$$\mathcal{T}' = \mathcal{F}^{-1} \left(\frac{\partial}{\partial r} - m\omega r \right) (-\sqrt{-g}) \left(\frac{\partial}{\partial r} + m\omega r \right) \mathcal{F}^{-1} + \left(\left(\frac{\ell^2}{r^2} \sqrt{-g} \right) + \frac{m^2 - 2\xi\Omega^2}{g^{00}} - \left(\frac{\Omega\ell}{g^{00}} \sqrt{-g} \right)^2 \right) \mathcal{F}^{-2}. \quad (72)$$

Using similar steps, we obtain the radial equation:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - m^2 \omega^2 r^2 - \frac{\sigma^2}{r^2} + \delta \right] \psi(r) = 0, \quad (73)$$

where

$$\sigma = \ell, \quad \delta = E^2 - m^2 + 2i\Omega^2 + (\Omega\ell)^2 + 2m\omega. \quad (74)$$

For $\gamma = m\omega r^2$, Eq. (73) becomes:

$$\left[\frac{d^2}{d\gamma^2} + \frac{1}{\gamma} \frac{d}{d\gamma} - \frac{\sigma}{4\gamma^2} + \frac{\delta}{4m\omega\gamma} - \frac{1}{4} \right] \psi(\gamma) = 0. \quad (75)$$

Assuming a solution of the type:

$$\psi(\gamma) = \gamma^{|\sigma|/2} e^{-\gamma/2} F(\gamma). \quad (65)$$

Substituting this into Eq. (75), we get:

$$\gamma \frac{d^2 F(\gamma)}{d\gamma^2} + (|\sigma| + 1 - \gamma) \frac{dF(\gamma)}{d\gamma} - \left(\frac{|\sigma|}{2} - \frac{\delta}{4m\omega} + \frac{1}{2} \right) F(\gamma) = 0. \quad (76)$$

This is the confluent hypergeometric equation [64], with solutions given in terms of the confluent hypergeometric function:

$$F(\gamma) = {}_1F_1 \left(\frac{|\sigma|}{2} - \frac{\delta}{4m\omega} + \frac{1}{2}, |\sigma| + 1, \gamma \right). \quad (77)$$

The solution (77) must be a polynomial of degree n :

$$\frac{|\sigma|}{2} - \frac{\delta}{4m\omega} + \frac{1}{2} = -n \quad n = 0, 1, 2, \dots \quad (78)$$

The quantized energy spectrum of KGO in Gürses space-time is:

$$E^\pm(n) = \pm \sqrt{4m\omega n + 2m\omega|\ell| + m^2 - 2\xi\Omega^2 - (\Omega\ell)^2}. \quad (79)$$

The corresponding wave function is:

$$\Psi(\mathbf{r}) = |\mathcal{A}_2| (m\omega r^2)^{|\ell|/2} e^{-\frac{m\omega r^2}{2}} {}_1F_1 \left(\frac{|\ell|}{2} - \frac{\delta}{4m\omega} + \frac{1}{2}, |\ell| + 1, m\omega r^2 \right). \quad (80)$$

The general eigenfunctions are:

$$\Psi(\mathbf{r}) = |\mathcal{A}_2| \begin{pmatrix} 1 + \frac{E}{\mathcal{N}} \\ 1 - \frac{E}{\mathcal{N}} \end{pmatrix} (m\omega r^2)^{|\ell|/2} e^{-\frac{m\omega r^2}{2}} e^{-i(Et - \ell\varphi)} {}_1F_1 \left(\frac{|\ell|}{2} - \frac{\delta}{4m\omega} + \frac{1}{2}, |\ell| + 1, m\omega r^2 \right), \quad (81)$$

where $|\mathcal{A}_2|$ is the normalization constant.

4. The Feshbach-Villars oscillator under potentials in Gürses Space-Time

We now study the KGO within Gürses space-time under a Coulomb-type potential. This section examines the behavior of a heavy, relativistic spin-0 particle whose wave function Ψ satisfies the KG equation (15) in the space-time created by a $(2+1)$ -dimensional Gürses with Coulomb-type potentials ($S(r) = \frac{\lambda}{r}$).

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2} + (E - S(r))^2 - m^2 + 2\xi\Omega^2 + (\Omega\ell)^2 \right] \varphi(r) = 0, \quad (82)$$

where

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2} + \left(E - \frac{\lambda}{r} \right)^2 - m^2 + 2\xi\Omega^2 + (\Omega\ell)^2 \right] \varphi(r) = 0. \quad (83)$$

After simple algebraic manipulations, we obtain:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(\ell^2 - \lambda^2)}{r^2} - 2\frac{E\lambda}{r} + E^2 - m^2 + 2\xi\Omega^2 + (\Omega\ell)^2 \right] \varphi(r) = 0. \quad (84)$$

Solving this yields the radial function $\varphi(r)$:

$$\varphi(r) = |\mathcal{A}_3| r^{-\frac{1}{2}} \text{Whittaker}M \left[-\frac{E\lambda}{\sqrt{m^2 - 2\xi\Omega^2 - \Omega^2\ell^2 - E^2}}, \sqrt{\ell^2 - \lambda^2}, 2r \left(\sqrt{m^2 - 2\xi\Omega^2 - \Omega^2\ell^2 - E^2} \right) \right]. \quad (85)$$

By simplifying the relationship between Whittaker and confluent hypergeometric functions [61], we obtain:

$$\varphi(r) = |\mathcal{A}_3| r^{-\frac{1}{2}} e^{-\left(\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}\right)r} \left(2\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}r \right)^{\frac{1}{2} + \sqrt{\ell^2 - \lambda^2}} g(r), \quad (86)$$

where

$$g(r) = {}_1F_1 \left(\frac{2\sqrt{\ell^2 - \lambda^2} \left(\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2} + 1 \right) + 2E\lambda}{2\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}}, 1 + 2\sqrt{\ell^2 - \lambda^2}, 2r \left(\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2} \right) \right). \quad (87)$$

The solution must be a polynomial of degree n . However, taking $n \rightarrow \infty$ imposes a divergence issue. We have a finite polynomial only if:

$$\frac{2\sqrt{\ell^2 - \lambda^2} \left(\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2} + 1 \right) + 2E\lambda}{2\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}} = -n \quad n = 0, 1, 2, \dots \quad (88)$$

The quantized energy spectrum of KGO in Gürses space-time is:

$$E^\pm(n) = \pm \frac{(2\sqrt{\ell^2 - \lambda^2} + 2n + 1) \left[\sqrt{(8n\sqrt{\ell^2 - \lambda^2} + 4\sqrt{\ell^2 - \lambda^2} + 4\left(\ell^2 + n^2 + n + \frac{1}{4}\right)) (m^2 - 2\xi\Omega^2 - \Omega^2\ell^2)} \right]}{(8n + 4)\sqrt{\ell^2 - \lambda^2} + 4\left(\ell^2 + n^2 + n + \frac{1}{4}\right)}. \quad (89)$$

The complete eigenstates are

$$\begin{aligned} \Psi(\mathbf{r}) &= |\mathcal{A}_3| \begin{pmatrix} 1 + \frac{E}{N} \\ 1 - \frac{E}{N} \end{pmatrix} e^{-i(Et - \ell\varphi)} r^{-\frac{1}{2}} e^{-\left(\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}\right)r} \\ &\times \left(2\sqrt{-\Omega^2\ell^2 - E^2 + m^2 - 2\xi\Omega^2}r \right)^{\frac{1}{2} + \sqrt{\ell^2 - \lambda^2}} g(r). \end{aligned} \quad (90)$$

Appendix

A. Feshbach-Villars Oscillator in Gürses Space-time under Coulomb-type Potential

To study the Feshbach-Villars Oscillator in Gürses space-time, we replace the momentum operator $\partial_r \longrightarrow \partial_r + m\omega r$. Thus, we obtain:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - m^2 \omega^2 r^2 - \frac{\ell^2 - \lambda^2}{r^2} - \frac{2\lambda E}{r} + E^2 - m^2 - 2\xi \Omega^2 + (\Omega \ell)^2 + 2m\omega \right] \varphi(r) = 0. \quad (\text{A.1})$$

Applying GFVT, we obtain:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - m^2 \omega^2 r^2 - \frac{\vartheta^2}{r^2} - \frac{2\lambda E}{r} + \delta \right] \varphi(r) = 0, \quad (\text{A.2})$$

where

$$\vartheta^2 = \ell^2 - \lambda^2, \quad \beta^2 = E^2 - m^2 + 2\xi \Omega^2 + 2m\omega + (\Omega \ell)^2. \quad (\text{A.3})$$

Using $\mathcal{P} = \sqrt{m\omega}r$, we rewrite the radial Eq. (A.1) as:

$$\left[\frac{d^2}{d\mathcal{P}^2} + \frac{1}{\mathcal{P}} \frac{d}{d\mathcal{P}} - \frac{\vartheta^2}{\mathcal{P}^2} - \frac{\delta}{\mathcal{P}} - \mathcal{P}^2 + \frac{\beta^2}{m\omega} \right] \varphi(\mathcal{P}) = 0, \quad (\text{A.4})$$

where

$$\delta = 2\lambda (m\omega^{-1})^{\frac{1}{2}}. \quad (\text{A.5})$$

Assuming a solution of the form:

$$\varphi(\mathcal{P}) = \mathcal{P}^{|\gamma|} e^{-\frac{\mathcal{P}^2}{2}} \mathcal{O}(\mathcal{P}), \quad (\text{A.6})$$

we get:

$$\frac{d^2 \mathcal{O}(\mathcal{P})}{d\mathcal{P}^2} + \left[\frac{2|\gamma| + 1}{\mathcal{P}} - 2\mathcal{P} \right] \frac{d\mathcal{O}(\mathcal{P})}{d\mathcal{P}} + \left[\frac{\beta^2}{m\omega} - 2(2|\gamma| + 1) - \frac{\delta}{\mathcal{P}} \right] \mathcal{O}(\mathcal{P}) = 0. \quad (\text{A.7})$$

The solution $\mathcal{O}(\mathcal{P})$ is the Heun biconfluent function [75, 78] :

$$\mathcal{O}(\mathcal{P}) = \text{HeunB} \left(2|\gamma|, 0, \frac{\beta^2}{m\omega}, 2\delta, \mathcal{P} \right). \quad (\text{A.8})$$

To obtain bound state solutions, we express $\mathcal{O}(\mathcal{P})$ as a power series expansion around the origin:

$$\mathcal{O}(\mathcal{P}) = \sum_{s=0}^{\infty} a_s \mathcal{P}^s. \quad (\text{A.9})$$

Substituting this series into yields the recurrence relation:

$$a_{s+2} = \frac{\delta}{(s+2)(s+1+\zeta)} a_{s+1} - \frac{(\Theta - 2s)}{(s+2)(s+1+\zeta)} a_s, \quad (\text{A.10})$$

where $\zeta = 2|\gamma| + 1$ and $\Theta = (\beta^2/m\omega) - 2(|\gamma| + 1)$.

Starting with $a_0 = 1$, we get:

$$a_1 = \frac{\delta}{\zeta} = 2\sqrt{\frac{m}{\omega}} \lambda \left(\frac{1}{2|\gamma| + 1} \right) \quad (\text{A.11})$$

$$a_2 = \frac{\delta^2}{2\zeta(1+\zeta)} - \frac{\Theta}{2(1+\zeta)} = \left(\frac{m}{\omega} \right) \lambda^2 \frac{1}{(2|\gamma| + 1)(|\gamma| + 1)} - \frac{\Theta}{4(|\gamma| + 1)}. \quad (\text{A.12})$$

The wave function must be normalizable. We suppose that $\mathcal{O}(\mathcal{P})$ vanishes at $\mathcal{P} \rightarrow 0$ and $\mathcal{P} \rightarrow \infty$. Thus, bound state solutions can be obtained by ensuring that the power series expansion (88) or the Heun biconfluent series becomes a polynomial of degree n :

$$\Theta = 2n, \quad \text{and} \quad a_{n+1} = 0, \quad (\text{A.13})$$

where $n = 1, 2, 3, \dots$. The quantized energy spectrum of KGO in Gürses space-time is:

$$E^\pm(n) = \pm \sqrt{2m(n + |\gamma|) + m^2 - 2\xi\Omega^2 - 2\xi\Omega^2}. \quad (\text{A.14})$$

The corresponding wave function is:

$$\varphi(r) = |\mathcal{A}_4| \mathcal{P}^{|\gamma|} e^{-\frac{\mathcal{P}^2}{2}} \text{HeunB} \left(2|\gamma|, 0, \frac{\beta^2}{m\omega}, 2\delta, \mathcal{P} \right). \quad (\text{A.15})$$

The final expression for the wave function of the spinless FVO propagating in the Gürses background is:

$$\Psi(\mathbf{r}) = |\mathcal{A}_4| \left(\frac{1 + \frac{E}{\mathcal{N}}}{1 - \frac{E}{\mathcal{N}}} \right) e^{-i(Et - \ell\varphi)} \mathcal{P}^{|\gamma|} e^{-\frac{\mathcal{P}^2}{2}} \text{HeunB} \left(2|\gamma|, 0, \frac{\beta^2}{m\omega}, 2\delta, \mathcal{P} \right), \quad (\text{A.16})$$

where the parameters ϑ and δ are defined in (A.3).

5. Conclusion

This research investigates the relativistic behavior of spinless quantum particles using the Feshbach-Villars approach in two models: one involving the interaction between KGO and the gravitational field generated by the background geometry of Gürses space-time, and the other involving Gürses space-time with a Coulomb-like potential. The FV formulation of scalar fields in Minkowski space-time is adapted for these models. The study provides exact solutions for both systems and presents energy spectra that depend on the parameters defining the space-time topology. The wave-functions of the quantum system are expressed as confluent hypergeometric functions for the free Feshbach-Villars equation in Gürses space-time, consistent with the underlying geometry.

Acknowledgments

We would like to express our sincere gratitude to the anonymous referees for their invaluable comments and constructive feedback on this paper. Their insightful suggestions have significantly enhanced the quality and clarity of our work. We also appreciate the time and effort they devoted to reviewing the manuscript. Their expertise and attention to detail have played a crucial role in shaping the final version of this paper.

-
1. A. Einstein, *Annalen Phys.* **49** (1916) 769.
 2. B. P. Abbott *et al.*, *Phys Rev Lett* **116** (2016) 061102.
 3. K. Akiyama *et al.*, *Astrophys J Lett* **875** (2019) L1.
 4. R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, Emended Edition (Dover Books on Physics 1965).
 5. M. D. Schwartz, *Quantum field theory and the standard model*, Cambridge University Press, (2013).
 6. A. Ashtekar and J. J. Stachel, *Conceptual problems of quantum gravity*, (1991).
 7. N. D. Birrell and P. Davies, *Quantum fields in curved space*, Cambridge University Press, (1980).
 8. L. Parker and D. J. Toms, *Quantum field theory in curved space-time*, Cambridge University Press, 2009.
 9. S. W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.
 10. W. G. Unruh and R. M. Wald, *Phys. Rev. D* **25**, (1982) 942.
 11. G. L. Sewell, *Ann. Physics* **141** (1982) 201.
 12. T. W. B. Kibble, *J. Phys. A* **9** (1976) 1387.
 13. T. W. B. Kibble, *Phys. Rep.* **67** (1980) 183.
 14. A. Vilenkin, *Phys. Lett. B* **133** (1983) 177.
 15. Y. B. Zel'dovich, *Mon. Not. R. Astron. Soc.* **192** (1980) 663.
 16. A. Vilenkin, *Phys. Rep.* **121** (1985) 263.
 17. A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects*, (1985).
 18. M. Moshinsky and Y. F. Smirnov, *The harmonic oscillator in modern physics*, (1996).
 19. A. Ushveridze, *Quasi-exactly solvable models in quantum mechanics*, (1994).

20. D. Itô, K. Mori, and E. W. Carriere, *Il Nuovo Cimento A* **51** (1967) 1119.
21. M. Moshinsky and A. P. Szczepaniak, *J. Phys. A* **22** (1989) L817.
22. O. Mustafa, S. H. Mazharimousavi, *Int. J. Theor. Phys.* **46** (2007) 1786.
23. O. Mustafa, Z. Algadhi, *Eur. Phys. J. Plus* **134** (2019) 288.
24. O. Mustafa, *Phys. Lett. A* **384** (2020) 126265.
25. O. Mustafa, *Eur. Phys. J. C* **82** (2022) 82.
26. O. Mustafa, *Ann. Phys.* **446** (2022) 169124.
27. O. Mustafa, *Ann. Phys.* **440** (2022) 168857.
28. O. Mustafa, *Phys. Scr.* **98** (2023) 015302.
29. O. Mustafa, *Phys. Lett. B.* **839** (2023) 137793.
30. S. A. Bruce and P. C. Minning, *Il Nuovo Cimento A* **106** (1993) 711.
31. V. V. Dvoeglazov, *Il Nuovo Cimento A* **107** (1994) 1785.
32. J. Carvalho, A. M. de M. Carvalho, E. Cavalcante, and C. Furtado, *Eur. Phys. J. C* **76** (2016) 365.
33. L. C. dos Santos and C. de Camargo Barros, *Eur. Phys. J. C* **78** (2017) 1.
34. R. L. L. Vitória and K. Bakke, *Eur. Phys. J. C* **78** (2018) 1.
35. R. R. Cuzinato, M. de Montigny, and P. Pompeia, *Class. Quant. Grav* **39** (2022) 075007.
36. F. Ahmed, *Europhys. Lett.* **131** (2020) 30002.
37. K. M. Case, *Phys Rev* **95** (1954) 1323.
38. L. L. Foldy, *Phys Rev* **102** (1956) 568.
39. L. L. Foldy and S. A. Wouthuysen, *Phys Rev* **78** (1950) 29.
40. H. Feshbach and F. M. H. Villars, *Rev. Modern Phys.* **30** (1958) 24.
41. B. A. Robson and D. S. Staudte, *J. Phys. A : Math. Gen* **29** (1996) 157.
42. D. S. Staudte, *J. Phys. A* **29** (1996) 169.
43. M. Merad, L. Chetouani, and A. Bounames, *Phys. Lett. A* **267** (2000) 225.
44. A. Bounames and L. Chetouani, *Phys. Lett. A* **279** (2001) 139.
45. S. Haouat and L. Chetouani, *Eur. Phys. J. C* **41** (2005) 297.
46. N. Brown, Z. Papp, and R. M. Woodhouse, *Few-Body Systems* **57** (2015) 103.
47. B. Motamedi, T. Shannon, and Z. Papp, *Few-Body Systems* **60** (2019) 65.
48. D. Wingard, A. Garcia Vallejo and Z. Papp, *Few-Body Syst* **65** (2024) 30.
49. A. Bouzenada, A. Boumali, E.O. Silva, *Ann. Phys.* **458** (2023) 169479.
50. A. Bouzenada, A. Boumali, R. Vitória, F. Ahmed, M. Al-Raei, *Nucl. Phys. B* **994** (2023) 116288.
51. A. Bouzenada, A. Boumali, F. Ahmed, *Nucl. Phys. B* **1007** (2024) 116682.
52. A. Bouzenada and A. Boumali, *Ann. Physics* **27** (2023) 169302.
53. Q. Wen-Chao, *Chinese Phys.* **12** (2003) 1054.
54. D. S. Staudte, Ph.D. thesis, The Australian National University (1993).
55. H. Motavalli and A. R. Akbarieh, *Mod. Phys. Lett. A* **25** (2010) 2523.
56. F. Yasuk, A. Durmus and I. Boztosun, *J. Math. Phys* **47** (2006) 082302.
57. A. L. Cavalcanti de Oliveira and E. R. Bezerra de Mello, *Class. Quantum Grav.* **23** (2006) 5249.
58. W. Greiner, *Relativistic Quantum Mechanics: Wave Equations*, 3rd Edition (Springer, Berlin, 2000).
59. M. Gürses, *Class. Quantum Grav.* **11** (1994) 2585.
60. F. L. Gross, *Relativistic quantum mechanics and field theory*, Wiley-VCH Verlag GmbH & Co. KGaA, 1993.
61. O. Klein, *Z. Phys* **37** (1926) 895.
62. W. Gordon, *Z. Phys* **40** (1926) 117.
63. A. J. Silenko, *Phys. Rev. A* **77** (2008) 012116.
64. W.A. Hiscock, *Phys. Rev. D* **31** (1985) 3288.
65. I. Gott, J. R., *Astrophys J* **288** (1985) 422.
66. K. Bakke, L. R. Ribeiro, C. Furtado, and J. R. Nascimento, *Phys. Rev. D* **79** (2009) 024008.
67. J. Carvalho, A. M. de M. Carvalho, E. Cavalcante, and C. Furtado, *Eur. Phys. J. C* **76** (2016) 1.
68. A. Boumali and N. Messai, *Can. J. Phys.* **92**, (2014) 1460.
69. A. J. Silenko, *Phys. Rev. D* **88** (2013) 045004.
70. A. Mostafazadeh, *J. Phys. A* **31** (1998) 7829.
71. O. Mustafa, M. Znojil, *J. Phys. A* **35** (2002) 8929.
72. M. Znojil, F. Gemperle, O. Mustafa, *J. Phys. A* **35** (2002) 5781.
73. A. Accioly and H. Blas, *Phys Rev D*, **66** (2002) 0675019.
74. A. Accioly and H. Blas, *Modern Physics Letters A*, **18** (2003) 867-873.
75. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, **volume 55**, Dover Publications, New York, 1970.
76. E. Bragança, H. S. Mota, and E. B. de Mello, *Int J Mod Phys D* **24** (2015) 1550055.
77. F. Ahmed, *Sci Rep* **12** (2022) 8794.
78. G. Arfken, H. Weber, and F. Harris, *Mathematical Methods for Physicists : A Comprehensive Guide*, (Elsevier Science, 2012).