Phase shift analysis with half-shell quantities for nucleus-nucleus systems using a simplified potential model

P. Sahoo[a], B. Swain[b,*], D. Naik[b,⋆] and U. Laha[b,†]

[a]*Department of Physics, Dharanidhar University, Keonjhar, 758001, Odisha, India,*
e-mail: nlphy.pati7@gmail.com; https://orcid.org/0000-0003-4691-9576.
[b]*Department of Physics, National Institute of Technology, Jamshedpur, 831014, Jharkhand, India,*
[*]*e-mail: biswanathswain73@gmail.com; https://orcid.org/0000-0002-9149-8857;*
[⋆]*e-mail: n.dibakar2001@gmail.com; https://orcid.org/0009-0001-1465-8511;*
[†]*e-mail: ujjwal.laha@gmail.com; https://orcid.org/0000-0003-4544-2358.*

Received 18 July 2024; accepted 9 September 2024

In this work we propose a combined interaction, nuclear and electromagnetic in origin, to treat the charged hadronic systems by considering the same range of both the potentials under consideration. Since the effect due to charges becomes screened at a certain distance, we believe it desirable to include the effect of very short-range electromagnetic interaction. It is worthwhile to mention that in such situations the effect of the combined potential is often examined within the nuclear domain. Keeping this in view we consider nuclear Manning-Rosen plus the atomic Hulthén potential of equal range parameter for simplicity of calculation. The validity of our conjecture is examined through some model calculations with respect to their on- and off-shell effects.

Keywords: Manning-Rosen plus the Hulthén potential; off shell Jost solution; T-matrix; scattering phase shifts.

DOI: https://doi.org/10.31349/RevMexFis.71.021203

1. Introduction

The interaction between two-nucleons is basic for all of nuclear physics. The primary aim of nuclear physics is to understand the properties of atomic nuclei in terms of the exposed interaction between pairs of nucleons. With the discovery of quantum chromo dynamics (QCD), it became clear that the nucleon-nucleon (NN) interaction is not fundamental. Nevertheless, even today one assumes the nucleons to be elementary particles for nuclear structure problems. The nucleon-nucleon interaction has been investigated by a large number of researchers all over the world for the past few decades. Even though, the meson theory is not supposed to be the fundamental anymore in the light of QCD model, the meson exchange concept continues to symbolize the best working model for a quantitative nucleon-nucleon potential. The nuclear scattering of two charged particles usually occurs under the combined influence of additive interactions, one is electromagnetic in nature, and the other is nuclear in origin. In general, for a short-range local nuclear plus the electromagnetic potential the Schrödinger equation does not admit an exact analytical solution. In the recent past, one of us [1,2] treated the problems containing nuclear Hulthén plus the atomic Hulthén potential to treat the alpha-proton and alpha-alpha systems by calculating approximate analytical solutions of the concerned Schrödinger equations and the formalism of super-symmetric algebra. However, to obtain exact analytical solutions of the Schrödinger equation people usually replace the short-range local nuclear potential by a non-local separable one. This is no loss of generality. The results for such problems have been advocated by a number of researchers [3-10]. Unlike the approaches adapted in our previous articles [1,2], here we take recourse to a different point of view. The electromagnetic potential is in principle well known and is the longest-range part (infinite) of the interaction. But, in reality, it is not the case. The infinite long-range part of the interaction becomes screened at some distance and thus short-range electromagnetic potentials are applied to observe the effect of screening in real situation. Relatively recently, we considered an interaction model with equal range of both the nuclear and electromagnetic parts and obtained good results with respect to binding energies and scattering phase shifts for nucleon-nucleus and nucleus-nucleus systems [11]. In physical processes which involve charged hadrons, the scattering generally takes place under the combined influence of electromagnetic and nuclear interactions. In such a situation the effect of the combined potential is examined within the nuclear domain. Since we found that the electromagnetic terms reproduced significant effect within few KeV and beyond that they became insignificant, we deem it desirable to include the effect of very short-range electromagnetic terms rigorously. The major goal of the present work is to construct a non-relativistic potential that can be used easily in nuclear many-body calculations. The description of the multi-particle systems requires knowledge of the two-body interaction off the energy shell which is encountered in atomic and nuclear physics [12-21]. We study the off-shell Jost functions as well as half-off shell T-matrix within the nuclear Manning-Rosen plus the Hulthén potential model of interaction in all partial waves by adapting pure quantum mechanical approach to the problem.

Here we consider a differential equation approach to the problem by using nuclear Manning-Rosen plus atomic Hulthén potential and construct the irregular solution and in turn the off-shell Jost function, half-shell T-matrix and off-shell extension function. The T-matrices with different off-shell behaviours are indispensible because the on-shell one does not specify the potential uniquely. The half-shell T-matrix, which is very much sensitive to the many-body calculations, has emerged as a fundamental quantity as the fully off-shell transition matrix can be calculated from it. Information regarding the wave function along with the phase parameters are inbuilt in the half-shell transition matrix and hence may be parameterized in terms of the short-range character of the wave function of the concerned system. In this report we compute the half-off-shell transition matrix and scattering phase shifts for the ($\alpha-{}^3$H) and ($\alpha-{}^3$He) systems with the potential model under consideration to examine the feasibility of our methodology. The organization of the text is as follows. Section 2 is devoted to construct the off-shell Jost solution, the half-shell T-matrix and the off-shell extension function. In Sec. 3 we discuss our results and finally we conclude in Sec. 4.

2. Methods for the off-shell Jost solution

In this section we adapt two different approaches to the problem, with judicious applications of boundary conditions, interacting Green's functions and their integral transforms together with certain properties of special functions of mathematical physics for construction of exact analytical expression for the off-shell Jost solution for motion in Manning Rosen plus Hulthén potential by exploiting the theory of ordinary differential equations. The off-shell physical wave function which is connected to the off-shell T-matrix, can be simply described in terms of the off-shell Jost/irregular solution $f_\ell(\xi, q, s)$ and the Jost function $f_\ell(\xi, q)$ [22,23]. The Jost function $f_\ell(\xi, q)$ is determined by the behaviour of the irregular solution $f_\ell(\xi, q, s)$ of the radial Schrödinger equation near the origin, and it plays a significant role in investigating the analytic characteristics of partial wave scattering amplitude. The Jost function has been developed for the off-energy-shell by Fuda and Whiting [24]. In the same manner that the on-shell Jost function is generated from its on-shell solution, the off-shell Jost function is derived from the irregular solution of an inhomogeneous Schrödinger equation. The Jost functions $f_\ell(\xi, q)$ can also be used to express the half-shell T-matrix. For the computation of transition matrices, knowledge of off-shell Jost functions and Jost solutions is necessary. The off-shell Jost functions can also be derived directly from the expression of off-shell Jost solutions. Therefore, it is of some importance to have explicit expressions in the literature for the off-shell Jost solutions relating to Manning Rosen plus Hulthén interaction which are encountered in the charged particle scattering [23,25].

The significance of the Jost function $f_\ell(\xi, q)$ in scattering theory may be viewed on two levels. On one hand it has been useful in helping establish a relativistic theory of the S-matrix by allowing the conjectured analyticity properties of the S-matrix to be tested in potential scattering; there the Schrödinger theory provides an alternate reliable standard against which the plausibility of the arguments of the new theory are to be measured. On the other hand it has become an invaluable tool strictly in the domain of potential scattering where it has been instrumental in the study of bound and resonant states, and in the general analysis of low energy scattering data. In particular, the Jost solution is holomorphic in the upper complex ξ-plane, where Im$\xi > 0$. For our approach, the most important quantity is the Jost function and the roots of the Jost function in the upper complex ξ-plane. Those correspond to the bound state energies of the related system. In general, the Jost function $f_\ell(\xi) = f_\ell(\xi, q = \xi)$ is a complex quantity and the phase of the Jost function is the negative of the scattering phase shift. Therefore, it provides a convenient expression for computing scattering phase shifts.

2.1. Differential equation approach

The nuclear Manning-Rosen potential is defined by [26-31]

$$V_N(s) = \frac{1}{\mathrm{a}^2}\left[\mu(\mu-1)\frac{\exp(-2s/\mathrm{a})}{[1-\exp(-s/\mathrm{a})]^2} - E\frac{\exp(-s/\mathrm{a})}{[1-\exp(-s/\mathrm{a})]}\right], \qquad (1)$$

with

$$\mu = \frac{1}{2}\left[1 \pm \sqrt{1+4\left\{\beta(\beta-1)+\ell(\ell+1)\right\}}\right], \qquad (2)$$

and ℓ takes the values $0, 1, 2, 3, \ldots$. Here the centrifugal barrier term is considered as

$$\frac{1}{\mathrm{a}^2}\frac{\ell(\ell+1)\exp(-2s/\mathrm{a})}{[1-\exp(-s/\mathrm{a})]^2}.$$

The parameters E, β are dimensionless quantities and a is the screening radius for nuclear potential having dimension of length. As an electromagnetic interaction we adapt the screened atomic Hulthén potential [32]

$$V_A(s) = E_0\frac{\exp(-s/b)}{1-\exp(-s/b)}, \qquad (3)$$

with E_0, the strength and b, the screening radius of the potential. For our present analysis we have considered the situation where $b = \mathrm{a}$. Thus, the effective potential is given as

$$V_{eff}(s) = \frac{1}{\mathrm{a}^2}\left\{\mu(\mu-1)\frac{\exp(-2s/\mathrm{a})}{[1-\exp(-s/\mathrm{a})]^2} - (E-E_0\mathrm{a}^2)\frac{\exp(-\mathrm{s}/\mathrm{a})}{[1-\exp(-\mathrm{s}/\mathrm{a})]}\right\}. \qquad (4)$$

At a centre of mass energy $E_{CM} = \xi^2 + \mathrm{i}\epsilon$ the off-shell Jost solution $f_\ell(\xi, q, s)$ for the above effective potential satisfies an inhomogeneous Schrödinger-like equation, written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{ds}^2}+\xi^2-V_{eff}(s)\right]f_\ell(\xi,q,s)=(\xi^2-q^2)\ \hat{h}_\ell^{(+)}(qs). \tag{5}$$

The quantity

$$\hat{h}_\ell^{(+)}(qs)=\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iqs)^L L!(\ell-L)!}e^{iqs},$$

is the Riccati Hankel function.

Introducing the following transformation

$$f_\ell(\xi,q,s)=\mathrm{a}^\mu\left[1-\exp\left(-\mathrm{s}/\mathrm{a}\right)\right]^\mu\exp\left(\mathrm{i}\xi\mathrm{s}\right)\Theta_\ell(\xi,\mathrm{q},\mathrm{s}), \tag{6}$$

Eq. (5) takes the form

$$\exp(s/\mathrm{a})\mathrm{a}^2(1-\exp\left(-s/\mathrm{a}\right))\Theta_\ell^{''}(\xi,q,s)+\left\{2\mu\mathrm{a}+2\mathrm{i}\xi\mathrm{a}^2\exp(\mathrm{s}/\mathrm{a})(1-\exp\left(-\mathrm{s}/\mathrm{a}\right))\right\}\Theta_\ell^{'}(\xi,\mathrm{q},\mathrm{s})$$
$$+\left\{2\mathrm{i}\xi\mathrm{a}\mu-\mu+(\mathrm{E}-\mathrm{E}_0\mathrm{a}^2)\right\}\Theta_\ell(\xi,\mathrm{q},\mathrm{s})=(\xi^2-\mathrm{q}^2)\sum_{\mathrm{L}=0}^{\ell}\frac{(\mathrm{i})^{2\mathrm{L}-\ell}(\ell+\mathrm{L})!}{(2\mathrm{iqs})^{\mathrm{L}}\mathrm{L}!(\ell-\mathrm{L})!}\exp(\mathrm{i}(\mathrm{q}-\xi)\mathrm{s})\exp(\mathrm{s}/\mathrm{a})$$
$$\times(1-\exp\left(-s/\mathrm{a}\right))^{1-\mu}\mathrm{a}^{2-\mu}, \tag{7}$$

where $\Theta_\ell(\xi,q,s)$ is a newly defined function of momenta and position. If we rewrite Eq. (7) by changing a new variable of the form $(1-\exp\left(-s/\mathrm{a}\right))=r$ and substituting the approximation $s^L\approx\mathrm{a}^{\mathrm{L}}(1-\mathrm{e}^{-\mathrm{s}/a})^{\mathrm{L}}$ in the expansion of $\hat{h}_\ell^{(+)}(qs)$ with $\mu=\eta+1$, it yields

$$r(1-r)\frac{\mathrm{d}^2\Theta_\ell}{\mathrm{dr}^2}+\{2\eta+2-(3+2\eta-2\mathrm{i}\xi\mathrm{a})\mathrm{r}\}\frac{\mathrm{d}\Theta_\ell}{\mathrm{dr}}-(1+\eta-\mathrm{E}+\mathrm{E}_0\mathrm{a}^2-2\eta\mathrm{i}\xi\mathrm{a}-2\mathrm{i}\xi\mathrm{a})\Theta_\ell$$
$$=\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{1-\eta-\mathrm{L}}(\xi^2-\mathrm{q}^2)\mathrm{r}^{-\eta-\mathrm{L}}(1-\mathrm{r})^{\mathrm{ia}(\xi-\mathrm{q})-1}. \tag{8}$$

On comparing Eq. (8) with the following standard differential equation for Gaussian hypergeometric function [33-36]

$$r(1-r)\frac{\mathrm{d}^2\Theta_\ell}{\mathrm{dr}^2}+\{R-(1+M+N)\mathrm{r}\}\frac{\mathrm{d}\Theta_\ell}{\mathrm{dr}}-MN\Theta_\ell=\mathrm{r}^{\sigma-1}(1-\rho\mathrm{r})^{\tau-1}, \tag{9}$$

we obtain

$$M=1+\eta-\mathrm{i}\xi\mathrm{a}+\sqrt{\eta^2+\eta+\mathrm{E}-\mathrm{E}_0\mathrm{a}^2-\xi^2\mathrm{a}^2}, \tag{10}$$

$$N=1+\eta-\mathrm{i}\xi\mathrm{a}-\sqrt{\eta^2+\eta+\mathrm{E}-\mathrm{E}_0\mathrm{a}^2-\xi^2\mathrm{a}^2}, \tag{11}$$

$$R=2+2\eta,\quad \rho=1,\quad \sigma=1-\eta-L, \tag{12}$$

and

$$\tau=\mathrm{ia}(\xi-\mathrm{q}). \tag{13}$$

Two linearly independent solutions of the homogeneous part of Eq. (9) namely $v_1(r)$ and $v_2(r)$ are given by [33,34]

$$v_1(r)={}_2F_1(M,N;R;r)=\frac{\Gamma(R)}{\Gamma(M)\Gamma(N)}\sum_{n=0}^{\infty}\frac{\Gamma(M+n)\Gamma(N+n)}{\Gamma(R+n)}\frac{r^n}{n!};\quad R>0, \tag{14}$$

and

$$v_2(r)={}_2F_1(M,N;M+N-R+1;1-r);\quad M+N-R+1\neq 0,-1,\ldots \tag{15}$$

With the following transformation [33,34] on $v_2(r)$,

$$ {}_2F_1(M,N;R;r) = (1-r)^{R-N-M}{}_2F_1(R-M,R-N;R;r), \tag{16} $$

one gets the expression for $v_2(r)$ as

$$ v_2(r) = (1-e^{-s/\mathrm{a}})^{-2\eta-1}{}_2F_1(N-R+1,M-R+1;M+N-R+1;e^{-s/\mathrm{a}}). \tag{17} $$

The particular solution [36] of Eq. (9) is written as

$$ F_P(r) = (\xi^2-q^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{1-\eta-\mathrm{L}}\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mathrm{n}+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}\mathrm{f}_{\mathrm{n}+1-\eta-\mathrm{L}}(\mathrm{M},\mathrm{N};\mathrm{R};\mathrm{r}), \tag{18} $$

with

$$ f_n(a,b;c;r) = \frac{r^n}{n(n+c-1)}{}_3F_2(1,n+a,n+b,n+1,n+c;r). \tag{19} $$

The complete primitive of $f_\ell(\xi,q,s)$ is obtained from Eq. (6) in conjunction with Eqs. (10)-(19) as

$$ \begin{aligned} f_\ell(\xi,q,s) = {}& \mathrm{a}^{\eta+1}\left[1-\exp(-\mathrm{s}/\mathrm{a})\right]^{\eta+1}\exp(\mathrm{i}\xi\mathrm{s})\Bigg[\mathrm{A}_1{}_2\mathrm{F}_1(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp(-\mathrm{s}/\mathrm{a}))+\mathrm{A}_2(1-\exp(-\mathrm{s}/\mathrm{a}))^{-2\eta-1} \\ & \times {}_2F_1(1-M^*,1-N^*,1-2\mathrm{i}\xi\mathrm{a};\exp(-\mathrm{s}/\mathrm{a}))+(\xi^2-\mathrm{q}^2)\sum_{\mathrm{L}=0}^{\ell}\frac{(\mathrm{i})^{2\mathrm{L}-\ell}(\ell+\mathrm{L})!}{(2\mathrm{iq})^{\mathrm{L}}\mathrm{L}!(\ell-\mathrm{L})!}\mathrm{a}^{1-\eta-\mathrm{L}} \\ & \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}f_{n+1-\eta-L}(M,N;R;1-\exp(-s/\mathrm{a}))\Bigg]. \end{aligned} \tag{20} $$

To determine the unknown constants A_1 and A_2 we shall apply the boundary conditions judiciously at $s=0$ and $s\to\infty$. One gets the off-shell Jost function $f_\ell(\xi,q)$ [5] by considering the limiting behaviour of the off-shell Jost solution $f_\ell(\xi,q,s)$ at the origin. Thus, for $s\to 0$ one gets

$$ A_2 = \frac{f_\ell(\xi,\mathrm{q})}{\mathrm{a}^{\eta+1}\mathrm{J}_\ell(\xi)}, \tag{21} $$

where the on-shell Jost function $J_\ell(\xi)$ [37] is

$$ J_\ell(\xi) = \mathrm{a}^{\eta}\frac{\Gamma(2+2\eta)\Gamma(1-2\mathrm{i}\xi\mathrm{a})}{\Gamma(\mathrm{M})\Gamma(\mathrm{N})}. \tag{22} $$

To obtain the other constant A_1 we use the limit when $s\to\infty$. The quantity $F_P(s)$ in Eq. (20) is related to the regular Manning-Rosen plus Hulthén Green's function $G_\ell^{(R)}(s,s')$ as

$$ \begin{aligned} & (\xi^2-q^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{2-\mathrm{L}}\exp(\mathrm{i}\xi\mathrm{s})\left[1-\exp(-\mathrm{s}/\mathrm{a})\right]^{\eta+1}\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mathrm{n}+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}\mathrm{f}_{\mathrm{n}+1-\eta-\mathrm{L}} \\ & \times(M,N;R;1-\exp(-s/\mathrm{a})) = (\xi^2-q^2)\int_0^s G_\ell^{(R)}(s,s')\,\hat{h}_\ell^{(+)}(qs')ds', \end{aligned} \tag{23} $$

with

$$ G_\ell^{(R)}(s,s') = \frac{1}{J_\ell(\xi)}\Big[\phi_\ell(\xi,s)f_\ell(\xi,s')-\phi_\ell(\xi,s')f_\ell(\xi,s)\Big]. \tag{24} $$

Here $\phi_\ell(\xi,s)$ and $f_\ell(\xi,s)$ are the regular and irregular solutions of the Manning-Rosen plus Hulthén potential [11],

$$ \phi_\ell(\xi,s) = \mathrm{a}^{\eta+1}\left[1-\exp(-\mathrm{s}/\mathrm{a})\right]^{\eta+1}\exp(\mathrm{i}\xi\mathrm{s}){}_2\mathrm{F}_1(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp(-\mathrm{s}/\mathrm{a})), \tag{25} $$

and

$$ f_\ell(\xi,s) = \left[1-\exp(-s/\mathrm{a})\right]^{-\eta}\exp(\mathrm{i}\xi\mathrm{s}){}_2F_1(1-M^*,1-N^*;1-2\mathrm{i}\xi\mathrm{a};\exp(-s/\mathrm{a})). \tag{26} $$

Fuda and Whiting [24] and, Laha and Bhoi [38] were able to prove that only the particular solution of Eq. (5) gives the off-shell Jost solution as

$$f_\ell(\xi,q,s) = (\xi^2-q^2)\int_s^\infty G_\ell^{(I)}(s,s')\,\hat{h}_\ell^{(+)}(qs')ds', \tag{27}$$

with the irregular Green's Function [23] for Manning-Rosen plus Hulthén potential

$$G_\ell^{(I)}(s,s') = \frac{1}{J_\ell(\xi)}\Big[\phi_\ell(\xi,s')f_\ell(\xi,s) - \phi_\ell(\xi,s)f_\ell(\xi,s')\Big]. \tag{28}$$

Under the limit $s\to\infty$ Eq. (20) together with Eqs. (23)-(28) yields

$$A_1 = \frac{\mathrm{i(q-\xi)}}{J_\ell(\xi)}\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{-\mathrm{L}}\frac{\Gamma(1-\eta-\mathrm{L})\Gamma(1-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}$$
$$\times\,{}_3F_2(M-1-2\eta, N-1-2\eta, -\mathrm{i}(\xi+\mathrm{q})\mathrm{a}, 1-2\mathrm{i}\xi\mathrm{a}, 1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1). \tag{29}$$

In evaluating the above constant we have applied the following standard integral [33,34,39]

$$\int_0^s z^{\rho-1}(s-z)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;cz)dz = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}s^{\rho+\sigma-1}{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;cs), \tag{30}$$

with Re $\sigma>0$, Re $\rho>0$, Re$(\gamma+\sigma-\alpha-\beta)>0$.

Having the constants A_1 and A_2 one obtains the compact expression for $f_\ell(\xi,q,s)$ as

$$f_\ell(\xi,q,s) = \mathrm{a}^{\eta+1}\left[1-\exp\left(-\mathrm{s/a}\right)\right]^{\eta+1}\exp\left(\mathrm{i}\xi\mathrm{s}\right)\Bigg[\frac{\mathrm{i(q-\xi)}}{\mathrm{J}_\ell(\xi)}\sum_{\mathrm{L}=0}^{\ell}\frac{(\mathrm{i})^{2\mathrm{L}-\ell}(\ell+\mathrm{L})!}{(2\mathrm{iq})^{\mathrm{L}}\mathrm{L}!(\ell-\mathrm{L})!}\mathrm{a}^{-\mathrm{L}}\frac{\Gamma(1-\eta-\mathrm{L})\Gamma(1-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}$$
$$\times\,{}_3F_2(M-1-2\eta, N-1-2\eta, -\mathrm{i}(\xi+\mathrm{q})\mathrm{a}, 1-2\mathrm{i}\xi\mathrm{a}, 1-\eta-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1)\,{}_2\mathrm{F}_1(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp\left(-\mathrm{s/a}\right))$$
$$+\frac{f_\ell(\xi,\mathrm{q})}{\mathrm{a}^{\eta+1}\mathrm{J}_\ell(\xi)}(1-\exp\left(-s/\mathrm{a}\right))^{-2\eta-1}{}_2F_1(1-M^*, 1-N^*; 1-2\mathrm{i}\xi\mathrm{a}; \exp\left(-\mathrm{s/a}\right))$$
$$+(\xi^2-q^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{1-\eta-\mathrm{L}}\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mathrm{n}+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}\mathrm{f}_{\mathrm{n}+1-\eta-\mathrm{L}}(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp\left(-\mathrm{s/a}\right))\Bigg]. \tag{31}$$

2.2. Integral transform method

For physical boundary condition Eq. (5) takes the form [40,41]

$$\left[\frac{\mathrm{d}^2}{\mathrm{ds}^2}+\xi^2-V_{eff}(s)\right]\psi_\ell^{(+)}(\xi,q,s) = (\xi^2-q^2)\,\hat{j}_\ell^{(+)}(qs), \tag{32}$$

where $\hat{j}_\ell^{(+)}(qs) = (\hat{h}_\ell^{(+)}(qs)-\hat{h}_\ell^{(-)}(qs))/2i$ and $\hat{h}_\ell^{(-)}(qs) = (\hat{h}_\ell^{(+)}(qs))^*$. According to Refs. [24,38] the particular solution of the above equation also represents the off-shell physical solutions $\psi_\ell^{(+)}(\xi,q,s)$ written as

$$\psi_\ell^{(+)}(\xi,q,s) = \frac{(\xi^2-q^2)}{2\mathrm{i}}\Big[\bar{G}_\ell^{(+)}(s,q)-\bar{G}_\ell^{(+)}(s,-q)\Big], \tag{33}$$

where

$$\bar{G}_\ell^{(+)}(s,q) = \int_0^\infty G_\ell^{(+)}(s,s')\,\hat{h}_\ell^{(+)}(qs')ds'. \tag{34}$$

The quantity $\bar{G}_\ell^{(+)}(s,-q)$ and $\bar{G}_\ell^{(+)}(s,q)$ is related by

$$\bar{G}_\ell^{(+)}(s,-q) = \Big[\bar{G}_\ell^{(+)}(s,q)\Big]_{q\to-q}. \tag{35}$$

To evaluate $\bar{G}_\ell^{(+)}(s,q)$ in Eq. (34) we follow the ordinary differential equation approach. The inhomogeneous differential equation [23,42] satisfied by $G_\ell^{(+)}(s,s')$ is

$$\left[\frac{\mathrm{d}^2}{\mathrm{ds}^2}+\xi^2-V_{eff}(s)\right]G_\ell^{(+)}(s,s')=\delta(s-s'). \tag{36}$$

We use the same kind of transformation as in Eq. (6) reads

$$G_\ell^{(+)}(s,s')=\mathrm{a}^{\mu}\left[1-\exp\left(-\mathrm{s/a}\right)\right]^{\mu}\exp\left(\mathrm{i}\xi\mathrm{s}\right)\Omega_\ell(\mathrm{s},\mathrm{s}'), \tag{37}$$

in the above equation to have

$$\exp(s/a)\mathrm{a}^2(1-\exp\left(-s/a\right))\Omega_\ell^{''}+\left\{2\mu\mathrm{a}+2\mathrm{i}\xi\mathrm{a}^2\exp(\mathrm{s/a})(1-\exp\left(-\mathrm{s/a}\right))\right\}\Omega_\ell^{'}$$
$$+\left\{2\mathrm{i}\xi\mathrm{a}\mu-\mu+(\mathrm{E}-\mathrm{E}_0\mathrm{a}^2)\right\}\Omega_\ell=\exp(-\mathrm{i}\xi\mathrm{s})\exp(\mathrm{s/a})(1-\exp\left(-\mathrm{s/a}\right))^{1-\mu}\mathrm{a}^{2-\mu}\delta(\mathrm{s}-\mathrm{s}'). \tag{38}$$

Taking the integral transform of $\Omega_\ell(s,s')$ with respect to s' in Eq. (38) and changing the independent variable by $\mathrm{y}=(1-\exp\left(-\mathrm{s/a}\right))$ and substituting $\mu=\eta+1$, we get

$$\mathrm{y}(1-\mathrm{y})\frac{\mathrm{d}^2\bar{\Omega}_\ell}{\mathrm{dy}^2}+\{2\eta+2-(3+2\eta-2\mathrm{i}\xi\mathrm{a})\mathrm{y}\}\frac{\mathrm{d}\bar{\Omega}_\ell}{\mathrm{dy}}-(1+\eta-\mathrm{E}+\mathrm{E}_0\mathrm{a}^2-2\eta\mathrm{i}\xi\mathrm{a}-2\mathrm{i}\xi\mathrm{a})\bar{\Omega}_\ell$$
$$=\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{2-\eta-\mathrm{L}}\mathrm{y}^{1-\eta-\mathrm{L}}(1-\mathrm{y})^{\mathrm{i}\delta(\xi-\mathrm{q})-1}, \tag{39}$$

with

$$\bar{\Omega}_\ell(s,q)=H\left\{\Omega_\ell(s,s');s'\to q\right\}. \tag{40}$$

Comparison of Eqs. (9) and (39) yields

$$\bar{\Omega}_\ell(s,q)=\Bigg[B_1\,{}_2F_1(M,N;R;1-\exp\left(-s/\mathrm{a}\right))+B_2\,(1-\exp\left(-s/\mathrm{a}\right))^{-2\eta-1}{}_2F_1(1-M^*,1-N^*;1-2\mathrm{i}\xi\mathrm{a};\exp\left(-\mathrm{s/a}\right))$$
$$+\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{1-\eta-\mathrm{L}}\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mathrm{n}+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}\mathrm{f}_{\mathrm{n}+1-\eta-\mathrm{L}}(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp\left(-\mathrm{s/a}\right))\Bigg]. \tag{41}$$

In Eq. (41) B_1 and B_2 are two unknown constants and will be determined from the boundary conditions at $s=0$ and $s=\infty$. At $s=0$, $\bar{G}_\ell^{(+)}(s,q)=0$ and we obtain $B_2=0$. The all partial-wave physical Green's function for Manning-Rosen plus Hulthén potential is written as [23]

$$G_\ell^{(+)}(s,s')=-\frac{\phi_\ell(\xi,s_<)f_\ell(\xi,s_>)}{J_\ell(\xi)}. \tag{42}$$

The quantities $s_>$ and $s_<$ have usual meaning. For the limit s$\to\infty$ we combine Eqs. (37), (41) and (42) with the judicious application of Eq. (30) to obtain

$$B_1=\frac{1}{\mathrm{i}(\xi+\mathrm{q})\mathrm{J}_\ell(\xi)}\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\frac{\Gamma(1-\eta-L)\Gamma(1-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(1-\eta-L-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}\mathrm{a}^{-\mathrm{L}}$$
$$\times\,{}_3F_2(M-1-2\eta,N-1-2\eta,-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1-2\mathrm{i}\xi\mathrm{a},1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1). \tag{43}$$

From Eqs. (37), (41) and (43) with $B_2=0$, we have

$$\bar{G}_\ell^{(+)}(s,q)=\mathrm{a}^{\eta+1}\left[1-\exp\left(-\mathrm{s/a}\right)\right]^{\eta+1}\exp\left(\mathrm{i}\xi\mathrm{s}\right)\Bigg[\frac{1}{\mathrm{i}(\xi+\mathrm{q})\mathrm{J}_\ell(\xi)}\sum_{\mathrm{L}=0}^{\ell}\frac{(\mathrm{i})^{2\mathrm{L}-\ell}(\ell+\mathrm{L})!}{(2\mathrm{iq})^{\mathrm{L}}\mathrm{L}!(\ell-\mathrm{L})!}\frac{\Gamma(1-\eta-\mathrm{L})\Gamma(1-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}$$
$$\times\,{}_3F_2(M-1-2\eta,N-1-2\eta,-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1-2\mathrm{i}\xi\mathrm{a},1-\eta-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1)_2\mathrm{F}_1(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp\left(-\mathrm{s/a}\right))$$
$$+\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{1-\eta-\mathrm{L}}\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mathrm{n}+1-\mathrm{ia}(\xi-\mathrm{q}))}{\Gamma(1-\mathrm{ia}(\xi-\mathrm{q}))\mathrm{n}!}\mathrm{f}_{\mathrm{n}+1-\eta-\mathrm{L}}(\mathrm{M},\mathrm{N};\mathrm{R};1-\exp\left(-\mathrm{s/a}\right))\Bigg]. \tag{44}$$

The expression for $\bar{G}_\ell^{(+)}(s,-q)$ is obtained by replacing q by $-$q in the Eq. (44). Therefore, by utilizing the Eqs. (33) and (44) one is able to write an expression for the off-shell physical solution for motion in Manning-Rosen plus Hulthén potential. There exists a relation between the off-shell physical and Jost solutions [18,43] expressed as

$$\psi_\ell^{(+)}(\xi,q,s)=\frac{\pi \mathrm{q}}{2}T_{\ell h}(\xi,q,\xi^2)f_\ell(\xi,s)+\frac{1}{2\mathrm{i}}\Big[f_\ell(\xi,q,s)-f_\ell(\xi,-q,s)\Big], \tag{45}$$

where $T_{\ell h}(\xi,q,\xi^2)$ stands for the half off-shell T-matrix written as

$$T_{\ell h}(\xi,q,\xi^2)=\frac{f_\ell(\xi,q)-f_\ell(\xi,-q)}{\mathrm{i}\pi \mathrm{q} f_\ell(\xi)}. \tag{46}$$

The quantity $f_\ell(\xi,q)$ represents the off-shell Jost function for Manning-Rosen plus Hulthén potential. The off-shell Jost function $f_\ell(\xi,q)$ is obtained from $f_\ell(\xi,q,s)$ as

$$f_\ell(\xi,q)=\lim_{s\to 0}f_\ell(\xi,q,s)=(\xi^2-q^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{\eta+2-\mathrm{L}}\frac{\Gamma(\eta+2-\mathrm{L})\Gamma(-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(\eta+2-\mathrm{L}-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}$$
$$\times\, {}_3F_2(M,N,\eta+2-L;R,\eta+2-L-\mathrm{i}(\xi+\mathrm{q})\mathrm{a};1). \tag{47}$$

Exploiting two times the following transformation [44]

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;1)=\frac{\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_2-a_3)\Gamma(b_1+b_2-a_1-a_2)}{}_3F_2(b_1-a_1,b_1-a_2,a_3,b_1,b_1+b_2-a_1-a_2;1). \tag{48}$$

Equation (47) reads as

$$f_\ell(\xi,q)=\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2iq)^L L!(\ell-L)!}\mathrm{a}^{\eta-\mathrm{L}}\frac{\Gamma(1+\mathrm{i}(\xi-\mathrm{q})\mathrm{a})\Gamma(1-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})}{\Gamma(\mathrm{i}(\xi-\mathrm{q})\mathrm{a}-\eta-\mathrm{L}+\mathrm{M})}\times\frac{\Gamma(\eta+2)}{\Gamma(2\eta+2-\mathrm{i}(\xi+\mathrm{q})\mathrm{a})-\mathrm{M})}$$
$$\times\, {}_3F_2(2\eta+2-M,N,\eta+L,2\eta+2,2\eta+2-\mathrm{i}(\xi+\mathrm{q})\mathrm{a}-\mathrm{M};1). \tag{49}$$

Equations (47) and (49) are equivalent. However, Eq. (49) is the most suitable one for checking limiting values and numerical treatment. In the on-shell limit i.e $q\to\xi$, $f_\ell(\xi,q)=f_\ell(\xi)$.

2.3. Off-shell extension function

The half-shell T-matrix may be rewritten as

$$T_{\ell h}(\xi,q,\xi^2)=\left(\frac{\xi}{q}\right)^{\ell}\frac{|f_\ell(\xi,q)|2\sin\Delta_\ell(\xi,q)e^{i\delta_\ell(\xi)}}{\pi q|f_\ell(\xi)|}, \tag{50}$$

where $\Delta_\ell(\xi,q)$ is the quasi phase and the quantity $\delta_\ell(\xi)$ stands for the scattering phase shift. After some algebric manipulation one can get

$$T_{\ell h}(\xi,q,\xi^2)=T_{\ell h}(\xi,\xi,\xi^2)H_\ell(\xi,q), \tag{51}$$

where $T_{\ell h}(\xi,\xi,\xi^2)$ is on-shell T-matrix and is expressed as

$$T_{\ell h}(\xi,\xi,\xi^2)=-\frac{2}{\pi\xi}\sin\delta_\ell(\xi)e^{i\delta_\ell(\xi)}. \tag{52}$$

The off-shell extension function $H_\ell(\xi,q)$ reads as

$$H_\ell(\xi,q)=\left(\frac{\xi}{q}\right)^{\ell-1}\frac{|f_\ell(\xi,q)|}{|f_\ell(\xi)|}\frac{\sin\Delta_\ell(\xi,q)}{\sin\delta_\ell(\xi)}. \tag{53}$$

For $q\to\xi$, $\Delta_\ell(\xi,\xi)=\delta_\ell(\xi)$ and $H_\ell(\xi,\xi)=1$. The off-shell quantities are used in many particle systems. In this connection one may consider the off-shell T-matrix as an important quantity for the study of scattering theory because its on-shell limit is directly related to the scattering amplitude. The half off-shell T-matrix related to the scattering phase shifts can be expressed in terms of the on- and off-shell Jost function as already mentioned in Eq. (46). Thus, having the compact analytical expressions for the on- and off-shell Jost functions one will be in a position to calculate half-off-shell T-matrix. In the next section we will compute scattering phase shifts, half-shell T-matrices and quasi phases for the $\alpha-^3$H and $\alpha-^3$He systems.

TABLE I. Parameters for the ($\alpha-^3$H) and ($\alpha-^3$He) systems.

States	($\alpha-^3$H)			States	($\alpha-^3$He)		
	β	E	a (in fm)		β	E	a (in fm)
$1/2^+$	0.005	1.954	0.57	$1/2^+$	0.05	1.89	0.671
$1/2^-$	-20.005	53.839	0.43	$1/2^-$	-65.05	155.892	0.384
$3/2^-$	-1.985	6.456	0.252	$3/2^-$	-16.05	34.622	0.289
$3/2^+$	0.005	1.554	0.207	$3/2^+$	0.055	1.05	0.341
$5/2^+$	1.055	0.955	0.227	$5/2^+$	0.55	1.02	0.430

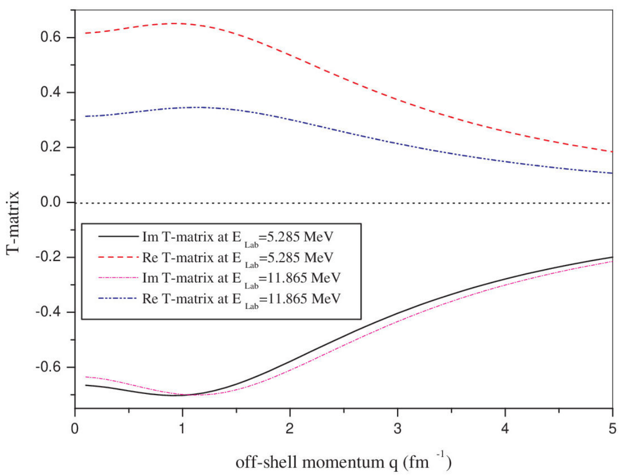

FIGURE 1. $T_{\ell h}(\xi, q, \xi^2)$ for $1/2^+$ state of $\alpha{-}^3$H system.

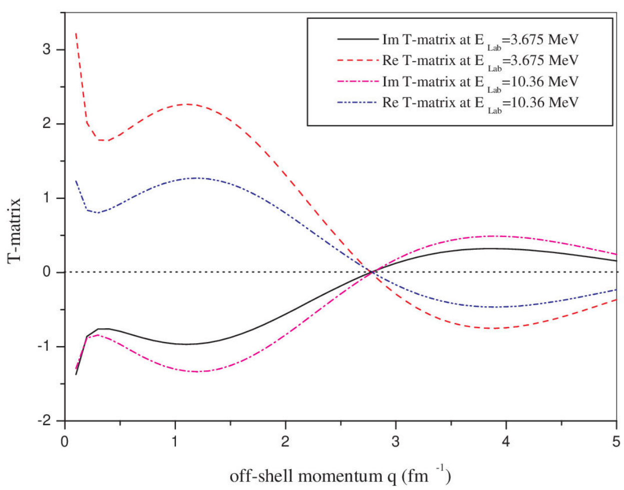

FIGURE 2. $T_{\ell h}(\xi, q, \xi^2)$ for $1/2^-$ state of $\alpha{-}^3$H system.

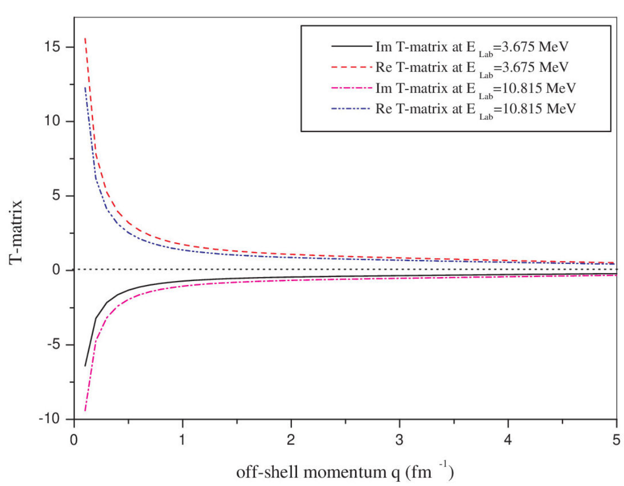

FIGURE 3. $T_{\ell h}(\xi, q, \xi^2)$ for $3/2^-$ state of $\alpha{-}^3$H system.

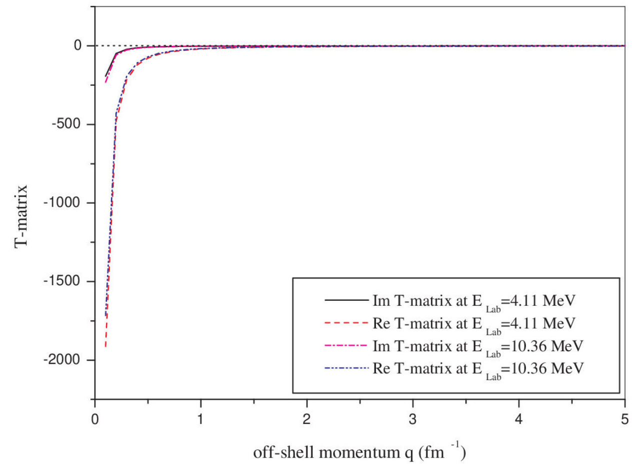

FIGURE 4. $T_{\ell h}(\xi, q, \xi^2)$ for $3/2^+$ state of $\alpha{-}^3$H system.

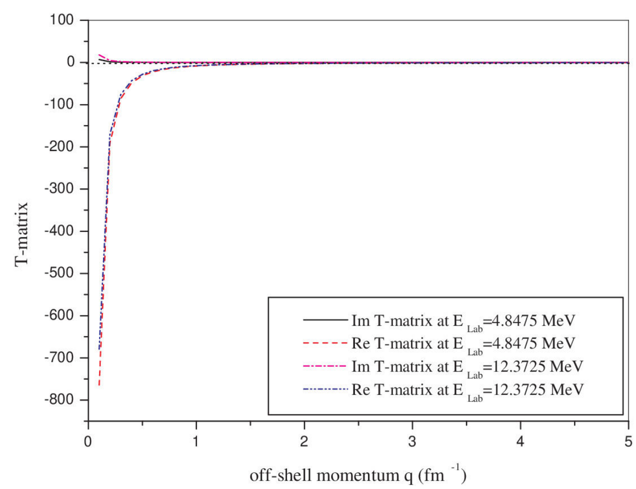

FIGURE 5. $T_{\ell h}(\xi, q, \xi^2)$ for $5/2^+$ state of $\alpha{-}^3$H system.

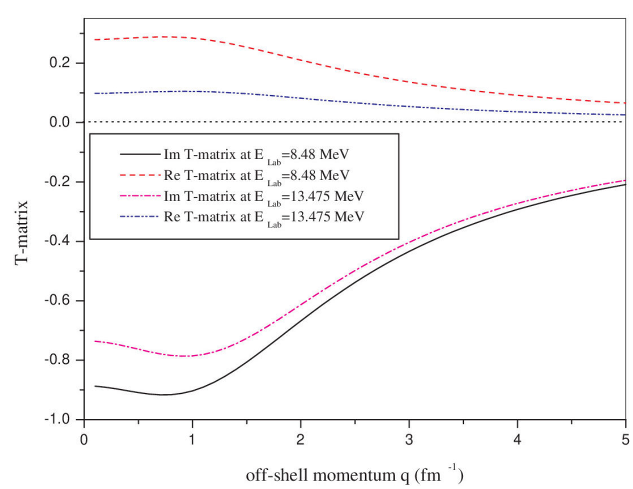

FIGURE 6. $T_{\ell h}(\xi, q, \xi^2)$ for $1/2^+$ state of $\alpha{-}^3$He system.

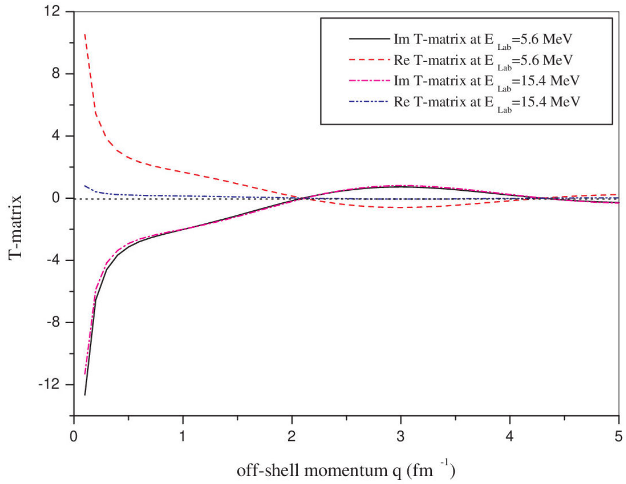

FIGURE 7. $T_{\ell h}(\xi, q, \xi^2)$ for $1/2^-$ state of $\alpha-^3$He system.

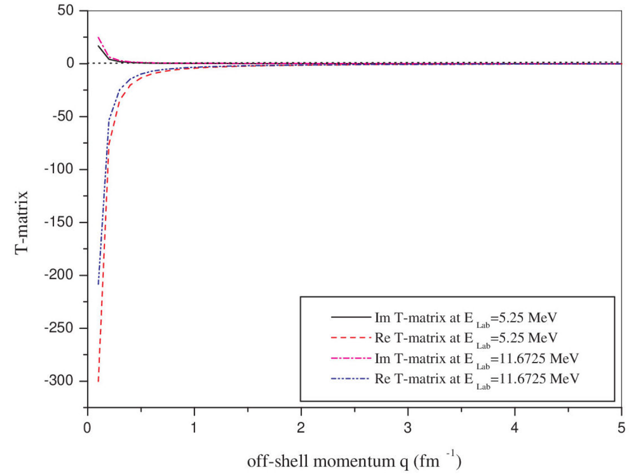

FIGURE 10. $T_{\ell h}(\xi, q, \xi^2)$ for $5/2^+$ state of $\alpha-^3$He system.

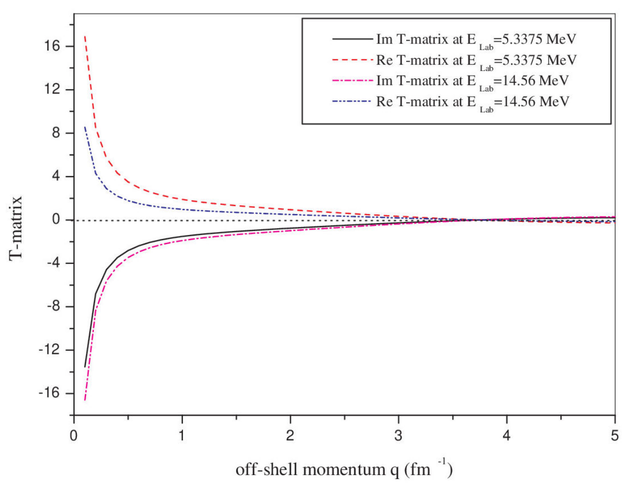

FIGURE 8. $T_{\ell h}(\xi, q, \xi^2)$ for $3/2^-$ state of $\alpha-^3$He system.

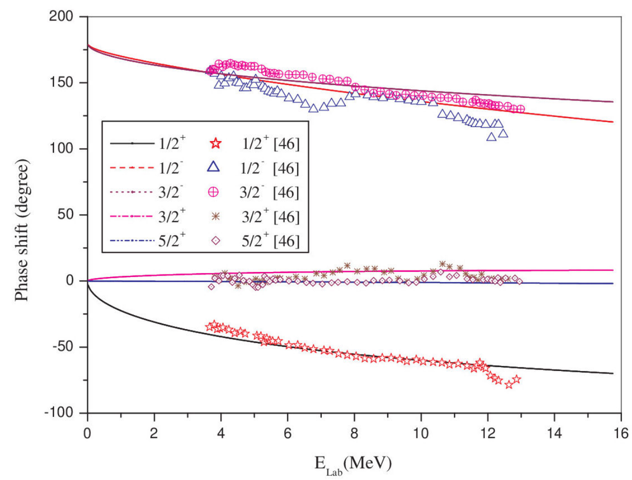

FIGURE 11. Phase shifts for $\alpha-^3$H system as a function of E_{Lab}.

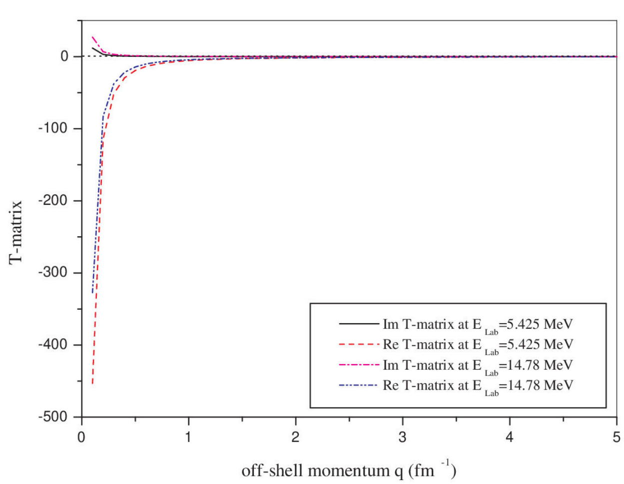

FIGURE 9. $T_{\ell h}(\xi, q, \xi^2)$ for $3/2^+$ state of $\alpha-^3$He system.

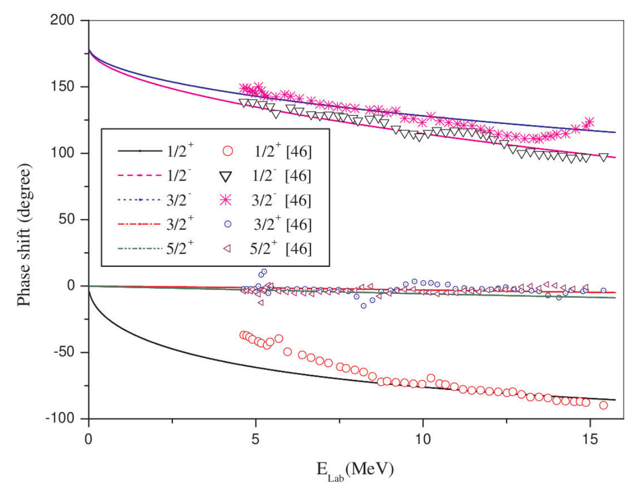

FIGURE 12. Phase shifts for $\alpha-^3$He system as a function of E_{Lab}.

TABLE II. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $1/2^+$ state of $\alpha{-}^3$H system at $E_{\rm Lab} = 5.285$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-0.035089	-0.179929	-11.035	—
0.2	-0.070508	-0.178718	-21.530	—
0.3	-0.106524	-0.176443	-31.120	—
0.4	-0.143291	-0.172761	-39.672	—
0.45	-0.161964	-0.170266	-43.568	—
0.46	-0.165721	-0.169708	-44.318	—
0.47	-0.169485	-0.169130	-45.060	—
0.48	-0.173255	-0.168530	-45.792	—
0.49	-0.177032	-0.167909	-46.515	-46.21
0.5	-0.180816	-0.167266	-47.229	—
0.6	-0.218948	-0.159553	-53.918	—
1.0	-0.370196	-0.101080	-74.727	—
2.0	-0.610239	0.187300	72.937	—
3.0	-0.638436	0.458157	54.335	—
4.0	-0.588290	0.634321	42.843	—
5.0	-0.524985	0.743770	35.216	—

TABLE III. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $1/2^+$ state of $\alpha{-}^3$H system at $E_{Lab} = 11.865$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-0.031095	-0.145959	-12.026	—
0.2	-0.062560	-0.145511	-23.264	—
0.3	-0.094718	-0.144529	-33.239	—
0.4	-0.127803	-0.142690	-41.849	—
0.5	-0.161924	-0.139588	-49.236	—
0.6	-0.197049	-0.134786	-55.626	—
0.7	-0.233002	-0.127858	-61.244	—
0.72	-0.240267	-0.126183	-62.292	—
0.73	-0.243906	-0.125307	-62.808	—
0.74	-0.247549	-0.124406	-63.318	—
0.75	-0.251196	-0.123478	-63.823	-63.73
0.8	-0.269472	-0.118436	-66.274	—
1.0	-0.342236	-0.091157	-75.085	—
2.0	-0.598183	0.176555	73.555	—
3.0	-0.636993	0.450663	54.721	—
4.0	-0.589052	0.630498	43.053	—
5.0	-0.525934	0.741785	35.337	—

TABLE IV. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $1/2^-$ state of $\alpha{-}^3$H system at $E_{Lab} = 11.865$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-9.126×10^{-8}	-1.012×10^{-5}	179.483	—
0.2	-4.703×10^{-7}	-6.225×10^{-6}	175.679	—
0.3	-1.081×10^{-6}	-5.303×10^{-6}	168.472	—
0.35	-1.465×10^{-6}	-5.111×10^{-6}	164.006	—
0.38	-1.717×10^{-6}	-5.027×10^{-6}	161.143	—
0.39	-1.804×10^{-6}	-5.002×10^{-6}	160.165	—
0.40	-1.893×10^{-6}	-4.978×10^{-6}	159.176	—
0.41	-1.983×10^{-6}	-4.954×10^{-6}	158.177	158.98
0.42	-2.076×10^{-6}	-4.931×10^{-6}	157.168	—
0.5	-2.859×10^{-6}	-4.731×10^{-6}	148.854	—
1.0	-7.590×10^{-6}	-5.953×10^{-7}	94.484	—
2.0	1.368×10^{-6}	9.854×10^{-6}	-7.905	—
3.0	8.611×10^{-6}	-1.203×10^{-6}	-82.042	—
4.0	2.372×10^{-7}	-6.241×10^{-6}	2.176	—
5.0	-3.774×10^{-6}	-1.399×10^{-6}	110.338	—

TABLE V. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $3/2^-$ state of $\alpha{-}^3$H system at $E_{Lab} = 10.815$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-0.030225	-0.249785	173.100	—
0.2	-0.030518	-0.126647	166.451	—
0.3	-0.031001	-0.086328	160.246	—
0.4	-0.031668	-0.066656	154.587	—
0.5	-0.032509	-0.055180	149.495	—
0.6	-0.033512	-0.047735	144.929	—
0.65	-0.034069	-0.044920	142.821	—
0.68	-0.034420	-0.043440	141.607	—
0.70	-0.034661	-0.042526	140.817	—
0.71	-0.034783	-0.042089	140.428	139.54
0.72	-0.034907	-0.041664	140.043	—
0.8	-0.035940	-0.038649	137.080	—
1.0	-0.038806	-0.033061	130.429	—
2.0	-0.053570	-0.012909	103.547	—
3.0	-0.057712	0.010000	80.169	—
4.0	-0.049670	0.028943	59.769	—
5.0	-0.035992	0.039600	42.267	—

TABLE VI. Quasi-elastic phase shifts $\Delta_\ell(\xi, q)$ for $3/2^+$ state of $\alpha-^3$H system at $E_{Lab} = 4.11$ MeV.

q(fm^{-1})	Im$f_\ell(\xi, q)$	Re$f_\ell(\xi, q)$	$\Delta_\ell(\xi, q)$ (degree)	$\delta_\ell(\xi), q = \xi$ (degree) Ref. [48]
0.1	-3.213799	131.819992	1.396	—
0.2	-1.607243	32.949940	2.792	—
0.3	-1.071876	14.640685	4.187	—
0.40	-0.804307	8.232463	5.580	—
0.42	-0.766095	7.466468	5.858	—
0.44	-0.731362	6.802535	6.136	—
0.45	-0.715154	6.503268	6.275	6.00
0.46	-0.699652	6.223305	6.414	—
0.5	-0.643855	5.266390	6.970	—
0.6	-0.536961	3.655210	8.357	—
0.7	-0.460672	2.683739	9.740	—
0.8	-0.403509	2.053238	11.118	—
1.0	-0.323609	1.311827	13.857	—
2.0	-0.164953	0.324030	26.979	—
3.0	-0.113019	0.142278	38.462	—
4.0	-0.087390	0.079786	47.604	—
5.0	-0.072006	0.051793	54.272	—

TABLE VII. Quasi-elastic phase shifts $\Delta_\ell(\xi, q)$ for $5/2^+$ state of $\alpha -^3 H$ system at $E_{\text{Lab}} = 12.3725$ MeV.

q(fm^{-1})	Im$f_\ell(\xi, q)$	Re$f_\ell(\xi, q)$	$\Delta_\ell(\xi, q)$ (degree)	$\delta_\ell(\xi), q = \xi$ (degree) Ref. [48]
0.1	-0.005597	-1.772713	-0.180	—
0.2	-0.002799	-0.443274	-0.361	—
0.3	-0.001867	-0.197081	-0.542	—
0.4	-0.001401	-0.110914	-0.723	—
0.5	-0.001122	-0.071031	-0.905	—
0.6	-0.000936	-0.049366	-1.086	—
0.70	-0.000804	-0.036303	-1.268	—
0.72	-0.000782	-0.034321	-1.304	—
0.74	-0.000761	-0.032497	-1.340	—
0.76	-0.000741	-0.030816	-1.377	-1.17
0.78	-0.000722	-0.029263	-1.413	—
0.80	-0.000704	-0.027824	-1.450	—
1.0	-0.000566	-0.017853	-1.814	—
2.0	-0.000292	-0.004558	-3.663	—
3.0	-0.000204	-0.002095	-5.575	—
4.0	-0.000164	-0.001231	-7.568	—
5.0	-0.000141	-0.000830	-9.648	—

TABLE VIII. Quasi-elastic phase shifts $\Delta_\ell(\xi, q)$ for $1/2^+$ state of $\alpha-^3$He system at $E_{Lab} = 8.48$ MeV.

q(fm^{-1})	Im$f_\ell(\xi, q)$	Re$f_\ell(\xi, q)$	$\Delta_\ell(\xi, q)$ (degree)	$\delta_\ell(\xi), q = \xi$ (degree) Ref. [48]
0.1	-0.042960	-0.128407	-18.498	—
0.2	-0.086270	-0.126091	-34.379	—
0.3	-0.130192	-0.121888	-46.886	—
0.4	-0.174817	-0.115348	-56.582	—
0.5	-0.220024	-0.105958	-64.285	—
0.60	-0.265469	-0.093233	-70.648	—
0.62	-0.274542	-0.090253	-71.802	—
0.63	-0.279072	-0.088707	-72.366	-69.65
0.64	-0.283597	-0.087122	-72.922	—
0.70	-0.310609	-0.076808	-76.110	—
0.8	-0.354766	-0.056498	-80.951	—
0.9	-0.397209	-0.032342	-85.345	—
1.0	-0.437237	-0.004593	-89.398	—
2.0	-0.646305	0.350216	61.547	—
3.0	-0.629879	0.615456	45.663	—
4.0	-0.566081	0.768876	36.362	—
5.0	-0.504826	0.860927	30.386	—

TABLE IX. Quasi-elastic phase shifts $\Delta_\ell(\xi, q)$ for $1/2^+$ state of $\alpha-^3$He system at $E_{Lab} = 13.475$ MeV.

q(fm^{-1})	Im$f_\ell(\xi, q)$	Re$f_\ell(\xi, q)$	$\Delta_\ell(\xi, q)$ (degree)	$\delta_\ell(\xi), q = \xi$ (degree) Ref. [48]
0.1	-0.038365	-0.094383	-22.120	—
0.2	-0.077150	-0.093078	-39.654	—
0.3	-0.116705	-0.090585	-52.181	—
0.4	-0.157249	-0.086467	-61.194	—
0.5	-0.198815	-0.080192	-68.033	—
0.6	-0.241229	-0.071201	-73.555	—
0.7	-0.284106	-0.058986	-78.270	—
0.75	-0.305547	-0.051545	-80.424	—
0.78	-0.318366	-0.046633	-81.666	—
0.79	-0.322626	-0.044921	-82.073	—
0.80	-0.326880	-0.043170	-82.476	-83.57
0.82	-0.335363	-0.039553	-83.273	—
1.0	-0.409310	-0.000165	-89.976	—
2.0	-0.639238	0.339399	62.034	—
3.0	-0.630046	0.610069	45.922	—
4.0	-0.566990	0.766445	36.492	—
5.0	-0.505577	0.859718	30.458	—

TABLE X. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $1/2^-$ state of $\alpha-^3$He system at $E_{Lab}=5.6$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	- 7.970×10^{-23}	-5.136×10^{-22}	171.179	—
0.2	- 8.424×10^{-23}	-2.612×10^{-22}	162.123	—
0.3	- 9.131×10^{-23}	-1.767×10^{-22}	152.678	—
0.4	- 1.001×10^{-22}	-1.320×10^{-22}	142.802	—
0.45	- 1.050×10^{-22}	-1.154×10^{-22}	137.712	—
0.50	- 1.099×10^{-22}	-1.008×10^{-22}	132.535	—
0.51	- 1.109×10^{-22}	-9.808×10^{-23}	131.490	131.09
0.52	- 1.118×10^{-22}	-9.536×10^{-23}	130.442	—
0.6	- 1.194×10^{-22}	-7.454×10^{-23}	121.960	—
0.8	- 1.336×10^{-22}	-2.412×10^{-23}	100.235	—
1.0	- 1.346×10^{-22}	2.815×10^{-23}	78.185	—
2.0	7.923×10^{-23}	1.331×10^{-22}	-30.752	—
3.0	1.114×10^{-22}	-9.453×10^{-23}	49.695	—
4.0	- 8.319×10^{-23}	-8.528×10^{-23}	-44.288	—
5.0	- 6.466×10^{-23}	5.988×10^{-23}	47.195	—

TABLE XI. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $3/2^-$ state of $\alpha-^3$He system at $E_{Lab}=14.56$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	6.403×10^{-20}	4.063×10^{-19}	171.045	—
0.2	6.395×10^{-20}	2.003×10^{-19}	162.293	—
0.3	6.382×10^{-20}	1.303×10^{-19}	153.909	—
0.4	6.364×10^{-20}	9.439×10^{-20}	146.012	—
0.5	6.341×10^{-20}	7.207×10^{-20}	138.658	—
0.6	6.312×10^{-20}	5.656×10^{-20}	131.859	—
0.7	6.279×10^{-20}	4.494×10^{-20}	125.592	—
0.80	6.240×10^{-20}	3.576×10^{-20}	119.817	—
0.82	6.232×10^{-20}	3.414×10^{-20}	118.717	118.21
0.83	6.228×10^{-20}	3.335×10^{-20}	118.173	—
1.0	6.148×10^{-20}	2.182×10^{-20}	109.539	—
2.0	5.394×10^{-20}	-1.628×10^{-20}	73.199	—
3.0	4.191×10^{-20}	-3.822×10^{-20}	47.634	—
4.0	2.619×10^{-20}	-5.310×10^{-20}	26.255	—
5.0	7.868×10^{-21}	-6.213×10^{-20}	7.217	—

3. Results and discussion

In this section, we turn our attention to compute the half-shell transition matrices for the ($\alpha-^3$H) and ($\alpha-^3$He) systems

TABLE XII. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $3/2^+$ state of $\alpha-^3$He system at $E_{Lab}=5.425$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-0.135601	-28.217365	-0.275	—
0.2	-0.067846	-7.058501	-0.550	—
0.3	-0.045281	-3.140192	-0.826	—
0.4	-0.034014	-1.768783	-1.101	—
0.45	-0.030263	-1.398720	-1.239	—
0.49	-0.027816	-1.180546	-1.349	—
0.50	-0.027265	-1.134015	-1.377	—
0.51	-0.026737	-1.090195	-1.404	-1.71
0.52	-0.026229	-1.048879	-1.432	—
0.6	-0.022777	-0.789203	-1.653	—
0.7	-0.019579	-0.581290	-1.929	—
0.8	-0.017188	-0.446346	-2.205	—
1.0	-0.013859	-0.287649	-2.758	—
2.0	-0.007377	-0.076014	-5.542	—
3.0	-0.005404	-0.036755	-8.363	—
4.0	-0.004546	-0.022938	-11.210	—
5.0	-0.004123	-0.016465	-14.057	—

TABLE XIII. Quasi-elastic phase shifts $\Delta_\ell(\xi,q)$ for $5/2^+$ state of $\alpha-^3$He system at $E_{Lab}=11.6725$ MeV.

q(fm^{-1})	Im$f_\ell(\xi,q)$	Re$f_\ell(\xi,q)$	$\Delta_\ell(\xi,q)$ (degree)	$\delta_\ell(\xi), q=\xi$ (degree) Ref. [48]
0.1	-0.707934	-46.629939	-0.869	—
0.2	-0.353973	-11.658231	-1.739	—
0.3	-0.235989	-5.181989	-2.607	—
0.4	-0.176999	-2.915304	-3.474	—
0.5	-0.141606	-1.866153	-4.339	—
0.6	-0.118012	-1.296244	-5.201	—
0.70	-0.101161	-0.952606	-6.061	—
0.72	-0.098353	-0.900473	-6.233	—
0.73	-0.097006	-0.875999	-6.319	-4.85
0.74	-0.095696	-0.852510	-6.404	—
0.75	-0.094421	-0.829954	-6.490	—
0.80	-0.088523	-0.729572	-6.918	—
1.0	-0.070833	-0.467284	-8.619	—
2.0	-0.035477	-0.117563	-16.792	—
3.0	-0.023718	-0.052795	-24.191	—
4.0	-0.017857	-0.030121	-30.662	—
5.0	-0.014356	-0.019619	-36.194	—

TABLE XIV. Scattering phase shifts for $1/2^+$ and $1/2^-$ states of $\alpha{-}^3$H system.

E_{Lab}(MeV)	Phase shift $1/2^+$ state (degree) (present work)	Phase shift $1/2^+$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $1/2^-$ state (degree) (present work)	Phase shift $1/2^-$ state (degree) Ref. [48]
3.657	-40.50	-34.80	3.675	157.56	158.98
3.797	-41.15	-32.9	3.815	157.00	156.85
3.885	-41.55	-36.36	3.937	156.51	147.85
3.9725	-41.94	-35.76	4.0075	156.23	155.02
4.06	-42.332	-35.15	4.1125	155.81	149.56
4.235	-43.02	-36.73	4.235	155.33	153.56
4.41	-43.81	-39.41	4.375	154.79	154.89
4.585	-44.52	-38.21	4.55	154.11	149.77
4.725	-45.08	-39.81	4.76	153.31	145.96
5.075	-46.41	-41.28	5.04	152.26	152.35
5.285	-47.18	-46.21	5.25	151.48	147.34
5.3725	-47.49	-43.37	5.32	151.22	145.58
5.46	-47.80	-44.97	5.425	150.84	143.58
5.7225	-48.70	-45.36	5.687	149.88	143.67
6.0375	-49.75	-48.55	6.037	148.64	138.25
6.30	-50.58	-48.41	6.3	147.71	137.22
6.51	-51.23	-50.51	6.527	146.92	133.47
6.7725	-52.02	-51.49	6.772	146.08	129.97
7.2625	-53.42	-52.89	7.315	144.26	134.2
7.525	-54.15	-54.99	7.595	143.33	134.45
8.05	-55.53	-56.94	8.032	141.91	141.37
8.575	-56.84	-58.88	8.575	140.18	139.21
9.1	-58.08	-58.07	9.135	138.43	139.57
9.5725	-59.14	-60.58	9.625	136.94	136.8
10.0625	-60.20	-60.87	10.01	135.79	135.65
10.36	-60.82	-61.28	10.36	134.75	134.69
10.85	-61.80	-63.24	10.885	133.22	129.87
11.1125	-62.32	-62.56	11.112	132.57	126.35
11.585	-63.21	-66.15	11.55	131.33	119.68
11.6725	-63.37	-63.94	11.6375	131.09	118.75
11.76	-63.53	-61.68	11.76	130.74	118.78
12.075	-64.11	-71.50	11.935	130.26	118.25
12.32	-64.54	-75.24	12.215	129.48	118.37
12.8625	-65.47	-74.40	12.46	128.81	110.89

using the parameters in Table I, which are illustrated in Figs. 1-10 respectively. For our calculation we use $\hbar^2/2\mu = 12.0954$ MeV fm^2, $E_0\text{a} = 0.2381$ fm^{-1} and $E_0\text{a} = 0.4762$ fm^{-1} for ($\alpha{-}^3$H) and ($\alpha{-}^3$He) systems respectively. We compute the numerical values of the half-shell T-matrices as a function of off-shell momenta q for the systems under consideration through MATLAB and present these results in Figures 1-10 for different laboratory energies. These numbers

TABLE XV. Scattering phase shifts for $5/2^+$ state of $\alpha-^3$He system.

E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]
4.6375	-2.77	-3.78	8.1725	-4.77	-0.81
4.725	-2.82	-3.48	8.435	-4.92	-0.19
4.8475	-2.89	-3.89	8.6975	-5.06	-7.66
4.9	-2.92	-2.46	8.995	-5.23	-5.59
4.987	-2.97	-4.34	9.2225	-5.35	-3.5
5.215	-3.09	-3.70	9.45	-5.48	-2.53
5.25	-3.11	-2.99	9.7125	-5.62	-2.29
5.3725	-3.18	0.61	9.975	-5.77	-4.61
5.6875	-3.36	-3.95	10.185	-5.88	-4.74
5.915	-3.49	-4.43	10.43	-6.02	-5.97
6.16	-3.63	-5.29	10.71	-6.17	-4.99
6.44	-3.79	-5.42	10.99	-6.32	-5.12
6.685	-3.93	-5.92	11.235	-6.45	-3.41
6.9125	-4.06	-4.20	11.4625	-6.58	-3.89
7.175	-4.21	-4.70	11.6725	-6.69	-4.38
7.4375	-4.36	-4.09	12.215	-6.98	-2.43
7.7	-4.50	-3.49	13.475	-7.64	-4.58
7.9625	-4.65	-3.24	14.49	-8.16	-4.09

TABLE XVI. Scattering phase shifts for $5/2^+$ state of $\alpha-^3$H system.

E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]
3.71	-0.34	-4.39	7.035	-0.73	-2.26
3.9375	-0.37	0.09	7.315	-0.76	-1.32
4.06	-0.38	4.08	7.5075	-0.79	-0.87
4.34	-0.41	-1.02	7.7875	-0.82	-0.44
4.41	-0.42	1.98	8.085	-0.86	-1.02
4.6725	-0.45	0.41	8.3475	-0.89	-1.6
4.8475	-0.47	-0.16	9.0475	-0.98	0.72
4.9525	-0.48	-2.7	9.625	-1.06	-1.43
5.04	-0.49	-4.74	10.0975	-1.13	1.45
5.25	-0.51	-2.76	10.36	-1.16	3.41
5.3725	-0.53	1.72	10.5875	-1.19	6.87
5.46	-0.54	-0.31	10.85	-1.23	3.26
5.565	-0.55	1.67	11.55	-1.33	1.08
5.635	-0.56	0.64	11.9875	-1.39	1.46
6.0375	-0.60	0.32	12.11	-1.40	0.43
6.3	-0.64	-0.42	12.215	-1.42	-0.12
6.5625	-0.67	-1.11	12.3725	-1.44	-1.17
6.755	-0.69	-1.68	12.53	-1.46	2.31

TABLE XVII. Scattering phase shifts for $1/2^+$ and $1/2^-$ states of $\alpha-{}^3$He system.

E_{Lab}(MeV)	Phase shift $1/2^+$ state (degree) (present work)	Phase shift $1/2^+$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $1/2^-$ state (degree) (present work)	Phase shift $1/2^-$ state (degree) Ref. [48]
4.6375	-59.68	-36.93	4.6375	136.57	138.65
4.90	-60.84	-40.19	4.90	135.29	138.34
5.1625	-61.93	-43.05	5.1975	133.88	136.84
5.3375	-62.64	-44.68	5.425	132.82	135.28
5.6875	-63.98	-39.60	5.60	132.02	130.09
5.95	-64.94	-49.54	6.0375	130.07	134.32
6.3875	-66.46	-51.91	6.30	128.94	131.98
6.65	-67.32	-53.94	6.65	127.45	129.26
6.9125	-68.15	-56.38	6.9125	126.36	128.54
7.175	-68.94	-58.98	7.175	125.28	129.06
7.525	-69.97	-60.84	7.4725	124.09	128.34
7.7	-70.46	-62.02	7.91	122.37	126.49
8.225	-71.87	-64.83	8.05	121.83	124.54
8.4875	-72.55	-68.10	8.4	120.50	125.87
8.75	-73.20	-72.21	8.6625	119.51	126.78
8.925	-73.63	-71.73	8.8375	118.87	123.23
9.1875	-74.25	-72.51	9.1875	117.59	118.45
9.45	-74.85	-72.86	9.45	116.64	115.31
9.7125	-75.44	-73.64	9.7125	115.71	114.60
9.975	-76.01	-73.88	9.8875	115.10	112.67
10.2375	-76.56	-69.35	10.15	114.18	115.19
10.5	-77.10	-73.46	10.4125	113.28	117.32
10.675	-77.46	-74.24	10.675	112.39	115.79
10.9375	-77.97	-75.84	10.9375	111.51	117.10
11.20	-78.48	-77.86	11.20	110.63	116.39
12.5125	-80.83	-80.48	12.425	106.69	107.53
12.95	-81.55	-81.61	12.95	105.05	102.05
13.475	-82.39	-83.57	13.475	103.45	99.40
13.7375	-82.79	-84.35	13.7375	102.66	99.09
14.2625	-83.58	-86.73	14.175	101.36	98.07
14.875	-84.46	-87.83	14.875	99.32	97.09
15.4	-85.18	-89.79	15.4	97.82	97.7

show that both Re$T_{\ell h}(\xi, q, \xi^2)$ and Im$T_{\ell h}(\xi, q, \xi^2)$ oscillate but approach zero as q becomes large. The function $f_\ell(\xi, q)$ also tends to zero as q increases. Interestingly, for low values of laboratory energies our potential shows large off-shell effects which is in conformity with the observations of earlier works [45-47] related to various kinds of potentials. These indicate that the off-shell behaviour of the potential in Eq. (4) is quite acceptable. This means that the action of the potential in producing a half-off-shell T-matrix $T_{\ell h}(\xi, q, \xi^2)$ depends also on q. It is well known that the phase of the half-shell transition matrix is the scattering phase shift. The set of curves for Re$T_{\ell h}(\xi, q, \xi^2)$ and Im$T_{\ell h}(\xi, q, \xi^2)$ satisfy this criterion. The phase parameters calculated from the half-

TABLE XVIII. Scattering phase shifts for $3/2^-$ and $3/2^+$ states of $\alpha-{}^3$He system.

E_{Lab}(MeV)	Phase shift $3/2^-$ state (degree) (present work)	Phase shift $3/2^-$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $3/2^+$ state (degree) (present work)	Phase shift $3/2^+$ state (degree) Ref. [48]
4.6375	144.27	148.97	4.6375	-1.18	-2.3
4.725	143.94	148.91	4.725	-1.21	-2.27
4.8125	143.62	146.70	4.8475	-1.24	-2.25
4.9	143.30	146.82	4.9875	-1.29	-2.22
5.075	142.66	150.07	5.075	-1.31	-0.18
5.3375	141.73	142.98	5.3375	-1.40	-3.78
5.6	140.82	142.46	5.425	-1.42	-1.71
5.8625	139.94	144.16	5.6875	-1.51	-3.27
6.0375	139.36	142.95	5.95	-1.59	-2.4
6.3	138.51	141.04	6.23	-1.68	-2.34
6.65	137.41	139.37	6.475	-1.76	-2.28
6.9125	136.60	137.04	6.737	-1.85	-2.63
7.175	135.81	136.13	6.9475	-1.92	-2.98
7.4375	135.03	135.18	7.21	-2.00	-2.92
7.7	134.27	134.23	7.4725	-2.09	-2.86
7.9625	133.53	133.75	7.77	-2.19	-4.03
8.4	132.31	132.56	8.4875	-2.43	-10.79
8.6625	131.60	132.85	8.75	-2.52	-4.2
8.925	130.90	130.56	9.03	-2.61	-3.33
9.1875	130.21	131.71	9.2225	-2.68	-2.86
9.45	129.53	126.41	9.485	-2.77	1.27
9.7125	128.86	125.98	9.7475	-2.86	3.37
10.2375	127.56	127.65	10.2375	-3.02	2.26
11.4625	124.65	120.68	11.4625	-3.44	-2.74
11.725	124.05	118.15	11.76	-3.55	-2.27
12.25	122.87	114.38	12.25	-3.71	-1.75
13.0375	121.16	111.01	12.95	-3.95	-3.62
13.3	120.60	111.21	13.2475	-4.06	-3.55
13.475	120.23	110.69	13.475	-4.13	-3.91
14.2625	118.61	114.68	14.2625	-4.40	-5.36
14.875	117.38	120.36	14.7875	-4.58	-4.02

shell T-matrix are depicted in Figs. 11 and 12 which are in reasonable agreement with those of R. J. Spiger and T. A. Tombrello [48].

The off-shell Jost function $f_\ell(\xi, q)$ in Eq. (49) is in the compact form and we calculate it for various laboratory energies for both the systems along with the inelastic scattering/quasi phases. It is well known that the phase of the on-shell Jost function $f_\ell(\xi)$ is the negative of the scattering phase shift $\delta_\ell(\xi)$. Further, we define a quantity, termed as quasi phase/inelastic scattering phase [49] $\Delta_\ell(\xi, q) = -\tan^{-1}$ (Im$f_\ell(\xi, q)$/Re$f_\ell(\xi, q)$). In Tables II-XIII we present the results of $f_\ell(\xi, q)$ and $\Delta_\ell(\xi, q)$ as a function of q and verify that $\Delta_\ell(\xi, q)$ produces the correct scattering phase shift $\delta_\ell(\xi)$ at $q = \xi$ at the respective laboratory energies.

The variable phase approach (VPA) to potential scattering is a direct numerical method for computing phase parameters without solving the wave equation [50]. It is regarded as an

TABLE XIX. Scattering phase shifts for $5/2^+$ state of $\alpha-^3$He system.

E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]	E_{Lab}(MeV)	Phase shift $5/2^+$ state (degree) (present work)	Phase shift $5/2^+$ state (degree) Ref. [48]
4.6375	-2.77	-3.78	8.1725	-4.77	-0.81
4.725	-2.82	-3.48	8.435	-4.92	-0.19
4.8475	-2.89	-3.89	8.6975	-5.06	-7.66
4.9	-2.92	-2.46	8.995	-5.23	-5.59
4.987	-2.97	-4.34	9.2225	-5.35	-3.5
5.215	-3.09	-3.70	9.45	-5.48	-2.53
5.25	-3.11	-2.99	9.7125	-5.62	-2.29
5.3725	-3.18	0.61	9.975	-5.77	-4.61
5.6875	-3.36	-3.95	10.185	-5.88	-4.74
5.915	-3.49	-4.43	10.43	-6.02	-5.97
6.16	-3.63	-5.29	10.71	-6.17	-4.99
6.44	-3.79	-5.42	10.99	-6.32	-5.12
6.685	-3.93	-5.92	11.235	-6.45	-3.41
6.9125	-4.06	-4.20	11.4625	-6.58	-3.89
7.175	-4.21	-4.70	11.6725	-6.69	-4.38
7.4375	-4.36	-4.09	12.215	-6.98	-2.43
7.7	-4.50	-3.49	13.475	-7.64	-4.58
7.9625	-4.65	-3.24	14.49	-8.16	-4.09

efficient tool for quantum mechanical problems. We shall also exploit the VPA to calculate the same parameters for the potential under consideration. Our phase parameters, computed via the VPA, are also presented in Tables XIV-XIX along with those of Ref. [48]. The phase equation [50] reads as

$$\delta_\ell^{'}(\xi, s) = -\xi^{-1}V_{eff}(s)\Big[\hat{j}_\ell(\xi s)\cos\delta_\ell(\xi s) - \hat{\eta}_\ell(\xi s)\sin\delta_\ell(\xi s)\Big]^2, \tag{54}$$

where $\hat{j}_\ell(\xi s)$ and $\hat{\eta}_\ell(\xi s)$ are the Riccati-Bessel functions. The reasonable agreement of our numerical values for phase shifts, obtained from half-shell T-matrix and VPA, with the standard ones [48] conclusively established that the constructed analytical expressions for off-shell Jost solution and function are correct.

4. Conclusions

The computation of the binding energy per particle in nuclear matter and the determination of the shell-model spectrum are carried out in terms of the transition matrices. Also the theoretical investigation of the (p-p) Bremsstrahlung is closely related to the study of the half-off-shell nucleon-nucleon T-matrix. Therefore, the expression for the T-matrix facilitates us to make best possible use of the available information about the two-nucleon wave function in coordinate space. The present text deals with three-parameter central nuclear potential instead of several parameter interactions with the inclusion of spin-orbit and tensor interactions. With this simple potential model our phase parameters agree quite well with the earlier works [48,51,52] except for the $1/2^+$ state of $\alpha-^3$He system at very low energies. This may be due to improper accountability of the electromagnetic interaction in this energy range. The behaviours of the half-shell transition matrices computed with the potential in Eq. (4) and those of Laha and Talukdar [45], J. Haidenbauer and W. Plessas [46], Sahoo *et al.* [47,53], Behera *et al.* [54] and Khirali *et al.* [55] indicate that low-energy part of the two-nucleon potentials appears to be nearly the same, indicating that one may have a common the low-momentum nucleon-nucleon potential. From the foregoing discussion it is noticed that our conjecture works quite satisfactorily with respect to on- and off-shell behaviour of the nucleon-nucleon potential under consideration. The present method can be applicable to the case of an arbitrary exponential type of nuclear local potential. The charged hadronic systems are generally represented by the screened/cut off Coulomb interaction as pure Coulomb potential has no existence in reality. The overall quality of the consistency between the theory and experiment, in the low energy region, is noteworthy. Therefore, the present approach may turn out to be interesting to theoretical and experimental physicists.

Funding

There is no funding source for this work.

Conflict of interest

The authors declare that they have no conflict of interest in this work.

1. J. Bhoi and U. Laha, Supersymmetry-inspired low-energy alpha-proton elastic scattering phases, *Theor. Math. Phys*. **190** (2017) 69, https://doi.org/10.1134/s0040577917010056.
2. J. Bhoi and U. Laha, Hulthén potential models for alpha-alpha and alpha-He 3 elastic scattering, *Pramana-J. Phys*. **88** (2017) 42, https://doi.org/10.1007/s12043-016-1352-1.
3. Y. Yamaguchi, Two-Nucleon Problem When the Potential Is Nonlocal but Separable. *I, Phys. Rev*. **95** (1954) 1628, https://doi.org/10.1103/physrev.95.1628.
4. D. K. Ghosh, *et al*., Laplace transform method for off-shell scattering on nonlocal potentials, *Czech. J. Phys*. **33** (1983) 528, https://doi.org/10.1007/bf01589791.
5. H. Van Haeringen, Charged Particle Interactions. Theory and Formulas (Coulomb Press Leyden, Leiden (Netherlands), 1985).
6. U. Laha and J. Bhoi, Hadron-Hadron Scattering Within the Separable Model of Interactions (OmniScriptum Publishing Group, Beau Bassin, 2018).
7. H. van Haeringen and R. van Wageningen, Analytic T matrices for Coulomb plus rational separable potentials, *Journal of Mathematical Physics* **16** (1975) 1441, https://doi.org/10.1063/1.522691.
8. W. Schweiger *et al*., Separable representation of the nuclear proton-proton interaction, *Phys. Rev. C* **27** (1983) 515, https://doi.org/10.1103/PhysRevC.27.515.
9. J. Haidenbauer and W. Plessas, Separable approximations of two-body interactions, *Phys. Rev. C* **27** (1983) 63, https://doi.org/10.1103/PhysRevC.27.63.
10. B. Talukdar, U. Laha, and T. Sasakawa, Green's function for motion in Coulomb-modified separable nonlocal potentials, *Journal of Mathematical Physics* **27** (1986) 2080, https://doi.org/10.1063/1.527028.
11. P. Sarkar *et al*., Exact solution of two-potential system under same range approximation and its implication, *Int. J. Mod. Phys. E* **30** (2021) 2150066, https://doi.org/10.1142/s021830132150066x.
12. W. Glöckle *et al*., A New Treatment of 2N and 3N Bound States in Three Dimensions, *Few-Body Syst.* **47** (2009) 25, https://doi.org/10.1007/s00601-009-0064-1.
13. T. Takemiya, Off-the-Energy-Shell TMatrix. I, *Progress Theor. Phys.* **48** (1972) 1547,https://doi.org/10.1143/ptp.48.1547.
14. H. J. Korsch and R. Mohlenkamp, Off-shell T matrix in the semiclassical limit, *J. Phys. B: At. Mol. Phys*. **15** (1982) 2187, https://doi.org/10.1088/0022-3700/15/14/012.
15. H. S. Picker, E. F. Redish, and G. J. Stephenson, Two-Nucleon T Matrix Half Off the Energy Shell: A Direct Approach, *Phys. Rev. C* **4** (1971) 287, https://doi.org/10.1103/physrevc.4.287.
16. P. Sauer, Parametrization of the 1S0 two nucleon transition matrix, *Ann. of Phys.* **80** (1973) 242, https://doi.org/10.1016/0003-4916(73)90106-1.
17. U. Laha and J. Bhoi, An Integral Transform of Coulomb Green's Function via Sturmian Representation and Off-Shell Scattering, *Few-Body Syst*. **54** (2013) 1973, https://doi.org/10.1007/s00601-013-0726-x.
18. U. Laha and J. Bhoi, Off-shell Jost solutions for Coulomb and Coulomb-like interactions in all partial waves, *J. Math. Phys.* **54** (2013) 013514, https://doi.org/10.1063/1.4776659.
19. U. Laha and J. Bhoi, Integral transform of the Coulomb Green's function by the Hankel function and off-shell scattering, *Phys. Rev. C* **88** (2013) 064001, https://doi.org/10.1103/physrevc.88.064001.
20. P. Sahoo and U. Laha, Nucleon-nucleus inelastic scattering by Manning-Rosen distorted nonlocal potential, *Canadian Journal of Physics* **100** (2022) 68, https://doi.org/10.1139/cjp-2021-0184.
21. P. Sahoo and U. Laha, Phase shift and cross section analysis of nucleon-nucleon and nucleon-nucleus scattering using second Pöschl-Teller potential, *Canadian Journal of Physics*, **101** (2023) 9, https://doi.org/10.1139/cjp-2022-0317.
22. R. Jost, Über die falschen Nullstellen der Eigenwerte der SMatrix, *Helvetica Physica Acta* **20** (1947) 256, https://doi.org/10.5169/seals-111803.
23. R. G. Newton, Scattering theory of waves and particles., 2nd ed. (Springer Verlag, New York, NY, 1982).
24. M. G. Fuda and J. S. Whiting, Generalization of the Jost Function and Its Application to Off-Shell Scattering, *Physical Review C* **8** (1973) 1255, https://doi.org/10.1103/physrevc.8.1255.
25. U. Laha and B. Kundu, Off-shell Jost solution for scattering by a Coulomb field, *Phys. Rev. A* **71** (2005) 032721, https://doi.org/10.1103/PhysRevA.71.032721.

26. M. F. Manning and N. Rosen, Exact Solutions of the Schrödinger Equation, *Phys. Rev.* **44** (1933) 951.

27. S.-H. Dong and J. García-Ravelo, Exact solutions of thes-wave Schrödinger equation with Manning-Rosen potential, *Physica scripta* **75** (2007) 307, https://doi.org/10.1088/0031-8949/75/3/013.

28. W.-C. Qiang, K. Li, and W.-L. Chen, New bound and scattering state solutions of the Manning-Rosen potential with the centrifugal term, *J. Phys. A: Math. Theor*. **42** (2009) 205306, https://doi.org/10.1088/1751-8113/42/20/205306.

29. B. Khirali *et al*., Regular and Jost states for the S-wave Manning- Rosen potential, *J. Phys. G: Nucl. Part. Phys.* **46** (2019) 115104, https://doi.org/10.1088/1361-6471/ab4118.

30. B. Khirali *et al*., Scattering with Manning-Rosen potential in all partial waves, *Ann. Phys*. **412** (2020) 168044, https://doi.org/10.1016/j.aop.2019.168044.

31. B. Khirali *et al*., Off-shell T-matrix for the Manning-Rosen potential, *Pramana- J. Phys.* **95** (2021) 179, https://doi.org/10.1007/s12043-021-02206-w.

32. L. Hulthén, On the characteristic solutions of the Schrödinger deuteron equation, *Arkiv for Matematik, Astronomi och Fysik* **28** (1942) 1.

33. L. J. Slater, Generalized hypergeometric functions (Cambridge Univ. Press, London, 1966).

34. A. Erdelyi, Higher Transcendental Functions (McGraw-Hill, New York, NY, 1953), p. 59.

35. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics (Chelsea Publishing Company, 1949).

36. A. W. Babister, Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations (MacMillan, New York, NY, 1967).

37. R. Jost, Uber diefalschen Nullstellen der Elgen werteder Smatrix, *Helv. Phys. Acta* **20** (1947) 256

38. U. Laha and J. Bhoi, On- and off-shell Jost functions and their integral representations, *Pramana- J. Phys.* **86** (2015) 947, https://doi.org/10.1007/s12043-015-1130-5.

39. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, London, 2000).

40. O. P. Bahethi and M. G. Fuda, The T Matrix for the Hulthén Potential, *J. Math. Phys.* **12** (1971) 2076, https://doi.org/10.1063/1.1665503.

41. P. Sahoo, U. Laha, and B. Khirali, Hulthén off-shell transition matrix for nuclear systems, *Chinese Journal of Physics* **73** (2021) 561, https://doi.org/10.1016/j.cjph.2021.07.008.

42. U. Laha, On the integral representations of the Jost function and Coulomb off-shell Jost solution, *Pramana J. Phys*. **67** (2006) 357, https://doi.org/10.1007/s12043-006-0080-3.

43. P. Sahoo *et al*., Half-Shell T-Matrix for Alpha-p and Alpha-12C Scattering, *Braz. J. Phys*. **51** (2021) 1478, https://doi.org/10.1007/s13538-021-00903-w.

44. W. N. Bailey, Generalized Hypergeometric Series (Cambridge University Press, London, 1935).

45. U. Laha and B. Talukdar, Half-shell T matrix for Coulombmodified Graz separable potential, *Pramana-J. Phys*. **36** (1991) 289, https://doi.org/10.1007/bf02846549.

46. J. Haidenbauer and W. Plessas, Separable representation of the Paris nucleon-nucleon potential, *Phys. Rev. C* **30** (1984) 1822, https://doi.org/10.1103/physrevc.30.1822.

47. P. Sahoo and U. Laha, Nucleon-nucleus inelastic scattering by Manning-Rosen distorted nonlocal potential, *Canadian Journal of Physics* **100** (2022) 68, https://doi.org/10.1139/cjp-2021-0184.

48. R. J. Spiger and T. A. Tombrello, Scattering of He3 by He4 and of He4 by Tritium, *Phys. Rev.* **163** (1967) 964, https://doi.org/10.1103/physrev.163.964.

49. J. Bhoi and U. Laha, Integral transforms and their applications to scattering theory, *Int. J. Appl. Phys. Math*. **4** (2014) 386, https://doi.org/10.17706/ijapm.2014.4.6.386-405.

50. F. Calogero, Variable Phase Approach to Potential Scattering (Academic Press, New York, 1967).

51. A. K. Behera *et al*., Fredholm determinants for the Hulthéndistorted separable potential, *Pramana- J. Phys.* **95** (2021) 1, https://doi.org/10.1007/s12043-021-02119-8.

52. J. Bhoi, R. Upadhyay, and U. Laha, Parameterization of Nuclear Hulthén Potential for Nucleus-Nucleus Elastic Scattering, *Commun. Theor. Phys.* **69** (2018) 203, https://doi.org/10.1088/0253-6102/69/2/203.

53. P. Sahoo and U. Laha, Treatment of hadronic systems involving two potentials under a new approximation scheme, *Pramana-J. Phys.* **96** (2022) 15, https://doi.org/10.1007/s12043-021-02267-x.

54. A. K. Behera *et al*., Hulthén Half-off-Shell T Matrix- Application to n-p and n-d Systems, *J. Korean Phys. Soc.* **76** (2020) 782, https://doi.org/10.3938/jkps.76.782.

55. B. Khirali *et al*., On- and off-shell Jost functions for the Manning-Rosen potential, *Physica Scripta* **95** (2020) 075308, https://doi.org/10.1088/1402-4896/ab95ae.

56. W. Magnus and F. Oberhettinger, For meln und Sätze für die speziellen Funktionen der mathematischen Physik (W. Braunbek), *Zeitschrift Naturforschung Teil A* **4** (1949) 319.

57. M. G. Fuda and J. S. Whiting, Generalization of the Jost Function and Its Application to Off-Shell Scattering, *Phys. Rev. C* **8** (1973) 1255, https://doi.org/10.1103/physrevc.8.1255.

58. U. Laha and J. Bhoi, *On- and off-shell Jost functions and their integral representations* **86** (2015) 947, https://doi.org/10.1007/s12043-015-1130-5.

59. G. Satchler *et al*., An optical model for the scattering of nucleons from 4He at energies below 20 MeV, *Nucl. Phys. A* **112** (1968) 1, https://doi.org/10.1016/0375-9474(68)90216-9.

60. A. K. Behera *et al*., Study of nucleon-nucleon and alphanucleon elastic scattering by the Manning-Rosen potential, *Communications in Theoretical Physics* **72** (2020) 075301, https://doi.org/10.1088/1572-9494/ab8a1a.