ON THE POSSIBILITY OF EXPERIMENTAL OBSERVATION OF DIFFRACTION IN TIME EFFECTS

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RESUMEN

Moshinsky's results[1] on "diffraction in time" are extended to the case of a rectangular pulse. Besides interference effects of the type studied by Moshinsky there appears also the well known quantum mechanical spreading effect. Rise time and monochromaticity requirements seem to preclude the possibility of experimental observation of the interference effects, so that they need not be taken into account in neutron velocity selectors. The detection of the spreading, wich is subject to weaker requirements, would provide a direct test of the uncertainty relation.

Let a monochromatic, non-relativistic beam of particles (mass m, wave number k) be confined to the half space $x < 0$ by means of a perfectly absorbing shutter. If, at the instant $t = 0$, the shutter is removed, the wave function $\Psi(x,t)$ for $t > 0$ will be a solution of Schrodinger's equation satisfying the initial conditions

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$$\Psi(x,0) = \begin{cases} \exp(ikx) & \text{if } x<0 \\ 0 & \text{if } x>0 \end{cases} \qquad (1)$$

The solution of this problem has been given by Moshinsky [1]:

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \exp(-i\,\pi/4)\ \Psi_o(x,t)\ [\,F(\infty) - F(u)\,] \qquad (2)$$

where

$$\Psi_o(x,t) = \exp i\,[\,kx - (\hbar k^2/2m)\,t\,] \qquad (3)$$

$$F(u) = \int_o^u \exp\left(i\frac{\pi}{2}\eta^2\right) d\eta$$

$$u = (2m/ht)^{1/2}\ (x - vt)\ ;\ v = \hbar k/m \qquad . \qquad (5)$$

The transient current $J(x,t)$ is the sum of two terms, one of which may be neglected if $x \gg \lambda = 2\pi/k$, giving [1]:

$$J(x,t)/J_o = \frac{1}{2}\ |\ F(\infty) - F(u)\ |^2 \quad , \qquad (6)$$

where $J_o = v$ is the incident current. Expression (6) is identical to the Fresnel diffraction pattern of a straight edge. This is the reason for the name of " di-ffraction in time". Fig. 1 shows a plot of J/J_o as a function of time, for given x. The unit step function starting at $T = x/v$ represents the effect according to classical mechanics ("geometrical optics" limit). As a measure of the "width" of the diffraction effect, we may take the difference $\tau = t_2 - t_1$ between the first two times at which J/J_o passes by its classical value (see Fig. 1). This is given by Moshinsky [1] as:

$$\tau \cong 0.85\,(\,x\hbar/mv^3)^{1/2} \qquad (7)$$

For 300 ° K neutrons at a distance $x = 1$ m, $\tau \cong 2.7 \times 10^{-9}$ sec. Since times of this order of magnitude may be measured, this raises the following questions:

1) Is the "diffraction in time" effect observable with present-day techniques?

2) If so, need it be taken into account in the operation of neutron velocity selectors ?

There are several effects, not examined in the preceding analysis, which ought to be considered before attempting to give an answer to these questions. The following seem to be among the most important ones:

a) Shape of the particle pulse.

b) Non-monochromaticity of the incident beam.

a) SHAPE OF THE PARTICLE PULSE

Moshinsky's results apply to a semi-infinite wave train. In practice, however, we always have to deal with wave packets of finite extension. Let us consider the case in which the initial shape of the packet is a rectangular pulse of length $l = v \Delta t$ (such as migth be produced by an infinitely fast shutter kept open during the time Δt). The initial conditions (1) are accordingly to be replaced by

$$\Psi(x,0) = \begin{cases} \exp(ikx) & \text{if } -l < x < 0 \\ 0 & \text{if } x < -l \text{ or } x > 0 \end{cases} . \quad (8)$$

This may be rewritten as

$$\Psi(x,0) = \Psi_1(x,0) + \Psi_2(x,0) \quad (9)$$

$$\Psi_1(x,0) = \begin{cases} \exp(ikx) & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases} ; \quad \Psi_2(x,0) = \begin{cases} -\exp(ikx) & \text{if } x < -l \\ 0 & \text{if } x > -l \end{cases} . \quad (10)$$

The wave functions $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are easily found with the help of Eqs. (1) to (5). The result is

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \exp(-i\,\pi/4)\;\Psi_0(x,t)\;[\,F(u_2) - F(u_1)\,] , \quad (11)$$

where

$$u_1 = (2m/ht)^{\frac{1}{2}} \ (x - vt) \quad ; \quad u_2 = (2m/ht)^{\frac{1}{2}} \ (x + l - vt) \qquad (12)$$

and the other symbols retain the meaning given in Eqs. (1) to (5). The transient current is again a sum of two terms, one of which may be neglected if $x \gg \lambda$,

$$[\ J(x,t)/v\ ] = \frac{1}{2} \ | \ F(u_2) - F(u_1) \ |^2 \ . \qquad (13)$$

As might have been expected, this is identical to the Fresnel diffraction pattern of a slit, of numerical width

$$\Delta u = u_2 - u_1 = (2m/ht)^{\frac{1}{2}} \ l \qquad (14)$$

In Optics, the numerical width is a measure of the number of "Fresnel zones" contained in the slit width. Its value determines the character of the diffraction pattern.

For $l \to \infty$ the above formulae reproduce Moshinsky's results. Figs. 2(a) to 2(f) illustrate the evolution of the diffraction pattern as a function of Δu. Since Δu decreases continuously as the time increases, they may also be thought of as depicting successive stages in the propagation of a rectangular wave packet.

Fig. 2(a) represent the classical ("geometrical optics") results, a replica of the initial pulse, which starts at $T_1 = x/v$ and ends at $T_2 = T_1 + \Delta t$. For $\Delta u \gg 1$ (Fig. 2(b)), we have essentially a superposition of two independent "straight edge" patterns, i.e., Moshinsky's effect (Fig. 1) is simply repeated at the beginning and at the end of the pulse. For $1 < \Delta u < 10$ (Figs. 2(c) 2(d)), the pattern shows large oscillations, which are due to interference between the two edges. These oscillations are perceptible even in the "geometrical shadow". However, the mean current does not deviate much from the classical value. For $\Delta u = 1$ (Fig. 2(e)), the wave packet begins to spread appreciably, with a consequent amplitude reduction. These effects become more marked as Δu decreases (Fig. 2(f)).

The broadening of the wave packet for $\Delta u \lesssim 1$ is a direct consequence of the uncertainty principle. In fact, the initial width Δt of the packet corresponds to an energy spread

$$\Delta E = mv\Delta v \gtrsim \hbar/\Delta t \quad , \tag{15}$$

and this gives rise, after a time t, to a broadening

$$\Delta x \sim (\Delta v)t \gtrsim \hbar/mv\,\Delta t = l/\pi(\Delta u)^2 , \tag{16}$$

which begins to be important for $\Delta u \lesssim 1$. This effect sets a limit to the degree of monochromaticity attainable with a velocity selector.

The lower limit to the broadening given by Eq. (16) obviously does not depend on the initial shape of the packet. It is well known, for instance, that a wave packet with an initial Gaussian probability distribution preserves its Gaussian shape in free propagation, but undergoes a spread the order of magnitude[2] of which is just that given by (16). Therefore, the appearance of spreading for $\Delta u \lesssim 1$ is a shape-independent effect.

On the other hand, the interference effects which appear in Figs. 2 (b), 2(c) and 2(d) are strongly shape-dependent. The oscillations in the pattern are closely connected with the existence of "straight edges" in the initial pulse i.e., with the assumption that its rise time and decay time are negligible. A "rounding" of the edges will smooth out the oscillations. This is clearly seen in the above cited example of a Gaussian wave packet.

To formulate these notions more precisely, let us call τ_R the rise time of the initial pulse. Then, in order to observe Moshinsky's diffraction width, we must have: $\tau_R \ll \tau$. Similarly, if we want to observe interference effects in the range $1 < \Delta u < 10$, we must have: $\tau_R \ll \Delta t/\Delta u$, i. e., τ_R must be small compared with the spacing between peaks. It is easily seen that either one of these conditions inplies that

$$\tau_R \ll \hbar^{\frac{1}{2}} m^{\frac{1}{4}} x^{\frac{1}{2}} E^{-\frac{3}{4}} \quad , \tag{17}$$

where E is the kinetic energy of the incident particles.

For the previously given example of 300° K neutrons at a distance $x = 1$ m, we must have, according to (17), $\tau_R < 2.7 \times 10^{-9}$ sec, which is impossible with a mechanical selector. Forming the neutron pulses by means of charged particles would be a change for the worse, since E would increase by a large factor. Results would be improved by decreasing the energy. "Cold" neutrons migth be obtained by several methods (neutron diffraction, use of low temperature thermalizing substance, etc.). Admitting that a 20° K neutron beam of sufficiently high intensity may be produced, this would mean a gain by a factor of $(15)^{\frac{3}{4}} \cong 7.6$. The distance x might be increased perhaps by factor of 20, giving an additional gain of $(20)^{\frac{1}{2}} \cong 4.5$. This would bring us to $\tau_R \ll 10^{-7}$ sec. It is still very doubtful whether such values of the rise time can be attained with a mechanical selector.

Another possibility would be to employ heavy ions in place of the neutrons. For 0.1 ev Rb ions at a distance $x = 1$ m, we find: $\tau_R \ll 3 \times 10^{-9}$ sec. Since the ion pulse can be formed by electronic devices, it might be possible to satisfy this condition.

b) NON-MONOCHROMATICITY OF THE INCIDENT BEAM

The incident beam will be, in general, a superposition of non-coherent components having a certain velocity distribution, e. g., a maxwellian distribution in the case of thermal neutron. The resulting "diffraction pattern" is then obtained by superposing the patterns given by (13) for each value (or small range of values) of the velocity. How does this affect the results previously obtained ?

Let us consider what comes out of superposing two patterns, corresponding to velocities v and $v + \delta v$. For $\delta v/v \ll 1$, the two patterns will have practically the same shape, and all we need consider is their relative shift in the time scale.

This shift is given by

$$\delta t = T(v) - T(v + \delta v) = x/v - x/(v + \delta v) \cong T(v)\, \delta v/v \quad , \quad (18)$$

If we do not want the oscillations to be "smoothed out" by overlapping, we must have $\delta t \ll \tau$ (for Moshinsky's effect) or $\delta t \ll \Delta t/\Delta u$ (for $1 < \Delta u < 10$). From this it follows, as in (17), that we must satisfy the condition

$$\delta v/v \cong \delta t/T(v) \ll \hbar^{\frac{1}{2}}\, m^{-\frac{1}{4}}\, x^{-\frac{1}{2}}\, E^{-\frac{1}{4}} \qquad (19)$$

For 300° K neutrons, $x = 1$ m, (19) gives: $\delta v/v \ll 6 \times 10^{-6}$. This would demand a degree of monochromaticity wich it seems wery difficult to attain. It would hardly be of any help to use "cold" neutrons, owing to the small exponent with which E appears in (19); even with 20° K neutrons, we should gain less than a factor of two. If, on the other hand, we try to increase $x^{-\frac{1}{2}}$ by a certain factor, (17) decreases by the same factor, which cannot be allowed, since τ_R for neutrons is already too small, as it is. For 0.1 ev Rb ions, $x = 1$ m, we find $\delta v/v \ll 1.4 \times 10^{-6}$, which is even worse. Thus we see that we cannot simultaneously satisfy requirements on rise time and monochromaticity expressed by conditions (17) and (19). Therefore, both Moshinsky's effect and the interference effect in the region $1 < \Delta u < 10$ are outside the range of present experimental possibilities.

What about the spreading effect? Let us find out, in the first place, whether the region $\Delta u < 1$ can be attained. Eq. (14) may be rewritten as follows:

$$\Delta u \cong 0.95\, \hbar^{-\frac{1}{2}} m^{-\frac{1}{4}} x^{-\frac{1}{2}} E^{\frac{3}{4}} \Delta t \quad . \qquad (20)$$

Taking the case of a very good mechanical selector[3], in which $x = 20$ m, $\Delta t = 5 \times 10^{-7}$ sec, we find: $\Delta u \cong 24$. Therefore, usual neutron velocity selectors operate in the region $\Delta u > 1$. To attain $\Delta u = 1$, for $x = 20$ m, with 20° K neutrons, we should have to make $\Delta t \cong 1.5 \times 10^{-7}$ sec. This would by a difficult problem, but it is not too far from what has already been done. On the other hand, for 300° K Rb ions, $x = 1$ m, we need $\Delta t \cong 1.4 \times 10^{-8}$ sec to get $\Delta u = 1$. The time of flight for this arrangement would be $T \cong 4.1 \times 10^{-3}$ sec, so that this delay would have to be measured with an accuracy of the order of 10^{-6}. These are rather stringent requirements, but they seem to be within the possibilities of modern electronic techniques. Of course, the above figures are only meant to be illustrative, and a better choice of parameters can probably be made.

Taking for granted that $\Delta u \leqslant 1$ may be attained, how will the problem be affected by the above effects? As we have seen, the initial shape is not critical. What about the non-monochromaticity of the incident beam? According to Eq. (16), the non-monochromaticity which originates from the experiment, due to the uncertainty principle, is

$$\Delta v/v \gtrsim \hbar/(E\Delta t) \sim \Delta t/[\, T(v)\,(\Delta u)^2\,] \quad . \qquad (21)$$

If we want to distinguish the spread due to the uncertainty principle from that due to the original non-monochromaticity of the incident beam, the latter has to be at most of the same order as the former, i. e., we must have: $\delta v/v \lesssim \Delta v/v$. In the case of our last example, we find from Eq. (21) that $\Delta v/v \gtrsim 3.4 \times 10^{-6}$. Eq. (21) also shows that $\Delta v/v$ increasses by decreasing either E or Δt. Thus, the detection of the spreading due to the uncertainty principle does not depend on such stringent requirements as the detection of the interference effects, but it still demands a highly monochromatic incident beam.

In conclusion, we see that: 1) Moshinsky's effect and similar interference effects for $1 < \Delta u < 10$ do not seem to be accessible to present techniques. Obviously, then, they need not be considered in the operation of neutron velocity electors. 2) To detect the spread of a wave packet due to the uncertainty principle with modern experimental techniques seems to be a difficult, but not perhaps an insurmountable problem. An experiment of this type would be very interesting, both as an observation of a transient solution of Schrödinger's equation, and as a direct test of Heisenberg's uncertainty relation.

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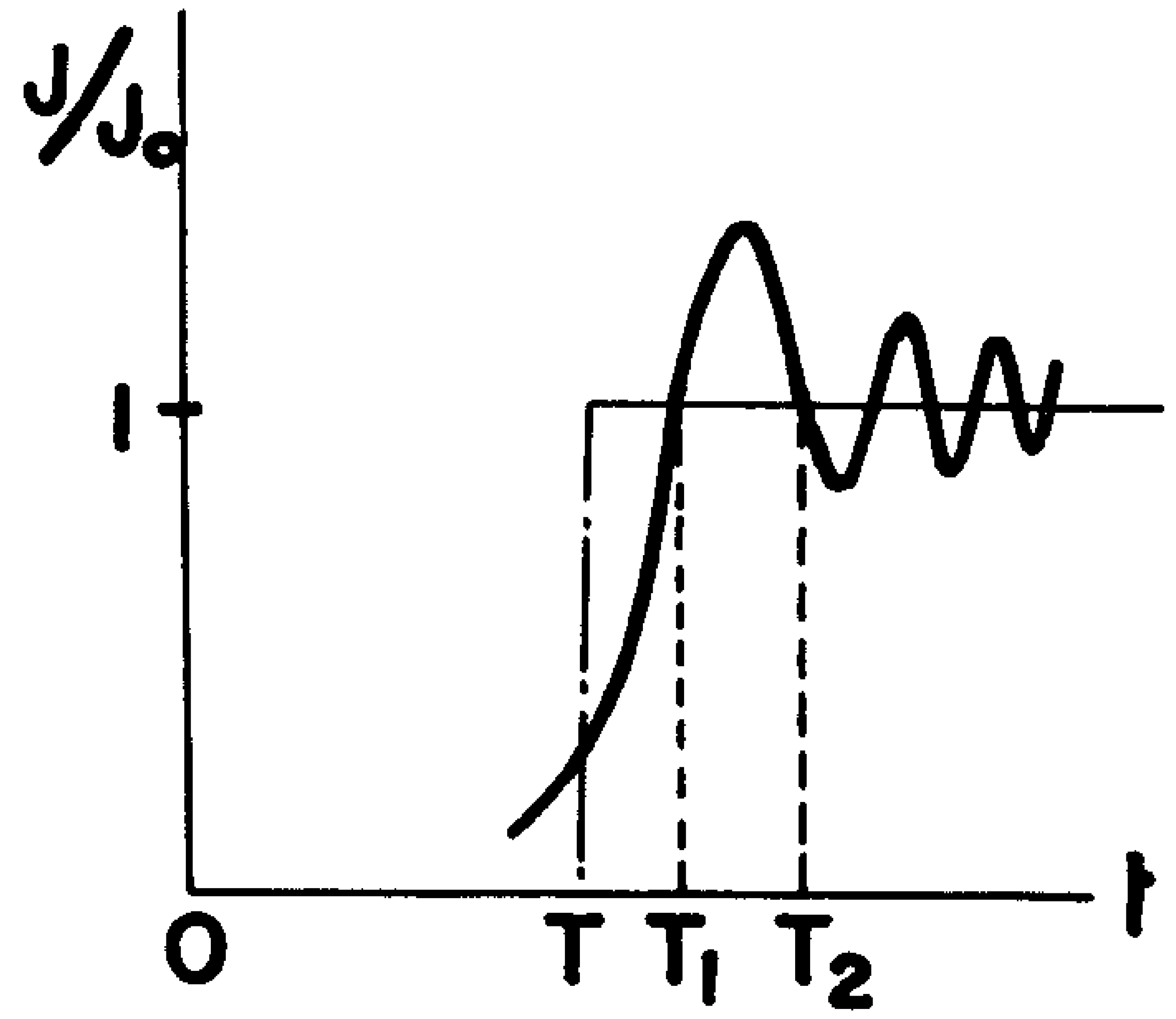

FIG. 1

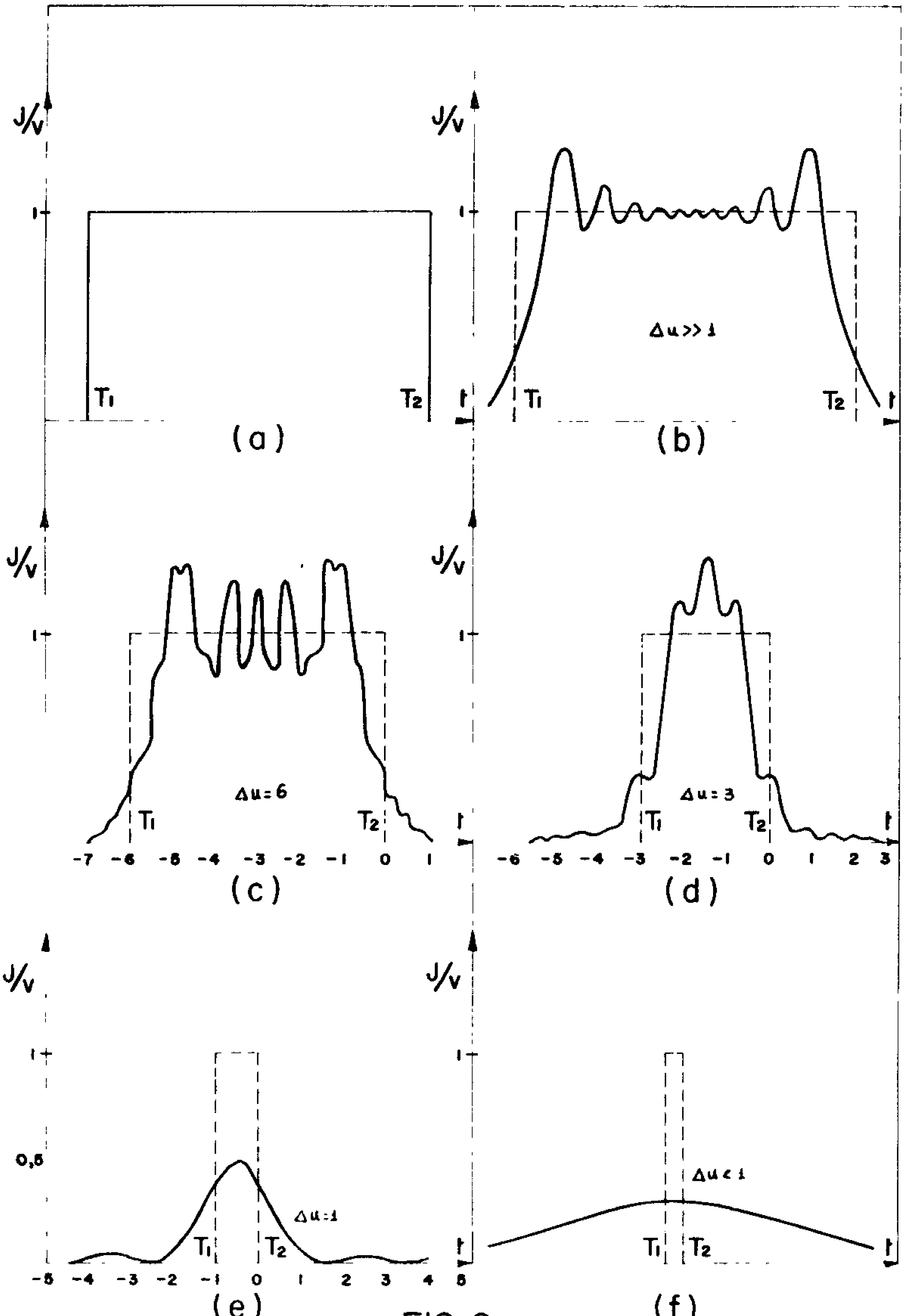
J/V
1
T1
T2
t
(a)
(b)
Δu>>1
(c)
Δu=6
-7 -6 -5 -4 -3 -2 -1 0 1
(d)
Δu=3
-6 -5 -4 -3 -2 -1 0 1 2 3
(e)
0,5
Δu=1
-5 -4 -3 -2 -1 0 1 2 3 4 5
(f)
Δu<1

FIG. 2