On the Euler collinear motion of three bodies interacting with the Newton gravitational force

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We present a set of mathematical properties that are very simple notwithstanding these properties are hidden in the literature, considering the Euler case of collinear motion of three bodies, including simple mathematical properties enabling the non-specialists to become very familiar with this classical pearl of the mechanics of the three body problem.

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1. Introduction

Since Isaac Newton formulation of the two-body problem, compatible with the three Kepler's laws for the motion of planets, it has been an interest in classical mechanics for performing a similar task in the case of three bodies. For over 250 years, there have been published hundreds of papers concerning this problem. In this paper, we highlight Euler's contribution to discover the collinear solution [1], for three different relative positions, according to the permutation order on the line. This work was extended by Lagrange [2], who added the other equilateral triangular configuration to the three-body problem.

The aim of our paper is to present a simple new set of mathematical properties hidden in the literature, so that even non-experts can become familiar with these Eulerian solutions.

The Euler's collinear solution was the first example of a central configuration. The Lagrange's equilateral triangle was the second known example of a central configuration.

A *central configuration* is a special configuration of the N-body problem, in which the position and acceleration vectors of each body with respect to the center of mass are proportional to the same parameter of proportionality, see [3]. One of the most important properties of the central configurations is related to the fact that they generate explicit solutions of the N-body problem, called the homographic solutions, where the configuration of the bodies is similar to the same central configuration for all time. It is well known that the central configurations are invariant under rotations, translations, reflections and dilatations.

On the other hand, a central configuration is said to be a *Dziobek's central configuration,* if it is a configuration of N bodies in a $(N-2)$-dimensional space. In Ref. [4], Dziobek formulated the central configuration problem for $N = 4$ in terms of mutual distances r_{ij}, obtaining algebraic equations that characterize such central configurations. For more details on this subject, the reader is addressed to [5] and references therein.

2. The Euler's three-body collinear solution

The collinear case of motion of three bodies interacting with Newton force, discovered by Euler, has the strong hypothesis that the three bodies remain collinear. In the frame of the center of mass, therefore, the three bodies have positions $\mathbf{r}_i$ along a time-dependent unit vector $\mathbf{u}$

$$\mathbf{r}_i = Z_i \mathbf{u}. \tag{1}$$

The velocity become

$$\dot{\mathbf{r}}_i = \dot{Z}_i \mathbf{u} + Z_i \dot{\mathbf{u}}, \tag{2}$$

where a dot on a variable denotes the time derivative.

The total angular momentum vector is therefore

$$\mathbf{J} = \sum_{i=1}^{3} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \left(\sum_{i=1}^{3} m_i Z_i^2 \right) \mathbf{u} \times \dot{\mathbf{u}}. \tag{3}$$

This vector is a constant of motion. Conservation of the total angular momentum is a consequence of the fact that the Newtonian force is derivable from a potential energy, which is a function of the distances between bodies. Avoiding the case in which both $\mathbf{u}$ and $\dot{\mathbf{u}}$ are parallel, this angular momentum vector is a constant vector orthogonal to both $\mathbf{u}$ and $\dot{\mathbf{u}}$ vectors, that remain in the fixed plane orthogonal to vector $\mathbf{J}$. Euler's collinear solution proves that the relative positions with respect to the center of mass Z_i are proportional to a common variable R

$$Z_i = R z_i, \tag{4}$$

where the lowercase z_i are three constants depending only on the values of the masses. They are the initial values of those relative positions to the center of mass if $R = 1$ is the initial value of coordinate R.

We choose the coordinate system with the third coordinate axis along the constant vector $\mathbf{J}$, therefore

$$\mathbf{J} = (0, 0, J). \tag{5}$$

The plane of motion of vector $\mathbf{u}$ (and $\dot{\mathbf{u}}$) is the coordinate plane orthogonal to $\mathbf{J}$, and then we give a coordinate ψ to vector $\mathbf{u}$

$$\mathbf{u} = (\cos\psi, \sin\psi, 0), \tag{6}$$

and then

$$\dot{\mathbf{u}} = \dot{\psi}(-\sin\psi, \cos\psi, 0). \tag{7}$$

The constant magnitude of the angular momentum is

$$J = \left(\sum_i m_i z_i^2\right) R^2 \dot{\psi}. \tag{8}$$

We have written the Cartesian coordinates of this three-body problem in terms of two coordinates R and ψ. We construct the Lagrangian function, equal to the kinetic energy minus the potential energy, namely

$$L(R, \dot{R}, \psi, \dot{\psi}) = \frac{1}{2}\left(\sum_i m_i z_i^2\right)\left(\dot{R}^2 + R^2\dot{\psi}^2\right) - \left(\sum_{i<j} \frac{G m_i m_j}{|z_i - z_j|}\right)\frac{1}{R}, \tag{9}$$

where G is the gravitational constant. With no loss of generality, we assume in what follows that its value is equal to one.

It follows the constant conjugate moment $J = \partial L/\partial\dot{\psi}$, to coordinate ψ that does not appear explicitly in this Lagrangian. Also, we have the constant total energy, because the Lagrangian is not an explicit function of the time.

This Lagrangian function is formally analogous to the Lagrangian function of the two-body problem that solves in polar coordinates the Kepler problem. The solutions are therefore the well known conic solutions. The three bodies move collinearly describing each one a conic. The three are homothetic and have the pole at the center of mass as center of homothecy.

3. The Euler three body collinear solution as a Dziobek central configuration

In this section we will write the equations for the collinear Euler's central configuration, relating the initial relative distances to the center of mass z_i, to the distances $r_{ij} = |z_i - z_j|$. The variational presentation of the central configurations is to use the property that a central configuration corresponds to an extreme of the potential energy subject to the condition of constant moment of inertia. The potential energy is

$$V = \sum_{i<j} \frac{m_i m_j}{r_{ij}}. \tag{10}$$

The total moment of inertia is

$$\mathcal{I} = \sum_{i<j} m_i m_j r_{ij}^2. \tag{11}$$

This prescription, assuming the independence of the three distances, gives the planar equilateral solution founded by Lagrange, deriving respect to r_{ij}^2

$$\frac{m_i m_j}{r_{ij}^3} = \nu m_i m_j \quad \rightarrow \quad r_{ij} = 1/\sqrt[3]{\nu}, \tag{12}$$

where ν is a Lagrange multiplier.

A collinear solution is obtained if we add to the previous variational formulation the prescription of zero area. According to the Henon's formula for the square of the area of the triangle, it is zero when

$$\begin{pmatrix} r_{23}^2 & r_{13}^2 & r_{12}^2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix} = 0. \tag{13}$$

Adding this constraint to the variational problem with another Lagrange multiplier, one has instead of (12)

$$\frac{m_i m_j}{r_{ij}^3} = \nu m_i m_j + \Lambda \frac{\partial}{\partial r_{ij}^2} \begin{pmatrix} r_{23}^2 & r_{13}^2 & r_{12}^2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix}, \tag{14}$$

which we write in matrix notation as

$$\begin{pmatrix} 1/r_{23}^3 \\ 1/r_{13}^3 \\ 1/r_{12}^3 \end{pmatrix} = \nu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix}, \tag{15}$$

where the new λ is $2\Lambda/(m_1 m_2 m_3)$.

The ν number is cancelled taking the $\times$ product of both sides of Eq. (15) with the vector factoring ν, that is,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1/r_{23}^3 \\ 1/r_{13}^3 \\ 1/r_{12}^3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix}. \tag{16}$$

Therefore both members are orthogonal to vector (15),

$$\begin{pmatrix} 1/r_{23}^3 \\ 1/r_{13}^3 \\ 1/r_{12}^3 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix} \right] = 0. \tag{17}$$

This equation was published in Ref. [6] as

$$\begin{pmatrix} 1/r_{23}^3 \\ 1/r_{13}^3 \\ 1/r_{12}^3 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} m_3 - m_2 & m_2 + m_3 & -m_2 - m_3 \\ -m_1 - m_3 & m_1 - m_3 & m_1 + m_3 \\ m_1 + m_2 & -m_1 - m_2 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix} = 0. \tag{18}$$

By interchangin the dot product and the $\times$ product in Eq. (17), we arrive at

$$\left[\begin{pmatrix} 1/r_{23}^3 \\ 1/r_{13}^3 \\ 1/r_{12}^3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix} = 0. \tag{19}$$

The three distances between bodies and the three masses obeys the simultaneous symmetric equations (13) and (17) that are invariant with respect to an interchange of two of the indices of the bodies. However, it is convenient, in order to reduce the number of variables, to assume the body 2 inside the segment connecting the bodies 1 and 3. Then the equation (13) is replaced by

$$r_{12} + r_{23} = r_{13}, \tag{20}$$

that give the equivalence

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_{23}^2 \\ r_{13}^2 \\ r_{12}^2 \end{pmatrix} = \begin{pmatrix} 2r_{12}r_{13} \\ -2r_{12}r_{23} \\ 2r_{23}r_{13} \end{pmatrix}. \tag{21}$$

A Dziobek's central configuration is a central configuration when N bodies are in a space of $N-2$ dimensions.

The case for $N = 4$ (four bodies in a fixed plane) has been considered in many publications, starting with Dziobek [4]. If we replace (21) in (15), it comes to an equation similar to a Dziobek's equation if we replace directed distances for directed areas. Something similar happens again for $N = 5$ if we replace now by directed volumes.

Substitution of the Eq. (21) in (19) and multiplication by $r_{12}^2 r_{23}^2 r_{13}^2$, it is transformed into

$$(r_{12}^3 - r_{13}^3) r_{23}^2 m_1 - (r_{23}^3 - r_{12}^3) r_{13}^2 m_2 + (r_{13}^3 - r_{23}^3) r_{12}^2 m_3 = 0, \tag{22}$$

that is generally written in terms of the ratio

$$x = \frac{r_{23}}{r_{12}}, \tag{23}$$

to have a five-order in x algebraic equation

$$[1 - (x+1)^3] x^2 m_1 + (1 - x^3)(x+1)^2 m_2 + [(x+1)^3 - x^3] m_3 = 0. \tag{24}$$

This equation is attributed to Euler and appears in many books where the three-body problem is considered, see for example [3, 8]. In what follows it will be called the Euler's quintic.

4. The Euler's quintic equation

Equation (24) is written in terms of three polynomials

$$P_1(x) = (1 - (x+1)^3) x^2, \tag{25}$$

$$P_2(x) = (1 - x^3)(x+1)^2, \tag{26}$$

$$P_3(x) = (x+1)^3 - x^3, \tag{27}$$

in the form

$$m_1P_1(x) + m_2P_2(x) + m_3P_3(x) = 0. \tag{28}$$

For a given choice of masses this Euler's quintic has a unique real root, that is always positive, [8].

The Eq. (24) will be expressed in different interesting forms related to the positions relative to the center of mass.

If z_1, z_2, z_3 are the positions of the bodies relative to the center of mass, they obey the equation

$$m_1z_1 + m_2z_2 + m_3z_3 = 0. \tag{29}$$

From this equation we obtain

$$\frac{1}{r_{12}}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{m}\begin{pmatrix} -m_2 - m_3(x+1) \\ m_1 - m_3x \\ m_1(x+1) + m_2x \end{pmatrix}, \tag{30}$$

where $m = m_1 + m_2 + m_3$.

The Euler's quintic (24) is rewritten in three different forms, by using a fourth symmetrical polynomial

$$P_4(x) = x^4 + 2x^3 + x^2 + 2x + 1, \tag{31}$$

namely

$$(m1 + m_2 + m_3)P_1(x) + (m_2 + m_3(x+1))P_4(x) = 0, \tag{32}$$

$$(m_1 + m_2 + m_3)P_2(x) + (m_3x - m_1)P_4(x) = 0, \tag{33}$$

$$(m_1 + m_2 + m_3)P_3(x) - (m_1(x+1) + m_2x)P_4(x) = 0. \tag{34}$$

These equations lead to express the relative positions (30) in terms of the rational functions

$$T_1(x) = \frac{P_1(x)}{P_4(x)}, \quad T_2(x) = \frac{P_2(x)}{P_4(x)}, \quad T_3(x) = \frac{P_3(x)}{P_4(x)}, \tag{35}$$

which are simply related

$$T_1(x) = T_2(x) - 1, \quad T_2(x) = T_3(x) - x. \tag{36}$$

One has

$$\frac{1}{r_{12}}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{m}\begin{pmatrix} -m_2 - m_3(x+1) \\ m_1 - m_3x \\ m_1(x+1) + m_2x \end{pmatrix} = \begin{pmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{pmatrix}. \tag{37}$$

The last equation is valid for $x = r_a$, the root of the Euler's quintic (24). We remark the non-trivial nature of Eq. (37). It comes, noting the middle term is equal to the extreme separated vectors. Albouy and Moeckel [7] in a general study of the inverse problem of the collinear motion of several bodies under the action of a general force, have considered the particular case we are considering here. They found this result as that they are looking for the position of the center of mass on the line, determined by the value of the root of the quintic, but independent of the actual values of the masses. In the same reference these authors take into account a demand of priority of this result from the Marchal's book of the three-body problem [8]. Actually the first entry of Eq. (37) coincides in such book as

$$\frac{z_1}{r_{12}} = \frac{P_1(r_a)}{P_4(r_a)}. \tag{38}$$

We recognize the priority of those authors, nevertheless compare the difference in presentation of the same result. In particular, our polynomial $P_4(x)$ appears in both Refs. [7, 8].

To simplify some proofs in the sequel, observe the middle vector in Eq. (37) is the $\times$ product

$$\frac{1}{m}\begin{pmatrix} -m_2 - m_3(x+1) \\ m_1 - m_3x \\ m_1(x+1) + m_2x \end{pmatrix} = \frac{1}{m}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} -x \\ x+1 \\ -1 \end{pmatrix}.$$

The Euler's quintic Eq. (24) has a property of symmetry. For an interchange of the two masses at the extreme, m_1 and m_3, the root changes from the value x to the value $1/x$. This comes from the symmetry on the line of the three bodies when it is reflected with respect to the center of mass.

By this trivial symmetry, we will consider only cyclic permutations of bodies.

Each of the three permutations of masses, gives, in general, a different quintic with a different root: r_a, r_b, r_c. We will have therefore the three cases

$$m_1P_1(r_a) + m_2P_2(r_a) + m_3P_3(r_a) = 0, \tag{39}$$

$$m_1P_3(r_b) + m_2P_1(r_b) + m_3P_2(r_b) = 0, \tag{40}$$

$$m_1P_2(r_c) + m_2P_3(r_c) + m_3P_1(r_c) = 0. \tag{41}$$

The relative positions with respect to the center of mass, corresponding to these three cases are

$$\begin{pmatrix} T_1(r_a) \\ T_2(r_a) \\ T_3(r_a) \end{pmatrix} = \frac{1}{m}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} -r_a \\ r_a+1 \\ -1 \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} T_3(r_b) \\ T_1(r_b) \\ T_2(r_b) \end{pmatrix} = \frac{1}{m}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} -1 \\ -r_b \\ r_b+1 \end{pmatrix}, \tag{43}$$

$$\begin{pmatrix} T_2(r_c) \\ T_3(r_c) \\ T_1(r_c) \end{pmatrix} = \frac{1}{m} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} r_c + 1 \\ -1 \\ -r_c \end{pmatrix}. \tag{44}$$

5. Permutation of masses and linear algebra

Mass permutation is here represented by a rotation of $\pi/3$ and its inverse

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

The three matrices in this product form a cyclic group generated by the first matrix at the left.

Vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{46}$$

is the axis of rotation of the group.

Equations (36) are represented also by the vectorial equation, valid for any x

$$\begin{pmatrix} P_1(x) \\ P_2(x) \\ P_3(x) \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -x \\ x + 1 \\ -1 \end{pmatrix} P_4(x). \tag{47}$$

Note the entries of the vector on the right are the directed distances $(-r_{23}, r_{13}, -r_{12})$ if r_{12} is the unit of distance.

Another auxiliary vector produces the interesting equation

$$\begin{pmatrix} P_1(x) \\ P_2(x) \\ P_3(x) \end{pmatrix} = \begin{pmatrix} (x+1)^2 \\ -x^2 \\ x^2(x+1)^2 \end{pmatrix} \times \begin{pmatrix} -x \\ x + 1 \\ -1 \end{pmatrix}. \tag{48}$$

Taking the $\times$ product of the axis of rotation with this equation leads to the property where appears the scalar product of the new vector with the axis of rotation

$$P_4(x) = \begin{pmatrix} (x+1)^2 \\ -x^2 \\ x^2(x+1)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{49}$$

where we used (47).

From (28) and (48) we see that the vector (48), computed at $x = r_a$ is orthogonal to the three vectors

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} (r_a+1)^2 \\ -r_a^2 \\ r_a^2(r_a+1)^2 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} -r_a \\ r_a + 1 \\ -1 \end{pmatrix}. \tag{50}$$

Therefore, they are linearly dependent

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \alpha \begin{pmatrix} (r_a+1)^2 \\ -r_a^2 \\ r_a^2(r_a+1)^2 \end{pmatrix} + \beta \begin{pmatrix} -r_a \\ r_a + 1 \\ -1 \end{pmatrix}, \tag{51}$$

where α will be determined now.

The scalar product of the axis of rotation (46) with the vector of masses is $m = m_1 + m_2 + m_3$. Then we have from (49) and (51)

$$m = \alpha P_4(r_a), \tag{52}$$

therefore

$$\frac{1}{m} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \frac{1}{P_4(r_a)} \begin{pmatrix} (r_a+1)^2 \\ -r_a^2 \\ r_a^2(r_a+1)^2 \end{pmatrix} - B \begin{pmatrix} -r_a \\ r_a + 1 \\ -1 \end{pmatrix}, \tag{53}$$

where $B = \beta/m$. This equation with a different notation was published by Álvarez and Piña in a book published in Spanish language to commemorate Poincaré and Hilbert [9].

This equation gives the masses as a function of the root r_a of the Euler's quintic and the parameter B. For any B value this expression of the masses give the same value of the root r_a, namely the same geometric configuration for any value of the masses in the open interval of the values of B that gives positive values of the masses. The interval ends when one of the masses has the zero value. Note any set of masses in (53) has the same center of mass (for any value of B).

A numerical example illustrates this result. If we choose the masses to have the values $m_1 = 5$, $m_2 = 2$, $m_3 = 34$, the root of the Euler's quintic is exactly the value 2. The same root is obtained for the masses defined by

$$\frac{1}{m} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \frac{1}{41} \begin{pmatrix} 9 \\ -4 \\ 36 \end{pmatrix} + B \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}, \tag{54}$$

for any real value of B, but for positive values of the masses, B is limited to be inside the open interval $(4/123, 9/82)$. For B equal to the extreme values of the interval, one of the masses is equal to zero.

The masses $(5, 2, 34)$ above correspond to the value $B = 2/41$. For $B = 3/41$ the masses are respectively $(3, 5, 33)$. For $B = 4/41$ the masses are respectively $(1, 8, 32)$. In these three cases the sum of the masses is 41.

A different expression for the masses in terms of the root of the Euler's quintic is

$$\frac{1}{m} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \mu_a \begin{pmatrix} T_2(r_a) \\ -T_1(r_a) \\ 0 \end{pmatrix} + \frac{1-\mu_a}{r_a + 1} \begin{pmatrix} T_3(r_a) \\ 0 \\ -T_1(r_a) \end{pmatrix}, \tag{55}$$

which give positive masses if μ_a is in the open interval $(0,1)$, and if $T_2(r_a) > 0$. Otherwise, the first vector should be replaced by

$$\begin{pmatrix} T_2(r_a) \\ -T_1(r_a) \\ 0 \end{pmatrix} \quad \rightarrow \quad \frac{1}{r_a}\begin{pmatrix} 0 \\ T_3(r_a) \\ -T_2(r_a) \end{pmatrix},$$

to have

$$\frac{1}{m}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \mu_a \frac{1}{r_a}\begin{pmatrix} 0 \\ T_3(r_a) \\ -T_2(r_a) \end{pmatrix} + \frac{1-\mu_a}{r_a+1}\begin{pmatrix} T_3(r_a) \\ 0 \\ -T_1(r_a) \end{pmatrix}.$$

The last expression for the masses is a linear combination of two vectors having a zero mass at the extrema of the interval, for μ_a equal to 0 or 1. Note the three vectors have a zero scalar product with vectors in (37). Also, we observe that the scalar product of the vectors in (55) with the axis of rotation give the value one, recovering our equations in the forms

$$\begin{aligned} T_2(r_a) - T_1(r_a) &= \frac{1}{r_a+1}[T_3(r_a) - T_1(r_a)] \\ &= \frac{1}{r_a}[T_3(r_a) - T_2(r_a)] = 1. \end{aligned}$$

Similar equations are obtained for the cases when the masses are cyclically permutated. We compute the permutation of the entries of the vector of masses using a rotation matrix, then we remember that the rotation of the $\times$ product is equal to the $\times$ product of the rotation, and we use that the axis of rotation (46) is an eigenvector, with eigenvalue equal one, of the rotation that represent the permutation; we find for example

$$\begin{aligned} \frac{1}{m}\begin{pmatrix} m_3 \\ m_1 \\ m_2 \end{pmatrix} &= \mu_b \begin{pmatrix} T_2(r_b) \\ -T_1(r_b) \\ 0 \end{pmatrix} \\ &+ \frac{1-\mu_b}{r_b+1}\begin{pmatrix} T_3(r_b) \\ 0 \\ -T_1(r_b), \end{pmatrix}, \end{aligned} \qquad (56)$$

which gives positive masses if μ_b is in the open interval $(0,1)$, and if $T_2(r_b) > 0$, otherwise a similar vector replacement should be made for the first vector

$$\begin{pmatrix} T_2(r_b) \\ -T_1(r_b) \\ 0 \end{pmatrix} \quad \rightarrow \quad \frac{1}{r_b}\begin{pmatrix} 0 \\ T_3(r_b) \\ -T_2(r_b) \end{pmatrix}.$$

For the third root

$$\begin{aligned} \frac{1}{m}\begin{pmatrix} m_2 \\ m_3 \\ m_1 \end{pmatrix} &= \mu_c \begin{pmatrix} T_2(r_c) \\ -T_1(r_c) \\ 0 \end{pmatrix} \\ &+ \frac{1-\mu_c}{r_c+1}\begin{pmatrix} T_3(r_c) \\ 0 \\ -T_1(r_c), \end{pmatrix}, \end{aligned} \qquad (57)$$

which gives positive masses if μ_c is in the open interval $(0,1)$, and if $T_2(r_c) > 0$, otherwise a similar replacement should be made for the first vector

$$\begin{pmatrix} T_2(r_c) \\ -T_1(r_c) \\ 0 \end{pmatrix} \quad \rightarrow \quad \frac{1}{r_c}\begin{pmatrix} 0 \\ T_3(r_c) \\ -T_2(r_c) \end{pmatrix}.$$

In the included figures (see Fig. 1) we ordered the masses, so the lower mass is measured in the horizontal from left to right. The middle mass is measured ascending on the right side of the triangle. The third-largest mass is measured descending on the left side of the triangle. With this order, $T_2(r_a) < 0$ and $T_2(r_b) > 0, T_2(r_c) > 0$. The segments on the figure labeled with the roots r_a and r_b cross the point representing the three masses, with one extreme on the left side, corresponding to $m_1 = 0$, and the other extreme on the horizontal side, corresponding to $m_2 = 0$. The third root r_c determines a segment connecting the left and right sides, corresponding to $m_1 = 0$ and $m_3 = 0$, respectively.

6. The masses in terms of the roots of the three quintics

We extract from equations (42-44) those including the three factions $T_2(r_a)$, $T_2(r_b)$ and $T_2(r_c)$, to form a linear system of three equations for the fractions of masses m_i/m namely

$$\begin{pmatrix} 1 & 0 & -r_a \\ -r_b & 1 & 0 \\ 0 & -r_c & 1 \end{pmatrix} \frac{1}{m}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} T_2(r_a) \\ T_2(r_b) \\ T_2(r_c) \end{pmatrix}. \qquad (58)$$

Assuming the determinant of the matrix of the system $1 - r_a r_b r_c$ to be different from zero, we solve for the masses in terms of the three roots

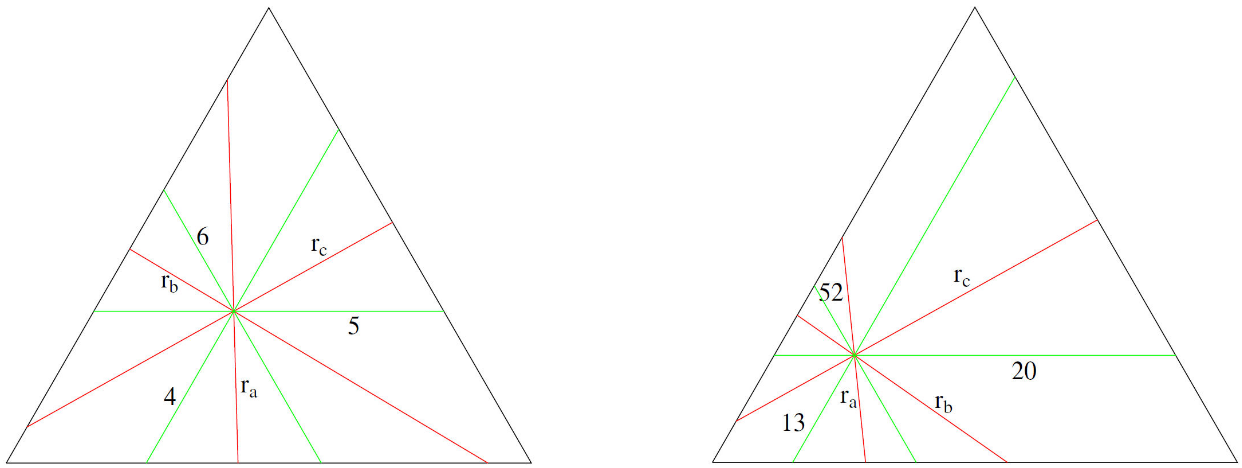

FIGURE 1. In the space of mass fractions we represent three masses and the masses corresponding to the same configuration for a given root of the Euler's quintic. The segment of mass ratios corresponding to a particular configuration has been identified by the name of the root.

$$\frac{1}{m}\begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix} = \frac{1}{1-r_ar_br_c}\begin{pmatrix} 1 & r_cr_a & r_a\\ r_b & 1 & r_ar_b\\ r_br_c & r_c & 1\end{pmatrix} \times \begin{pmatrix} T_2(r_a)\\ T_2(r_b)\\ T_2(r_c)\end{pmatrix}. \tag{59}$$

In the previous section, assuming one root we determine the fractions of masses up to an open interval. Here, if we know the three roots, the mass fractions are uniquely determined.

In fact only two of the three roots are independent. To prove it we note the sum of the mass fractions is equal to 1. It comes to the restriction on the three roots

$$(1+r_b+r_br_c)T_2(r_a) + (1+r_c+r_cr_a)T_2(r_b) + (1+r_a+r_ar_b)T_2(r_c) = 1-r_ar_br_c. \tag{60}$$

Which has been written independent from of the masses. It is a quintic equation for any root in terms of the other two. Actually, it is a compact form of writing the three quintics when we permute the masses.

Cancelling the root r_a between the last two equations we write the fractions of mass in terms of two roots of the Euler's quintics

$$\frac{1}{m}\begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix} = \frac{1}{1+r_b+r_br_c}\left[T_2(r_c)\begin{pmatrix} -1\\ -r_b\\ r_b+1\end{pmatrix} - T_2(r_b)\begin{pmatrix} r_c+1\\ -1\\ -r_c\end{pmatrix} + \begin{pmatrix} 1\\ r_b\\ r_br_c\end{pmatrix}\right]. \tag{61}$$

7. Conclusions

We have presented a review of the Euler's case of the collinear three-body problem from different perspectives. We present first the coordinates where the problem is more accessible. We prove the motion is in a fixed plane if it is not in a fixed line. We find the solutions are conics by using the Lagrangian formalism. We introduce the initial positions relative to the center of mass as characteristic constants of motion, that are determined by the root of a quintic, when substituted in the three polynomials, multiplying the masses in the Euler's quintic. We use a symmetric polynomial, our $P_4(x)$, that appears also in Marchal's book [8]. We relate this polynomial to the three polynomials of the Euler's quintic: we remark in our Eq. (47) the vector formed by the polynomials of the Euler's quintic computed for the value of their real root times the axis of rotation is that polynomial multiplying the vector of directed distances. Other auxiliar vector with entries $((x+1)^2, -x^2, x^2(1+x)^2)$ times the vector of directed distances produces the vector of three polynomials of the Euler's quintic. The scalar product of the same vector with the axis of rotation gives the symmetric polynomial.

The inverse problem for the masses as a function of the configuration is written as a linear combination of the previous vectors. This reproduces the same equation in our reference [9]. A new presentation of the same equation is modified to express the masses as a *compound central configuration* [10] of two $2+1$ configurations, each one with a different zero mass.

Last in this paper we find the three roots of the Euler quintic, obtained by permutation of the masses are not independent. Given two, the third is determined by the first two. The masses are now fixed by two of the roots.

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