Optical solitons and parameters stability of the coupled system in magneto-optical waveguides

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In this paper, optical solitons propagation in magneto-optical waveguides which is modeled by a coupled nonlinear Schrödinger equation system. A series of wave propagation patterns are given, including solitary wave solutions, periodic solutions and singular solutions. In addition, the physical realization of the optical wave modes is carried out under certain parameter values. In particular, the parameters stability of these optical wave modes is obtained.

Keywords: Optical solitons; nonlinear Schrödinger equation; waveguide; parameters stability.

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1. Introduction

The nonlinear Schrödinger equation (NLSE) is an essential branch in nonlinear science, which can model most nonlinear phenomena in physics. With the continuous deepening of research, people have extended the NLSE to more complex forms such as high-order, high-dimensional, variable coefficient, coupled, etc. It is widely used in many fields [1-6].

Solitons maintain their shape and velocity unchanged during spatiotemporal transmission. They are broadly applied in optics, fluid mechanics, astrophysics [7-9], etc. In 1973, Hasegawa and Tappert first proposed optical solitons [10]. From a physical perspective, optical solitons are the result of balancing self-phase modulation (SPM) and group velocity dispersion (GVD), and they can maintain waveform invariance during transmission in optical fibers. They have been widely studied in optical fiber communication [11-13]. From a mathematical perspective, optical solitons are closely related to the NLSE. The theory of optical solitons is derived from the mathematical model of the NLSE, which describes the behaviors of nonlinear optical waves. So, the optical solitons can be seen as integrable solutions to the NLSE. Researchers have studied the exact solution problem of nonlinear NLSE and have successively proposed some effective methods. For example, the inverse scattering transform method [14, 15], the Darboux transformations method [16,17], the Hirota method [18,19], the generalized extended tanh method [20], and the Jacobi elliptic function expansion method [21], etc.

In magneto-optic waveguides (MOW), the magnetic-optical effect can compel solitons to move from an attractive state to an isolated state, effectively avoiding information overflow from pulse-to-pulse. This is also the main role of magneto-optic effect [22]. For the coupling case for a magnetic waveguide, for example, for an external magnetic field, along z-axis, the dielectric tensor ε is related to the magnetic-optical coefficient Q, which leads to the polarization components being $P_x = -iQ\varepsilon E_y, P_y = iQ\varepsilon E_x$, and hence, the coupled NLSEs are given by [22, 23]. In conventional optical systems, solitons arise from a balance between dispersion and nonlinearity. In the case of magneto-optic effect, the existence of soliton also depends mainly on such balance from the view of computation point in the paper. Therefore, studying the dynamic problem of optical solitons in MOW is feasible and meaningful [24-28].

In 2019, Kudryashov proposed a novel refractive index structure for nonlinear optical fibers [29], and it is useful to study the soliton propagation models with this law [30]. Kudryashov [31] studied the generalized Kudryashov's equation in 2020, which can be used to describe pulse propagation in optical fibers with various polynomial nonlinearities.

In recent years, Zayed *et al*. [32-34] have studied the optical solitons in MOW for generalized NLSE with Kudryashov's law (KL). For example, for the cubic-quartic coupled system of NLSE [32], they combined bright-singular solitons, bright solitons, and singular solitons are given by using the addendum to Kudryashov's method. More complexly, Zayed *et al*. have studied the dynamics of optical solitons in MOW with NLSEs, they have used the $(\frac{Z'}{Z})$-expansion technique and enhanced Kudryashov's method, resulting in the formation of bright solitons, singular solitons, combined bright, singular solitons, and combinations thereof, as well as kink-type solutions [33]. On this basis, they combined the generalized ϕ^6-model expansion technique with the enhanced Kudryashov scheme, using Jacobi elliptic functions to obtain bright, dark, isolated solitons, as well as additional straddled soliton solutions [34]. In addition, Karim *et al.* [35] have used the modified extended direct algebraic method to solve the (2+1)-dimensional NLSE. They obtained dark solitons, bright solitons, singular solitons and so on.

Even more to the point, in 2022, Zayed *et al.* [36] studied the cubic-quartic optical solitons in MOW for NLSE. The one with KL arbitrary refractive index and generalized non-local laws of nonlinearity. At the same time, they also introduced the coupled system for the first time. They used the modified Kudryashov's methodology and the addendum to Kudryashov's methodology to provide dark, bright optical, candid, and combo bright-singular solutions for the system [36]. However, the research methods they applied are based on stricter conditions without providing richer solutions. Our goal is to get richer solutions. The trial equation method and complete discrimination system for polynomial are good methods, which have fewer constraints and more abundant solutions [37-40]. In addition, Kai *et al*. [41-43] used these two methods to study the topological stability of optical wave models.

In this paper, we use the trial equation method and the complete discrimination system for polynomial method to study the coupled system, and obtain some new solutions. More importantly, according to these two methods, we study the topological stability of the resulting optical wave modes, that is, the analysis of whether the optical wave modes change when the parameters change, which is important for physics.

2. Mathematical analysis

In this section, we study the coupled system as follows [36]

$$\begin{aligned} &iu_t + ia_1u_{xxx} + b_1u_{xxxx} + \left[c_1|u|^n + d_1|u|^{2n} + e_1|u|^{3n} + f_1|u|^{4n} + g_1\left(|u|^n\right)_{xx} + h_1\left(|u|^{2n}\right)_{xx}\right]u + \left[k_1|v|^n + l_1|v|^{2n}\right. \\ &\quad \left. + \sigma_1|v|^{3n} + s_1|v|^{4n} + \xi_1\left(|v|^n\right)_{xx} + \eta_1\left(|v|^{2n}\right)_{xx}\right]u = Q_1v + i\left[\lambda_1\left(|u|^{2m}u\right)_x + \mu_1\left(|u|^{2m}\right)_x u + \theta_1|u|^{2m}u_x\right], \end{aligned} \tag{1}$$

$$\begin{aligned} &iv_t + ia_2v_{xxx} + b_2v_{xxxx} + \left[c_2|v|^n + d_2|v|^{2n} + e_2|v|^{3n} + f_2|v|^{4n} + g_2\left(|v|^n\right)_{xx} + h_2\left(|v|^{2n}\right)_{xx}\right]v + \left[k_2|u|^n + l_2|u|^{2n}\right. \\ &\quad \left. + \sigma_2|u|^{3n} + s_2|u|^{4n} + \xi_2\left(|u|^n\right)_{xx} + \eta_2\left(|u|^{2n}\right)_{xx}\right]v = Q_2u + i\left[\lambda_2\left(|v|^{2m}v\right)_x + \mu_2\left(|v|^{2m}\right)_x v + \theta_2|v|^{2m}v_x\right], \end{aligned} \tag{2}$$

where x is spatial variables, t is temporal, all coefficients are nonzero constants. a_j, b_j $(J = 1,2)$ are the parameters of third-order, forth-order. c_j, d_j, e_j, f_j $(J = 1,2)$ are the coefficients of SPM. g_j, h_j, ξ_j, η_j $(J = 1,2)$ are the coefficients of the non-local generalized refractive index law. k_j, l_j, σ_j, s_j $(J = 1,2)$ represent the nonlinear parameters effects caused by cross-phase modulation. Besides, Q_j are the MOW terms coefficients, μ_j, λ_j and θ_j represent nonlinear dispersal coefficients. m and n represent the maximum intensity and power nonlinear parameters respectively, and $0 < n < 2$. Equation (1) and (2) are coupled NLSEs.

Firstly, we use the transformation as follows [36]

$$\begin{aligned} u(t,x) &= \psi_1(\xi)\exp[i\phi(x,t)], \\ v(t,x) &= \psi_2(\xi)\exp[i\phi(x,t)], \end{aligned} \tag{3}$$

and

$$\xi = x - Vt, \quad \phi(x,t) = -kx + wt + \theta_0, \tag{4}$$

where V, k, w and θ_0 represent velocity, frequency, wave number and phase constant, respectively.

Substituting Eq. (3) into Eq. (1) and Eq. (2), we get the real parts

$$\begin{aligned} &b_1\psi_1^{(4)} + 3k\left(a_1 - 2b_1k\right)\psi_1'' - \left(w + a_1k^3 - b_1k^4\right)\psi_1 + c_1\psi_1^{n+1} + d_1\psi_1^{2n+1} + e_1\psi_1^{3n+1} + f_1\psi_1^{4n+1} \\ &\quad + g_1[n\left(n-1\right)\psi_1^{-1}\psi_1'^2 + n\psi_1'']\psi_1^n + 2nh_1[(2n-1)\psi_1^{-1}\psi_1'^2 + \psi_1'']\psi_1^{2n} + k_1\psi_1\psi_2^n + l_1\psi_1\psi_2^{2n} + \sigma_1\psi_1\psi_2^{3n} \\ &\quad + s_1\psi_1\psi_2^{4n} + n\xi_1[(n-1)\psi_2^{-2}\psi_1\psi_2'^2 + \psi_2^{-1}\psi_2''\psi_1]\psi_2^n + 2n\eta_1[(2n-1)\psi_2^{-2}\psi_1\psi_2'^2 + \psi_2^{-1}\psi_2''\psi_1]\psi_2^{2n} \\ &\quad - Q_1\psi_2 - k(\lambda_1 + \theta_1)\psi_1^{2m+1} = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &b_2\psi_2^{(4)} + 3k\left(a_2 - 2b_2k\right)\psi_2'' - \left(w + a_2k^3 - b_2k^4\right)\psi_2 + c_2\psi_2^{n+1} + d_2\psi_2^{2n+1} + e_2\psi_2^{3n+1} + f_2\psi_2^{4n+1} + g_2[n\left(n-1\right)\psi_2^{-1}\psi_2'^2 \\ &\quad + n\psi_2'']\psi_2^n + 2nh_2[(2n-1)\psi_2^{-1}\psi_2'^2 + \psi_2'']\psi_2^{2n} + k_2\psi_2\psi_2^n + l_2\psi_2\psi_1^{2n} + \sigma_2\psi_2\psi_1^{3n} + s_2\psi_2\psi_1^{4n} + n\xi_2[(n-1)\psi_1^{-2}\psi_2\psi_1'^2 \\ &\quad + \psi_1^{-1}\psi_1''\psi_2]\psi_1^n + 2n\eta_2[(2n-1)\psi_1^{-2}\psi_2\psi_1'^2 + \psi_1^{-1}\psi_1''\psi_2]\psi_1^{2n} - Q_2\psi_1 - k(\lambda_2 + \theta_2)\psi_2^{2m+1} = 0, \end{aligned} \tag{6}$$

and the imaginary parts

$$\left(a_1 - 4b_1k\right)\psi_1''' - \left(V + 3a_1k^2 - 4b_1k^3\right)\psi_1' - \left[2m(\lambda_1 + \mu_1) + \lambda_1 + \theta_1\right]\psi_1^{2m}\psi_1' = 0, \tag{7}$$

$$\left(a_2 - 4b_2k\right)\psi_2''' - \left(V + 3a_2k^2 - 4b_2k^3\right)\psi_2' - \left[2m(\lambda_2 + \mu_2) + \lambda_2 + \theta_2\right]\psi_2^{2m}\psi_2' = 0. \tag{8}$$

From Eq. (7) and (8), we have the constraint conditions

$$a_1 - 4b_1k = 0, \qquad a_2 - 4b_2k = 0, \tag{9}$$

$$V + 3a_1k^2 - 4b_1k^3 = 0, \qquad V + 3a_2k^2 - 4b_2k^3 = 0, \tag{10}$$

$$2m(\lambda_1 + \mu_1) + \lambda_1 + \theta_1 = 0, \qquad 2m(\lambda_2 + \mu_2) + \lambda_2 + \theta_2 = 0. \tag{11}$$

From Eq. (9), we have

$$k = \frac{a_s}{4b_s}, \qquad s = 1,\ 2, \qquad a_1b_2 = a_2b_1. \tag{12}$$

From Eq. (10), we have

$$V = (-3a_s + 4b_sk)k^2, \qquad s = 1,\ 2. \tag{13}$$

From Eq. (11), we have

$$2m(\lambda_s + \mu_s) + \lambda_s + \theta_s = 0, \qquad s = 1,\ 2. \tag{14}$$

Setting

$$\psi_2(\xi) = \Gamma\psi_1(\xi), \tag{15}$$

where Γ is a real number, and $\Gamma \neq 0, 1$. So the Eq. (5) and Eq. (6) are rewritten as

$$\begin{aligned} b_1\psi_1^{(4)} &+ 3k\left(a_1 - 2b_1k\right)\psi_1'' - \left(w + a_1k^3 - b_1k^4 - Q_1\Gamma\right)\psi_1 + (c_1 + k_1\Gamma^n)\psi_1^{n+1} + (d_1 + l_1\Gamma^{2n})\psi_1^{2n+1} \\ &+ (e_1 + \sigma_1\Gamma^{3n})\psi_1^{3n+1} + (f_1 + s_1\Gamma^{4n})\psi_1^{4n+1} + n(g_1 + \Gamma^n\xi_1)[(n-1)\psi_1^{-1}\psi_1'^2 + \psi_1'']\psi_1^n \\ &+ 2n(h_1 + \Gamma^{2n}\eta_1)[(2n-1)\psi_1^{-1}\psi_1'^2 + \psi_1'']\psi_1^{2n} - k(\lambda_1 + \theta_1)\psi_1^{2m+1} = 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} b_2\Gamma\psi_1^{(4)} &+ 3k\Gamma\left(a_2 - 2b_2k\right)\psi_1'' - \left[\Gamma(w + a_2k^3 - b_2k^4) + Q_2\right]\psi_1 + \Gamma^n(k_2 + c_2\Gamma)\psi_1^{n+1} + \Gamma(l_2 + d_2\Gamma^{2n})\psi_1^{2n+1} \\ &+ \Gamma(\sigma_2 + e_2\Gamma^{3n})\psi_1^{3n+1} + \Gamma(s_2 + f_2\Gamma^{4n})\psi_1^{4n+1} + n\Gamma(\xi_2 + g_2\Gamma^n)[(n-1)\psi_1^{-1}\psi_1'^2 + \psi_1'']\psi_1^n \\ &+ 2n\Gamma(\eta_2 + h_2\Gamma^{2n})[(2n-1)\psi_1^{-1}\psi_1'^2 + \psi_1'']\psi_1^{2n} - k\Gamma^{2m+1}(\lambda_2 + \theta_2)\psi_1^{2m+1} = 0. \end{aligned} \tag{17}$$

(18)

Then we have the constraint conditions

$$b_1 = b_2\Gamma, \tag{19}$$

$$a_1 - 2b_1k = \Gamma\left(a_2 - 2b_2k\right), \tag{20}$$

$$c_1 + k_1\Gamma^n = \Gamma^n\left(k_2 + c_2\Gamma\right), \tag{21}$$

$$d_1 + l_1\Gamma^{2n} = \Gamma\left(l_2 + d_2\Gamma^{2n}\right), \tag{22}$$

$$e_1 + \sigma_1\Gamma^{3n} = \Gamma(\sigma_2 + e_2\Gamma^{3n}), \tag{23}$$

$$f_1 + s_1\Gamma^{4n} = \Gamma\left(s_2 + f_2\Gamma^{4n}\right), \tag{24}$$

$$g_1 + \Gamma^n\xi_1 = \Gamma(\xi_2 + g_2\Gamma^n), \tag{25}$$

$$h_1 + \Gamma^{2n}\eta_1 = \Gamma(\eta_2 + h_2\Gamma^{2n}), \tag{26}$$

$$\lambda_1 + \theta_1 = \Gamma^{2m+1}(\lambda_2 + \theta_2), \tag{27}$$

$$w + a_1k^3 - b_1k^4 - Q_1\Gamma = \Gamma(w + a_2k^3 - b_2k^4) + Q_2. \tag{28}$$

Thus, we have

$$w = \frac{Q_1\Gamma + Q_2}{(1 - \Gamma)}. \tag{29}$$

When $Q_1 = Q_2 = 0$, we have $w = 0$, that is, the frequency is equal to zero if the magnetic-optical coupled coefficients are equal to zeros. Therefore, this perhaps can be used to verify the effect of MOW.

Applying the transformation $\psi_1(\xi) = U^{\frac{1}{n}}(\xi)$ to the Eq. (16). We get

$$U^3U^{(4)} + \iota_0U'^4 + \iota_1UU'^2U'' + \iota_2[3U''^2 + 4U'U''']U^2 + \iota_3U^2U'^2 + \iota_4U^3U'' \\ - \iota_5U^4 + \iota_6U^5 + \iota_7U^6 + \iota_8U^7 + \iota_9U^8 + \iota_{10}U^4U'' - \iota_{11}U^{\frac{2m}{n}+4} + \iota_{12}(U^4U'^2 + U^5U'') = 0, \tag{30}$$

where

$$\iota_0 = \frac{(1-n)(1-2n)(1-3n)}{4n^3}, \quad \iota_1 = \frac{3(1-n)(1-2n)}{2n^2}, \quad \iota_2 = \frac{1-n}{4n}, \quad \iota_3 = \frac{3(1-n)k^2}{2n}, \quad \iota_4 = \frac{3k^2}{2},$$
$$\iota_5 = \frac{n(w + a_1k^3 - b_1k^4 - Q_1\Gamma)}{4b_1}, \quad \iota_6 = \frac{n(c_1 + k_1\Gamma^n)}{4b_1}, \quad \iota_7 = \frac{n(d_1 + l_1\Gamma^{2n})}{4b_1}, \quad \iota_8 = \frac{n(e_1 + \sigma_1\Gamma^{3n})}{4b_1},$$
$$\iota_9 = \frac{n(f_1 + s_1\Gamma^{4n})}{4b_1}, \quad \iota_{10} = \frac{n(g_1 + \xi_1\Gamma^n)}{4b_1}, \quad \iota_{11} = \frac{nk(\lambda_1 + \theta_1)}{4b_1}, \quad \iota_{12} = \frac{n(h_1 + \Gamma^{2n}\eta_1)}{2b_1}, \tag{31}$$

provided $b_1 \neq 0$.

Next, we apply the trial equation method to solve the Eq. (30).

We set the trial equation,

$$(U')^2 = \sum_{i=0}^{n} z_iU^i, \tag{32}$$

For the convenience of calculation, let the $2m = n$ in Eq. (29). By balancing the order of U^3U'' and ι_9U^8 , we can get $n = 4$.

The trial equation is written as follows

$$(U')^2 = z_4U^4 + z_3U^3 + z_2U^2 + z_1U + z_0, \tag{33}$$

then we derive

$$U'' = 2z_4U^3 + \frac{3}{2}z_3U^2 + z_2U + \frac{1}{2}z_1,$$
$$U'''' = [12z_4U + 3z_3](U')^2 + [6z_4U^2 + 3z_3U + z_2]U''. \tag{34}$$

substituting Eq. (32), Eq. (33) and Eq. (34) into Eq. (29). Then let each coefficient be zero to form an algebraic equations system, we can obtain the values of $z_i(i = 0, 1, 2, 3, 4)$ in Eq. (32) as follows

$$z_4 = \frac{-3\iota_{12} \pm \sqrt{9\iota_{12}^2 - 4\iota_0\iota_9 - 8\iota_1\iota_9 - 144\iota_2\iota_9 - 96\iota_9}}{2\iota_0 + 4\iota_1 + 72\iota_2 + 48}, \quad z_3 = \frac{-\iota_8 - 2\iota_{10}z_4}{2\iota_0z_4 + \frac{7}{2}\iota_1z_4 + 54\iota_2z_4 + \frac{5}{2}\iota_{12} + 30z_4},$$
$$z_2 = -\frac{\left(\frac{15}{2} + \iota_0 + \frac{3}{2}\iota_1 + \frac{75}{4}\iota_2\right)z_3^2}{20z_4 + 2\iota_0z_4 + 3\iota_1z_4 + 40\iota_2z_4 + 2\iota_{12}} - \frac{(\iota_3 + 2\iota_4)z_4 + \frac{3}{2}\iota_{10}z_3 + \iota_7}{20z_4 + 2\iota_0z_4 + 3\iota_1z_4 + 40\iota_2z_4 + 2\iota_{12}},$$
$$z_1 = -\frac{\left(\frac{15}{2} + 2\iota_0 + \frac{5}{2}\iota_1 + 25\iota_2\right)z_2z_3}{15z_4 + 2\iota_0z_4 + \frac{5}{2}\iota_1z_4 + 30\iota_2z_4 + \frac{3}{2}\iota_{12}} - \frac{\left(\iota_3 + \frac{3}{2}\iota_4\right)z_3 + \iota_{10}z_2 + \iota_6 - \iota_{11}}{15z_4 + 2\iota_0z_4 + \frac{5}{2}\iota_1z_4 + 30\iota_2z_4 + \frac{3}{2}\iota_{12}},$$
$$z_0 = \frac{\iota_5 - \frac{1}{2}\iota_{10}z_1 - \left(\frac{33}{2}\iota_2 + 2\iota_1 + 2\iota_0 + \frac{9}{2}\right)z_1z_3}{12z_4 + 2\iota_0z_4 + 2\iota_1z_4 + 24\iota_2z_4 + \iota_{12}} + \frac{-(\iota_4 + \iota_3)z_2 - (7\iota_2 + \iota_1 + \iota_0 + 1)z_2^2}{12z_4 + 2\iota_0z_4 + 2\iota_1z_4 + 24\iota_2z_4 + \iota_{12}}, \tag{35}$$

and satisfied the following constraints

$$2\iota_0z_0z_3 + \frac{3}{2}\iota_1z_0z_3 + 12\iota_2z_0z_3 + 2\iota_0z_1z_2 + \frac{3}{2}\iota_1z_1z_2 + 7\iota_2z_1z_2 + \iota_3z_1 + 3z_0z_3 + \frac{1}{2}z_1z_2 + \frac{1}{2}\iota_4z_1 = 0,$$
$$2\iota_0z_0z_2 + \iota_0z_1^2 + \frac{1}{2}\iota_1z_1^2 + \frac{3}{4}\iota_2z_1^2 + \iota_1z_0z_2 + 4\iota_2z_0z_2 + \iota_3z_0 = 0, 2\iota_0z_0z_1 + \frac{1}{2}\iota_1z_0z_1 = 0, \iota_0z_0^2 = 0. \tag{36}$$

Finally, taking the following transformation to Eq. (34)

$$p = (z_4)^{\frac{1}{4}}\left(U + \frac{z_3}{4z_4}\right), \qquad \xi_1 = (z_4)^{\frac{1}{4}}\xi. \tag{37}$$

We get

$$(p_{\xi_1})^2 = p^4 + l_1 p^2 + l_2 p + l_3, \tag{38}$$

where

$$l_1 = -\frac{3}{8}(z_3)^2(z_4)^{-\frac{3}{2}} + z_2(z_4)^{-\frac{1}{2}}, \quad l_2 = (z_4)^{-\frac{1}{4}}\left(\frac{(z_3)^3}{8(z_4)^2} - \frac{z_2 z_3}{2z_4} + z_1\right), \quad l_3 = \frac{-3(z_3)^4}{256(z_4)^3} + \frac{z_2(z_3)^2}{16(z_4)^2} - \frac{z_1 z_3}{4z_4} + z_0. \tag{39}$$

Simplifying Eq. (38) into the integral form

$$\pm(\xi_1 - \xi_0) = \int \frac{dp}{\sqrt{F(p)}}, \tag{40}$$

where

$$F(p) = p^4 + l_1 p^2 + l_2 p + l_3. \tag{41}$$

Now, we can use the complete discrimination system for quartic polynomial to solve Eq. (41). The discrimination system is as follows

$$D_1 = 1, \quad D_2 = -l_1, \quad D_3 = -2l_1^3 + 8l_1 l_3 - 9l_2^2,$$
$$D_4 = -l_1^3 l_2^2 + 4l_1^4 l_3 + 36 l_1 l_2^2 l_3 - 32 l_1^2 l_3^2 - \frac{27}{4} l_2^4 + 64 l_3^3, \quad E_2 = 9l_2^2 - 32 l_1 l_3. \tag{42}$$

In the next section, we will classify the roots of Eq. (41) which give all optical wave modes.

3. The optical wave modes

In this section, we obtain thirteen different optical wave modes in total. In addition, it is important to study the topological stability of the optical wave modes. The analysis method is as follows, these solutions are stable when conditions are given by inequalities, unstable when conditions are given by equations, and semi-stable when conditions are given by equations and inequalities. In fact, in general, inequalities can remain unchanged when their parameters are perturbed small, and when some parameters change, the equation becomes an inequality [44]. Then our results are as follows

Case 1. $D_2 = D_3 = D_4 = 0$, then $F(p) = p^4$, we have a singular rational solution as follows

$$u_1(t,x) = \left((z_4)^{-\frac{1}{4}}\left[-(z_4^{\frac{1}{4}}\xi - \xi_0)^{-1}\right] - \frac{z_3}{4z_4}\right)^{\frac{1}{n}} e^{\{i(-kx+wt+\theta_0)\}}. \tag{43}$$

This optical wave pattern is parameter unstable.

Case 2. $E_2 < 0, D_2 < 0, D_3 = D_4 = 0$, then $F(p) = [(p-\zeta_1)^2 + \zeta_2^2]^2$,we have a singular periodic solution as follows

$$u_2(t,x) = \left(z_4^{-\frac{1}{4}}\left[\zeta_2 \tan(\zeta_2 z_4^{\frac{1}{4}}\xi - \xi_0) + \zeta_1\right] - \frac{z_3}{4z_4}\right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{44}$$

where ζ_1 and ζ_2 are real constants.

This pattern is parameter semi-stable and changes form with parameter perturbations.

Case 3. $E_2 > 0, D_2 > 0, D_3 = D_4 = 0$, then $F(p) = (p-\zeta_1)^2(p-\zeta_2)^2$, we have two solitary wave solutions as follows

$$u_3(t,x) = \left(z_4^{-\frac{1}{4}}\left[\frac{\zeta_2 - \zeta_1}{2}\left\{\coth\frac{(\zeta_1 - \zeta_2)(z_4^{\frac{1}{4}}\xi - \xi_0)}{2} - 1\right\} + \zeta_2\right] - \frac{z_3}{4z_4}\right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{45}$$

$$u_4(t,x) = \left(z_4^{-\frac{1}{4}}\left[\frac{\zeta_2 - \zeta_1}{2}\left\{\tanh\frac{(\zeta_1 - \zeta_2)(z_4^{\frac{1}{4}}\xi - \xi_0)}{2} - 1\right\} + \zeta_2\right] - \frac{z_3}{4z_4}\right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}. \tag{46}$$

These two solitary modes of optical wave are parameters semi-stable.

Case 4. $D_2 > 0, D_3 > 0, D_4 = 0$, then $F(p) = (p-\zeta_1)^2(p-\zeta_2)(p-\zeta_3)$,
when $(\zeta_1-\zeta_3)/(\zeta_2-\zeta_1) < 0$, and ζ_1, ζ_2 and ζ_3 are real constants, we have two wave modes as follows

$$u_5(t,x) = \left(z_4^{-\frac{1}{4}} \frac{\zeta_2(\zeta_1-\zeta_3) - (\zeta_1-\zeta_2)\zeta_3 \coth^2 \frac{B}{2}}{(\zeta_1-\zeta_3) - (\zeta_1-\zeta_2)\coth^2 \frac{B}{2}} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{47}$$

and

$$u_6(t,x) = \left(z_4^{-\frac{1}{4}} \frac{\zeta_2(\zeta_1-\zeta_3) - (\zeta_1-\zeta_2)\zeta_3 \tanh^2 \frac{B}{2}}{(\zeta_1-\zeta_3) - (\zeta_1-\zeta_2)\tanh^2 \frac{B}{2}} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{48}$$

where $B = (z_4^{(1/4)}\xi - \xi_0)\sqrt{(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)}$, ζ_1, ζ_2 and ζ_3 are real constants.

When $(\zeta_1-\zeta_3)/(\zeta_2-\zeta_1) > 0$, and ζ_1, ζ_2 and ζ_3 are real constants. Then we have a singular periodic wave mode,

$$u_7(t,x) = \left(z_4^{-\frac{1}{4}} \frac{\zeta_2(\zeta_1-\zeta_3) - (\zeta_2-\zeta_1)\zeta_3 \tan^2 \frac{B}{2}}{(\zeta_1-\zeta_3) - (\zeta_2-\zeta_1)\tan^2 \frac{B}{2}} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{49}$$

where $B = (z_4^{(1/4)}\xi - \xi_0)\sqrt{(\zeta_2-\zeta_1)(\zeta_1-\zeta_3)}$.

These modes are parameters semi-stable when the parameters change.

Case 5. $E_2 = 0, D_2 > 0, D_3 = D_4 = 0$, then $F(p) = (p-\zeta_1)^3(p-\zeta_2)$, we have a rational singular solution,

$$u_8(t,x) = \left(z_4^{-\frac{1}{4}} \left[\zeta_1 + \frac{4(\zeta_1-\zeta_2)}{(\zeta_2-\zeta_1)^2[z^{\frac{1}{4}}\xi - \xi_0]^2 - 4} \right] - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{50}$$

where ζ_1 and ζ_2 are real constants.

In this case, the optical wave pattern is parameter semi-stable.

Case 6. $D_3 < 0, D_4 = 0$, then $F(p) = (p-\zeta_1)^2[(p-\zeta_2)^2 + \zeta_3^2]$, we have a solitary wave solution in exponential form,

$$u_9(t,x) = \left(z_4^{-\frac{1}{4}} \frac{(c-l) + \sqrt{[(\zeta_1-\zeta_2)^2 + \zeta_3^2](2-l)}}{c^2-1} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{51}$$

where $c = \exp[\pm\sqrt{(\zeta_1-\zeta_2)^2+\zeta_3^2}(z_4^{(1/4)}\xi - \xi_0)]$, $l = (\zeta_1 - 2\zeta_2)/\sqrt{(\zeta_1-\zeta_2)^2+\zeta_3^2}$, ζ_1, ζ_2 and ζ_3 are real constants.

This optical wave pattern is parameter semi-stable.

Case 7. $D_2 > 0, D_3 > 0, D_4 > 0$, then $F(p) = (p-\zeta_1)(p-\zeta_2)(p-\zeta_3)(p-\zeta_4)$, we have the elliptic function double periodic solutions,

$$u_{10}(t,x) = \left(z_4^{-\frac{1}{4}} \frac{\zeta_2(\zeta_1-\zeta_4)sn^2(C,M) - \zeta_1(\zeta_2-\zeta_4)}{(\zeta_1-\zeta_4)sn^2(C,M) - (\zeta_2-\zeta_4)} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{52}$$

$$u_{11}(t,x) = \left(z_4^{-\frac{1}{4}} \frac{\zeta_4(\zeta_2-\zeta_3)sn^2(C,M) - \zeta_3(\zeta_2-\zeta_4)}{(\zeta_2-\zeta_3)sn^2(C,M) - (\zeta_2-\zeta_4)} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{53}$$

where $C = (\sqrt{(\zeta_1-\zeta_3)(\zeta_2-\zeta_4)}/2)(z_4^{(1/4)}\xi - \xi_0)$, $M = \sqrt{(\zeta_1-\zeta_4)(\zeta_2-\zeta_3)/(\zeta_1-\zeta_3)(\zeta_2-\zeta_4)}$, ζ_1, ζ_2, ζ_3 and ζ_4 are real constants and $\zeta_1 > \zeta_2 > \zeta_3 > \zeta_4$.

These patterns are parameters stable.

Case 8. $D_4 < 0, D_2D_3 \geq 0$, then $F(p) = (p-\zeta_1)(p-\zeta_2)[(p-\zeta_3)^2 + \zeta_4^2]$, we have the elliptic function double periodic solutions,

$$u_{12}(x,t) = \left(z_4^{-\frac{1}{4}} \frac{\gamma_1 cn(C,M) + \gamma_2}{\gamma_3 cn(C,M) + \gamma_4} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{54}$$

where ζ_1, ζ_2, ζ_3 and ζ_4 are real constants, and

$$\begin{aligned} &\gamma_1 = \frac{1}{2}(\zeta_1+\zeta_2)\gamma_3 - \frac{1}{2}(\zeta_1-\zeta_2)\gamma_4, \qquad \gamma_2 = \frac{1}{2}(\zeta_1+\zeta_2)\gamma_4 - \frac{1}{2}(\zeta_1-\zeta_2)\gamma_3, \\ &\gamma_3 = \zeta_1 - \zeta_3 - \frac{\zeta_4}{m_1}, \quad \gamma_4 = \zeta_1 - \zeta_3 - \zeta_4 m_1, \qquad E = \frac{\zeta_4^2 + (\zeta_1-\zeta_3)(\zeta_2-\zeta_3)}{\zeta_4(\zeta_1-\zeta_2)}, \\ &m_1 = E \pm \sqrt{E^2+1}, \quad M = \sqrt{\frac{1}{1+m_1^2}}, \qquad C = \frac{\sqrt{-2\zeta_4 m_1(\zeta_1-\zeta_2)}}{2Mm_1}(z_4^{\frac{1}{4}}\xi - \xi_0). \end{aligned} \tag{55}$$

This pattern is parameter stable.

Case 9. $D_4 > 0, D_2D_3 \leq 0$, then $F(p) = [(p-\zeta_1)^2+\zeta_2^2][(p-\zeta_3)^2+\zeta_4^2]$. When $\zeta_2 \geq \zeta_4 > 0$, we have the elliptic function double periodic solutions,

$$u_{13}(x,t) = \left(z_4^{-\frac{1}{4}} \frac{\gamma_1 sn(C,M)+\gamma_2 cn(C,M)}{\gamma_3 sn(C,M)+\gamma_4 cn(C,M)} - \frac{z_3}{4z_4} \right)^{\frac{1}{n}} e^{i\{(-kx+wt+\theta_0)\}}, \tag{56}$$

where ζ_1, ζ_2, ζ_3 and ζ_4 are real constants, and

$$\gamma_1 = \zeta_1\gamma_3+\zeta_2\gamma_4, \quad \gamma_2 = \zeta_1\gamma_4-\zeta_2\gamma_3, \quad \gamma_3 = -\zeta_2-\frac{\zeta_4}{m_1}, \quad \gamma_4 = \zeta_1-\zeta_3,$$
$$E = \frac{(\zeta_1-\zeta_3)^2+\zeta_2^2+\zeta_4^2}{2\zeta_2\zeta_4}, \quad m_1 = E+\sqrt{E^2-1}, \quad M = \sqrt{\frac{m_1^2-1}{m_1^2}}, \tag{57}$$
$$C = \frac{\zeta_4\sqrt{(\gamma_3^2+\gamma_4^2)(m_1^2\gamma_3^2+\gamma_4^2)}}{\gamma_3^2+\gamma_4^2}(z_4^{\frac{1}{4}}\xi-\xi_0).$$

This pattern is parameter stable.

4. The numerical simulation of the solutions

In this section, we calculate different forms of several characteristic solutions under the conditions of given parameter values, and plot envelope graphs to get more intuitive conclusions.

Example 1: Singular solutions

Taking $n = w = \xi_0 = \theta_0 = \sigma_1 = 1$, $\Gamma = 2$, $\zeta_1 = \iota_0 = \iota_1 = \iota_2 = \iota_3 = 0$, $\zeta_2 = m = 1/2$, $b_1 = -1$, $V = 16\sqrt{6}/9$, $k = \sqrt{6}/3, h_1 = 4, \eta_1 = 7, g_1 = -2, \xi_1 = -3, f_1 = -16, s_1 = -5, e_1 = 8$, we get

$$u_2 = \frac{1}{2}\left(\tan\left[\frac{1}{2}\left\{ x - \frac{16\sqrt{6}}{9}t \right\} - 1 \right] \right) e^{i(-\frac{\sqrt{6}}{3}x+t+1)}. \tag{58}$$

The 3D plot of $|u_2|$ are shown in Fig. 1. This is a parameter unstable periodic solution with singularity.

Taking $n = w = \xi_0 = \theta_0 = k = 1, b_1 = -1, \Gamma = 2, \zeta_2 = 1/4, \zeta_3 = -(1/4), \zeta_1 = \iota_0 = \iota_1 = \iota_2 = \iota_3 = 0, m = 1/2, h_1 = 12, \eta_1 = 5, \sigma_1 = 3, \xi_1 = -5, f_1 = -16, s_1 = -5, V = e_1 = 8, g_1 = -6$, we get

$$u_7 = \left(\frac{\frac{1}{16}+\frac{1}{16}\tan^2[\frac{1}{8}(x-8t-1)]}{\frac{1}{4}-\frac{1}{4}\tan^2[\frac{1}{8}(x-8t-1)]} \right) e^{i(-x+t+1)}. \tag{59}$$

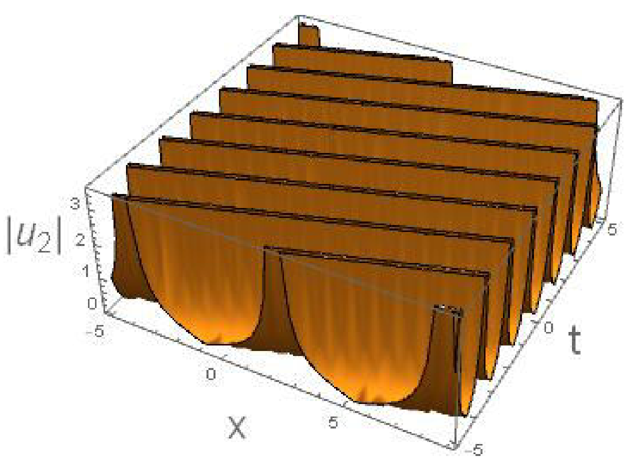

FIGURE 1. The modulus of $u_2(x,t)$.

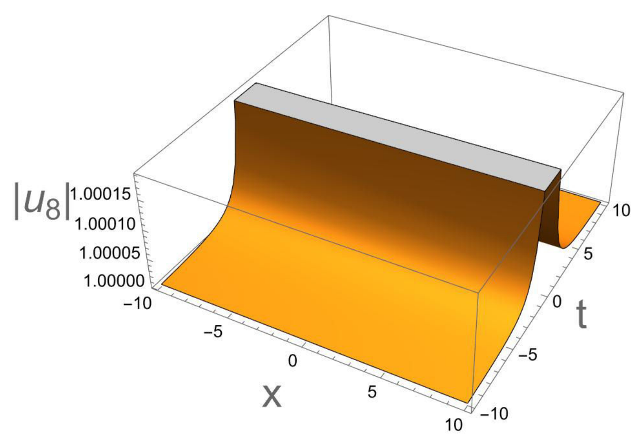

FIGURE 2. The modulus of $u_8(x,t)$.

This is also a parameter unstable periodic solution with singularity.

Taking $n=w=\xi_0=\theta_0=1, \zeta_1=b_1=-1, k=\Gamma=2, \zeta_2=3, m=1/2, \iota_0=\iota_1=\iota_2=\iota_3=0, h_1=8, \eta_1=6, \sigma_1=2, \xi_1=-4, f_1=-16, s_1=-5, e_1=16, g_1=-8, \xi_1=-4, V=64$, we get

$$u_8=\left(-\frac{16}{16(x-64t-1)^2-4}-1\right)e^{i(-2x+t+1)}. \tag{60}$$

This is an unstable rational solution with singularity. The 3D plot of $|u_8|$ is shown in Fig. 2.

Example 2: Solitary wave solutions

Taking $n=w=\xi_0=\theta_0=\zeta_1=b_1=1, \Gamma=2, \iota_0=\iota_1=\iota_2=\iota_3=0, \sigma_1=\zeta_2=-1, V=-(58\sqrt{174}/9), k=(\sqrt{174}/6), h_1=-4, \eta_1=-7, m=1/2, g_1=-2, \xi_1=-7, e_1=f_1=16, s_1=5$, we get

$$u_3=\left(-\coth(x+\frac{58\sqrt{174}}{9}t-1)+\frac{9}{4}\right)e^{i(-\frac{\sqrt{174}}{6}x+t+1)}, \tag{61}$$

$$u_4=\left(-\tanh(x+\frac{58\sqrt{174}}{9}t-1)+\frac{9}{4}\right)e^{i(-\frac{\sqrt{174}}{6}x+t+1)}. \tag{62}$$

They are solitary waves that can propagate stably, and the 3D plot of $|u_4|$ is shown in Fig. 3.

Example 3: Elliptic function double periodic solutions

Taking $n=b_1=w=\xi_0=\theta_0=\zeta_2=1, m=\sigma_1=\frac{1}{2}, \zeta_3=-1, \zeta_4=-2, \zeta_1=\Gamma=g_1=2, \iota_0=\iota_1=\iota_2=\iota_3=0, h_1=-4, e_1=4, \eta_1=-7, \xi_1=7, f_1=16, s_1=5,\ k=\sqrt{55614}/78,\ V=(-2852\sqrt{55614})/3042$, we get

$$u_{10}=\left(\frac{4sn^2(\frac{3}{2}(x-\frac{-2852\sqrt{55614}}{3042}t-1),\frac{2\sqrt{2}}{3})-6}{4sn^2(\frac{3}{2}(x-\frac{-2852\sqrt{55614}}{3042}t-1),\frac{2\sqrt{2}}{3})-3}-\frac{1}{4}\right)e^{i(-\frac{\sqrt{55614}}{78}x+t+1)}. \tag{63}$$

This is a periodic pattern of optical wave propagation.

Taking $n=b_1=w=\xi_0=\theta_0=\zeta_3=1, \zeta_4=\zeta_1=\Gamma=g_1=2, m=\sigma_1=1/2, \iota_0=\iota_1=\iota_2=\iota_3=0, \zeta_2=h_1=-4, e_1=4, \eta_1=-7, \xi_1=7, f_1=16, s_1=5, k=\sqrt{48945102}/2166, V=-(90388\sqrt{48945102})/2345778$, we get

$$u_{12}=\left(\frac{\gamma_1 cn(\sqrt{\frac{290+2\sqrt{145}}{2+2\sqrt{145}}}(x-Vt-1),\frac{12}{\sqrt{290+2\sqrt{145}}})+\gamma_2}{\gamma_3 cn(\sqrt{\frac{290+2\sqrt{145}}{2+2\sqrt{145}}}(x-Vt-1),\frac{12}{\sqrt{290+2\sqrt{145}}})+\gamma_4}-\frac{1}{4}\right)e^{i(-kx+t+1)}, \tag{64}$$

where $\gamma_1=(-13-2\sqrt{145})/3,\ \gamma_2=-(11+2\sqrt{145})/3,\ \gamma_4=(7+\sqrt{145})/6$.

The 3D plot of $|u_{12}|$ is shown in Fig. 4. This is also a periodic stable pattern of optical wave propagation.

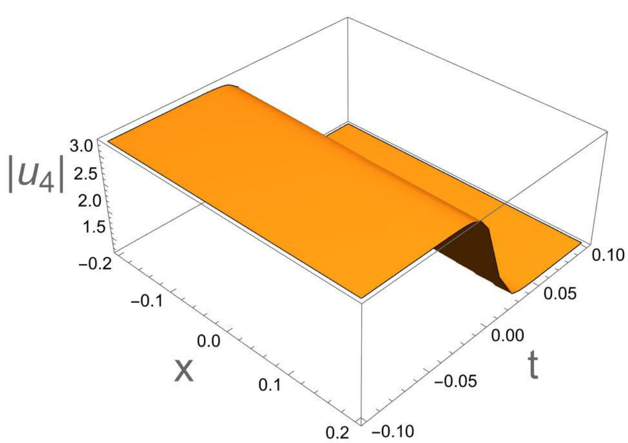

FIGURE 3. The modulus of $u_4(x,t)$.

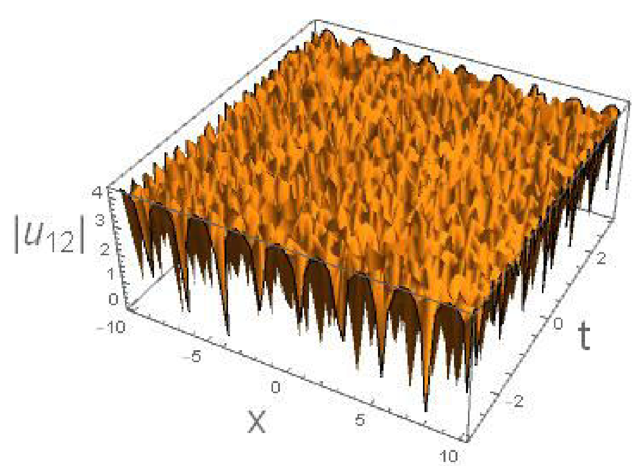

FIGURE 4. The modulus of $u_{12}(x,t)$.

5. Conclusion

In this paper, the coupled NLSE system in magneto-optical waveguides is studied by the trial equation method and the complete discrimination system for polynomial method, and thirteen different exact solutions are obtained. In particular, the elliptic function double periodic solutions is a new achievement. In addition, the types of different solutions can be determined according to the physical parameters, stable, semi-stable, or unstable. For similar NLSE problems, you can also use these two methods to solve, which are two powerful tools.

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