Numerical phase shift analysis of nucleon-nucleon systems with Hellmann plus spin dependence

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The scattering phase shifts for quantum mechanical potential scattering by local interaction can be computed without solving the Schrödinger equation. This can be done by numerically solving the phase equation from the origin to the asymptotic region. Phase Function Method (PFM) is regarded as a resourceful method for calculating scattering phase shifts in quantum mechanics. We utilize the PFM to handle the Hellmann plus spin-orbit interaction. Our approach uses a five-parameter potential model to compute the scattering phase shift. Our results for nucleon-nucleon systems closely match previous findings.

Keywords: Phase function method; Hellmann plus spin-orbit interaction; scattering phase shifts; nucleon-nucleon systems.

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1. Introduction

In the early 1940s, the search was on for the needed correction in the potential to give proper magic numbers. Finally, success came from Mayer [1], Haxel [2], and others, who showed in 1949 that separation of the subshells could be generated by including spin-orbit potential. At first, atomic physicists used the idea of a spin-orbit to explain the fine structure of spectral lines. This arises from the electromagnetic interaction between the magnetic field generated by electrons moving around the nucleus and the electrons' magnetic moment. Although electromagnetic spin-orbit effects [3] are typically minimal, of the order of one part in 10^5 in the spacing of atomic levels, it is not strong enough to support adequate changes in nuclear level spacing, required to generate the observed magic numbers. Therefore, we adopt a similar form for the atomic spin-orbit concept but for nuclear potential. One gluon exchange (OGE) [4] can be regarded as a viable source for velocity-dependent potential like spin-orbit coupling, which is exhibited by the strong polarization found in the scattering of nucleons by nuclei as well as by the substantial splitting between levels of a doublet explained in terms of a shell model. The precise dynamical origin of the strong nuclear spin-orbit force is not fully resolved [5], but the atomic physics analogy of the force hints at a relativistic effect. Empirical data implies the dominance of static-type (velocity-independent) nuclear potentials at somewhat low energies for the incident particles. However, at higher energies, forces depend on the relative nucleonic momentum $P = P_1 - P_2$. Here, P_1 and P_2 are the individual nucleons' momenta, and according to Galilean invariance, the interaction is determined solely by the relative motion P between the two particles, not their absolute motion at any given time during the interaction. To first order approximation in P, the possible scalars that could be formed are $(\vec{r}\cdot\vec{P})$, $(\vec{S}\cdot\vec{P})$ and $(\vec{r}\times\vec{P})\cdot\vec{S}$, out of what first violates time reversal invariance. In contrast, the second one violates parity [3]. Therefore, for two-body interactions, the only viable spin-orbit coupling is $V_{SO}(r)((\vec{r}\times\vec{P})\cdot\vec{S})$. Here, $V_{SO}(r)$ is the spin-orbit potential with the form factor of central potential with differing parameters, and r is the relative inter-nucleon distance. The expectation value of the spin-orbit interaction is proportional to $2<\vec{L}\cdot\vec{S}>= J(J+1)-L(L+1)-S(S+1)$ where the notations L, S and J stand for the orbital angular momentum, spin angular momentum and the total angular momentum respectively. Although traditionally, the spin-orbit potential is a surface term and is proportional to $1/r(dV/dr)$ [6-8], as its effect is piqued at the edge of the nucleus, and the spin density vanishes inside the nucleus. However, we adapt the Hellmann-like spin-orbit term, adding it to the central Hellmann potential, hypothesizing that it will take care of the varying effects of the interior and surface of the nucleus with suitable parameterization of the potential. It is also well known that the form of the potential in spin-orbit term $V_{SO}(r)(\vec{L}\cdot\vec{S})$ is not that vital [3], but it is the $(\vec{L}\cdot\vec{S})$ factor which contributes significantly to reordering the levels. A significant spin-orbit coupling potential appears advantageous in explaining high-energy data, according to Signell & Marshak [9] and Gammel & Thaler analyses [10].

The Hellmann potential, first proposed by Hellmann [11-13], is the superposition of the attractive electromagnetic potential, and the Yukawa potential is $V(r) = -a/r +$

$b(e^{-cr}/r)$. Here, c is the screening parameter [12], and a and b are the strength parameters of the Coulomb and Yukawa potentials. For the sake of the potential's significance in atomic physics, many works [14-21] have been carried out in all the limits of quantum mechanics. Relatively recent studies [15-17,22-24] focus on treating the Hellmann potential in quantum mechanics with non-relativistic and relativistic bound state problems. The Hellmann potential model finds many applications in physics, namely the electron-ion inner-shell ionization problem [25], the electron-core problem [26], solid-state physics [27], alkali hydride molecules [28], etc. The present text addresses the study of nucleon-nucleon phase shift analysis through the variable phase method [29], briefly highlighted in Sec. 2. Section 3 is for our results and discussion. We finally conclude in Sec. 4.

2. Methodology

The phase-function method (PFM) [29] is a powerful numerical technique that works well as an alternative to the traditional Schrödinger equation approach. Recent works by our group [30-32] deal with this methodology in a somewhat lower energy region. The methodology is based on the possible reduction of second-order linear homogeneous equations to first-order nonlinear Riccati or phase equations, as given below.

$$\delta'_\ell(k,r) = -k^{-1}V(r)[\cos\delta_\ell(k,r)\hat{j}_\ell(kr) - \sin\delta_\ell(k,r)\hat{\eta}_\ell(kr)]^2, \quad (1)$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati-Bessel functions [33]. The resulting Phase equations [29-32] for $\ell = 0, 1$ and 2 read as

$$\delta'_0(k,r) = -k^{-1}V(r)[\sin(\delta_0(k,r)+kr)]^2, \quad (2)$$

$$\delta'_1(k,r) = -\frac{V(r)}{k^3r^2}[\sin(\delta_1(k,r)+kr) - kr\cos(\delta_1(k,r)+kr)]^2, \quad (3)$$

and

$$\delta'_2(k,r) = -k^{-1}V(r)\left(\left[\frac{3}{k^2r^2}-1\right]\sin(\delta_2(k,r)+kr) - \frac{3}{kr}\cos[\delta_2(k,r)+kr]\right)^2. \quad (4)$$

Here k stands for centre of mass momentum and is related to the centre of mass energy E as $\sqrt{2mE}/\hbar$. The term $\delta_\ell(k,r)$ is called the phase function, which satisfies the phase equation given by Eq. (1). This has the meaning of the phase shift of the wave function for scattering by the potential at each point, finally truncated at a distance r. This implies that the step size of the variable 'r' is crucial in accumulating phase shifts within the interaction range. Thus, one needs to optimize the step size judiciously to obtain close phase parameters. With the initial condition $\delta_\ell(k,0) = 0$, phase equations given by Eqs. (2)-(4) are solved numerically for the potential under consideration to compute the phase shifts of the scattering in different states in line with the established data. The effective Hellmann potential [11-13,30] in all partial waves is written as

$$V_{HM}(r) = -\frac{a}{r} + b\left(\frac{e^{-cr}}{r}\right) + \frac{\ell(\ell+1)}{r^2}, \quad (5)$$

and the effective spin-orbit coupling is taken as

$$V_{SO}(r) = \left(-\frac{a'}{r} + b'\left[\frac{e^{-c'r}}{r}\right]\right)(\vec{L}\cdot\vec{S}). \quad (6)$$

The effective equivalent nuclear potential under investigation for uncharged hadrons is thus

$$V(r) = V_{HM}(r) + V_{SO}(r). \quad (7)$$

All the five parameters a, b, c, a' and b' of the total potential $V(r)$ have the unit of fm^{-1}. In order to treat the charged hadron systems, one needs to add an electromagnetic interaction with the nuclear potential

$$V(r) = V_{HM}(r) + V_{SO}(r) + V_C(r). \quad (8)$$

We consider the electromagnetic potential $V_C(r) = 2k\eta/r$ as the long-range part of the effective interaction, which is theoretically extended to infinity. Here, η represents the Sommerfeld parameter. The pure Coulomb potential, in theory, has long-range limitless interaction. In practice, it screens out after a certain distance, as the Coulomb potential becomes trivial after a finite distance.

3. Results and discussion

By exploiting Eqs. (2)-(4) in conjunction with Eqs. (7)-(8), we parameterize the nuclear Hellmann potential with the spin-orbit term by R^2 fitting method in our MATLAB programme to reproduce the existing standard phase parameters in the literature [34] of different states of the $(n-p)$ and $(p-p)$ systems. The phase shifts reproduced by sophisticated potential models, which yield an accurate fit to all experimental data currently accepted for elastic nucleon-nucleon scattering, are regarded as standard phase parameters in the literature. The best-fitted potential parameters for various states of the $(n-p)$ and $(p-p)$ systems are given in Table I. The optimized step sizes have also been included in Table I.

For our scattering phase shift calculation, we have used the exact values of $\hbar^2/2m = 41.47$ MeV fm^2 for both $(n-p)$ and $(p-p)$ systems, respectively. Here, m represents the reduced mass of the respective systems. This article studies the neutron-proton $(n-p)$ and proton-proton $(p-p)$ scattering phase shifts for $\ell = 1$ and 2. Figures 1-3 elucidate our values of phase shifts against the established phase shift results [34] for various $(n-p)$ and $(p-p)$ triplet scattering states, as mentioned in Table I.

TABLE I. List of parameters for $(n-p)$ and $(p-p)$ systems.

System	States	$a(fm^{-1})$	$b(fm^{-1})$	$a'(fm^{-1})$	$b'(fm^{-1})$	$c(fm^{-1})$	Step-size
$(n-p)$	$^3P_0$	1.10	-4.0	0.50	-0.7	0.46	0.0034
	$^3P_1$	0.80	-2.30	0.45	-0.6	0.6	0.0045
	$^3P_2$	0.15	-3.30	0.07	-4.5	0.91	0.0200
	$^3D_1$	0.90	0.20	0.20	1.2	0.30	0.0102
	$^3D_2$	0.02	-5.5	0.26	11	0.52	0.0290
	$^3D_3$	0.34	-7.5	0.35	-2.1	0.79	0.0340
$(p-p)$	$^3P_0$	1.10	-4.0	0.50	-0.7	0.44	0.0036
	$^3P_1$	0.80	-2.3	0.45	-0.60	0.58	0.0052
	$^3P_2$	0.15	-3.3	0.07	-4.5	0.90	0.0186

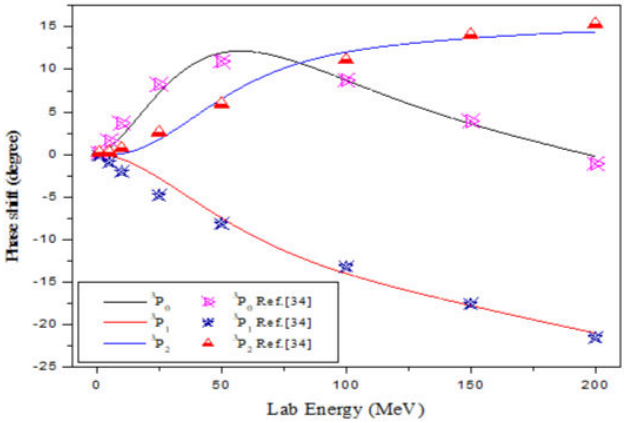

FIGURE 1. $(n-p)$ P-wave scattering phase shifts as a function of laboratory energy. Standard data are from [34] and `https://nn-online.org/NN/`.

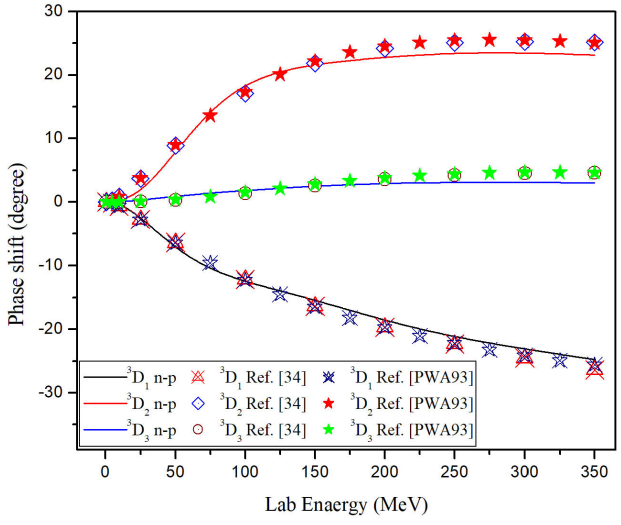

FIGURE 2. $(n-p)$ D-wave scattering phase shifts as a function of laboratory energy. Standard data are from [34] and `https://nn-online.org/NN/`.

It is evident from Figs. 1-3 that, up to a laboratory energy of 200 MeV, our phase parameters for various states of the P- and D-wave of the $(n-p)$ and $(p-p)$ systems closely match with those of Perez *et al.* [34]. Although the calculation in the energy region 200- 350 MeV may need an energy-dependent Hellmann-like spin-orbit interaction, we wish to check our current model calculation in this energy range to see to what extent our simplified model regenerates the scattering observables. The scattering phase shift data for P-waves in both the $(n-p)$ and $(p-p)$ systems within the energy range of 200-350 MeV, discern by approximately 4° from other established calculations [34]. However, they follow a trend similar to those reported in Ref. [34] in this energy range. But our scattering phase shift data for the D-wave of the $(n-p)$ system are in good agreement with the data of [34]. The goodness of fit can be assessed from the R^2 values for different states of the $n-p$ and $p-p$ systems. Finding phase shifts for different states of the $n-p$ system up to 200 MeV, the respective R^2 values are $^3P_0$ (0.953), $^3P_1$ (0.984), $^3P_2$ (0.987), $^3D_1$ (0.994), $^3D_2$ (0.990), and $^3D_3$ (0.923), whereas for the p-p system, the values are $^3P_0$ (0.969), $^3P_1$ (0.986), and $^3P_2$ (0.985).

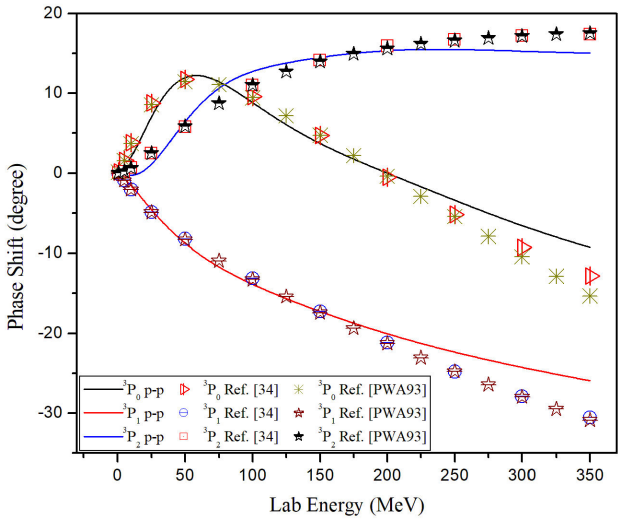

FIGURE 3. $(p-p)$ P-wave scattering phase shifts as a function of laboratory energy. Standard data are from [34] and `https://nn-online.org/NN/`.

The R^2 values for the mentioned fit, being very close to 1, show our curves' higher goodness of fit to the available data. However, in the energy range 200−350 MeV, the respective R^2 values of all the states of P-waves of both systems are not very close to 1. Still, these values also suggest our data agree with available data [34]. On the other hand, the R^2 values for D-wave $(n-p)$ scattering phase shifts are a little lower than those in the energy region up to 200 MeV but close to 1, indicating our model reproduces the standard phase shift data [34] in the higher energy region very well. The Hellmann plus Hellmann-like spin-orbit interaction successfully regenerates the phase shifts with respect to the previously established data [34] for the $n-p$ and $p-p$ systems, suggesting that our approach is robust and reliable. For $(p-p)$ triplet states $^3P_0$, $^3P_1$ and $^3P_2$, we keep all the strength parameters identical as those of $(n-p)$ states $^3P_0$, $^3P_1$ and $^3P_2$. However, a slight adjustment has been made only in the inverse range parameter c, which affects both the central nuclear and spin-orbit terms slightly but not considerably. The reason for not changing the strength values is that the nuclear force does not distinguish between $(n-p)$ and $(p-p)$ interactions, as neutrons and protons are treated the same as a nucleon. Therefore, similar parameters for the strong nuclear environment are expected for similar states of two-nucleon interactions independent of their charge. However, Coulomb force is associated with $(p-p)$ interaction, which we have taken care of by adding additional coulomb terms to existing nuclear and spin-orbit terms as in Eq. (8). In the case of the $^3P_0$ state of $(p-p)$, we only change $c = 0.44$ instead of 0.46 (in the case of $n-p$), whereas for the $(p-p)$ $^3P_1$ state, c is taken to be 0.58 instead of 0.6. For the $(p-p)$ $^3P_2$ state, we adopt $c = 0.9$ instead of 0.91 for the proper parameterization of the nuclear and coulomb effects.

We have chosen the values of the strength parameters a and a' to be positive as we want the first terms of both the central and spin-orbit terms of our potential in Eqs. (7)-(8) to be attractive only, the other strength parameters b and b' are allowed free run in the numerical routine for both positive and negative values to adjust the shape of the potential for adequate parameterization properly. The range of any potential has to be positive values only; thus, parameter c has been assigned only positive values.

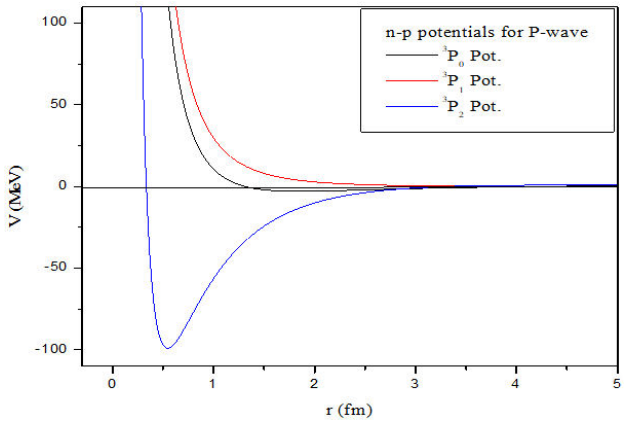

FIGURE 4. $(n-p)$ P-wave potentials as a function of r.

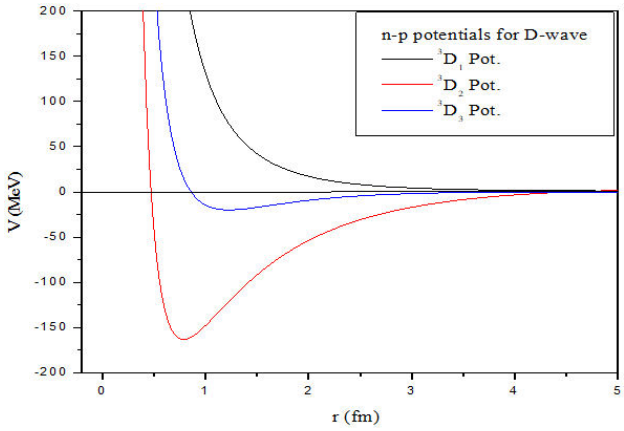

FIGURE 5. $(n-p)$ D-wave potentials as a function of r.

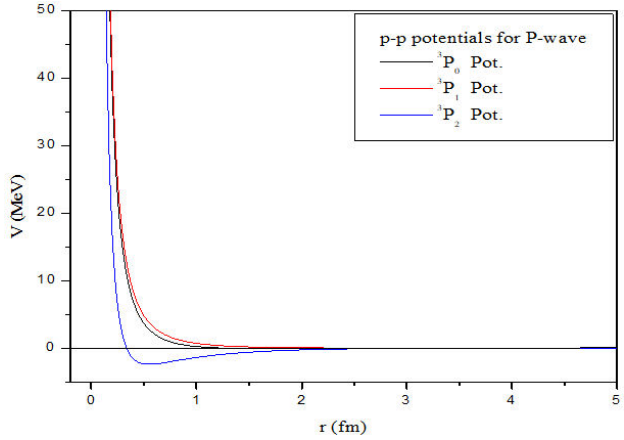

FIGURE 6. $(p-p)$ P-wave potentials as a function of r.

As nuclear potential depends on spin to a great extent and thus on different states, therefore each one of the states of P and D-waves of $(n-p)$ interaction and states of the P-wave of $(p-p)$ interaction has to be singularly parameterized in order to reproduce the actual inter-nucleon interaction field as shown in Figs. 4-6.

Looking at Figs. 4-6, one observes that the higher the inverse range parameter c for a certain state, the sooner the potential approaches zero for that state, as the interaction range r for an individual state is given by $1/c$. For all states, we draw the potential up to $r = 5$ fm in order to have a vivid view of the extensive nature of the potential.

With high energy, the probability of the projectile penetrating the nuclear range of interaction gets higher, and it adequately feels the spin-orbit effect, which helps in the correct regeneration of the actual nuclear phenomenological field. With even higher incident energy of the particles at about 200 MeV, we see a reversal of the phase shift for the $^3P_0$ state in both $(n-p)$ and $(p-p)$ scattering in Figs. 1 and 3, which suggests a repulsive hardcore is being encountered.

Our equivalent nuclear potential has starkly addressed this effect.

4. Conclusion

Phase shift analysis of the charged and uncharged fundamental nucleon-nucleon interactions for motion in the Hellmann [11-13] plus Hellmann-like spin-orbit interaction field [Eq. (6)] for higher incident particles' energy up to 350 MeV in higher partial waves has been put to test using phase function methodology [29]. With five parameter potential, we observe very good fitting in phase shift parameters for all the P-wave and D-wave triplet states of the $n-p$ scattering, and the P-wave triplet states of the $p-p$ scattering, have been reproduced [34]. Our findings demonstrate that the high energy scattering data up to 200 MeV can be explained if the nuclear interaction has two characteristics: a strong spin-orbit coupling, possibly in the outer region, which bears the same exchange character as that of pion theoretical potential, and a hard-core-like repulsive force in the inner region. Typically, np or pp data for various states are fitted to create nucleon-nucleon potentials. Phase parameters up to 350 MeV are reproduced using several sophisticated potential models, which include spin-orbit and tensor interactions. Since our model potential is essentially molecular, we should not claim that it is better than the more complex models that exist in the literature. However, when used in the nuclear realm, this simple-minded molecular potential may fairly accurately reproduce phase shifts throughout a broad energy range. We have just checked that our model has the capability of reproducing required scattering observables in the energy range of 200-350 MeV to a reasonable extent, which may excellently explain these data if the energy-dependent Hellmann-like spin-orbit interaction replaces the Hellmann-like spin-orbit interaction. Having success in the present work, we are motivated to upgrade to the polarization effect calculation of the scattered particles for unpolarized incident and target particles. However, calculating polarization will require phase parameter calculation for the triplet state ($^3S_1$ state), even though the specified state cannot have any spin-orbit contribution. We also look forward to working with spin-orbit coupling, including second-order approximation of momentum.

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Conflict of interest

The authors declare that they have no conflict of interest in this work.

1. M. G. Mayer, On closed shells in nuclei, *Physical Review* **74** (1948) 235, https://doi.org/10.1103/PhysRev.74.235.
2. O. Haxel, J. H. D. Jensen, and H. E. Suess, On the "magic numbers" in nuclear structure, *Physical Review* **75** (1949) 1766, https://doi.org/10.1103/PhysRev.75.1766.2.
3. K. S. Krane, Introductory nuclear physics (John-Wiley & Sons, 1991).
4. H. Hong-Xia *et al.*, Influence of Spin-Orbit Force on Nucleon-Nucleon Scattering in the Quark Delocalization Colour Screening Model, *Chinese Physics Letters* **25** (2008) 1617, https://doi.org/10.1088/0256-307X/25/5/026.
5. N. Kaiser, Nuclear spin-orbit interaction from chiral pion-nucleon dynamics, *Nuclear Physics A* **709** (2002) 251, https://doi.org/10.1016/S0375-9474(02)01044-8.
6. M. Sajedi and Z. Kargar, Shifted Deng-Fan potential and cluster structure in 19Ne, *Nuclear Physics A* **1015** (2021) 122314, https://doi.org/10.1016/j.nuclphysa.2021.122314.
7. L. Kumar, A. Khachi, and O. Sastri, Phase Shift Analysis for Neutron-Alpha Elastic Scattering Using Phase Function Method with Local Gaussian Potential, *Journal of Nuclear Physics, Material Sciences, Radiation and Applications* **9** (2022) 215, https://doi.org/10.15415/jnp.2022.92032.
8. F. Brieva and J. Rook, Nucleon-nucleus optical model potential:(III). The spin-orbit component, *Nuclear Physics A* **297** (1978) 206, https://doi.org/10.1016/0375-9474(78)90272-5.
9. P. Signell and R. E. Marshak, Semiphenomenological twonucleon potential, *Physical Review* **109** (1958) 1229, https://doi.org/10.1103/PhysRev.109.1229.
10. J. Gammel and R. Thaler, Spin-orbit coupling in the proton-proton interaction, *Physical Review* **107** (1957) 291, https://doi.org/10.1103/PhysRev.107.291.
11. H. Hellmann, A combined approximation procedure for calculation of energies in the problem of many electrons, *Acta Physicochim URSS* **1** (1935) 913.
12. H. Hellmann, A new approximation method in the problem of many electrons, *The Journal of Chemical Physics* **3** (1935) 61, https://doi.org/10.1063/1.1749559.
13. H. Hellmann and W. Kassatotschkin, Metallic binding according to the combined approximation procedure, *The Journal of Chemical Physics* **4** (1936) 324, https://doi.org/10.1063/1.1749851.
14. G. Koçak, O. Bayrak, and I. Boztosun, Arbitrary -state solution of the Hellmann potential, *Journal of Theoretical and Computational Chemistry* **6** (2007) 893, https://doi.org/10.1142/S0219633607003313.
15. A. K. Roy, A. F. Jalbout, and E. I. Proynov, Accurate calculation of the bound states of Hellmann potential, *Journal of math-*

ematical chemistry **44** (2008) 260, https://doi.org/10.1007/s10910-007-9308-9.

16. J. Adamowski, Bound eigenstates for the superposition of the Coulomb and the Yukawa potentials, *Physical Review A* **31** (1985) 43, https://doi.org/10.1103/PhysRevA.31.43.
17. S. M. Ikhdair and R. Sever, A perturbative treatment for the bound states of the Hellmann potential, *Journal of Molecular Structure: THEOCHEM* **809** (2007) 103, https://doi.org/10.1016/j.theochem.2007.01.019.
18. E. William, E. Inyang, and E. Thompson, Arbitrary l-solutions of the Schrödinger equation interacting with Hulthén-Hellmann potential model, *Rev. Mex. Fis* **66** (2020) 730, https://doi.org/10.24018/ejphysics.2021.3.3.83.
19. C. Edet *et al.*, Any l-state solutions of the Schrodinger equation interacting with Hellmann-Kratzer potential model, *Indian Journal of Physics* **94** (2020) 243, https://doi.org/10.1007/s12648-019-01467-x.
20. C. Onate *et al.*, Eigen solutions and entropic system for Hellmann potential in the presence of the Schrödinger equation, *The European Physical Journal Plus* **132** (2017) 1, https://doi.org/10.1140/epjp/i2017-11729-8.
21. R. L. Hall and Q. D. Katatbeh, Spectral bounds for the Hellmann potential, *Physics Letters A* **287** (2001) 183, https://doi.org/10.1016/S0375-9601(01)00497-2.
22. A. Arda and R. Sever, Pseudospin and spin symmetric solutions of the Dirac equation: Hellmann potential, Wei-Hua potential, Varshni potential, *Zeitschrift für Naturforschung A* **69** (2014) 163, https://doi.org/10.5560/ZNA.2014-0007.
23. P. Okoi, C. Edet, and T. Magu, Relativistic treatment of the Hellmann-generalized Morse potential, *Rev. Mex. Fis* **66** (2020) 1, https://doi.org/10.31349/revmexfis.66.1.
24. S. Hassanababdi *et al.*, Approximate solution of scattering states of the spinless Salpeter equation with the Yukawa potential, *Chinese journal of physics* **52** (2014) 1194, https://doi.org/10.6122/CJP.52.1194.
25. J. Das and S. Chakraborty, Atomic inner-shell ionization, *Physical Review A* **32** (1985) 176, https://doi.org/10.1103/physreva.32.176.
26. J. Callaway and P. Laghos, Application of the pseudopotential method to atomic scattering, *Physical Review* **187** (1969) 192, https://doi.org/10.1103/PhysRev.187.192.
27. A. J. Hughes and J. Callaway, Energy Bands in Body-Centered and Hexagonal Sodium, *Physical Review* **136** (1964) A1390, https://doi.org/10.1103/PhysRev.136.A1390.
28. Y. P. Varshni and R. C. Shukla, Alkali hydride molecules: Potential energy curves and the nature of their binding, *Reviews of Modern Physics* **35** (1963) 130, https://doi.org/10.1103/RevModPhys.35.130.
29. F. Calogero, Variable Phase Approach to Potential Scattering, *vol. 35* (Elsevier, 1967).
30. B. Khirali *et al.*, Phase function metod for elastic nucleon-nucleon scattering using Hellmann plus Coulomb potential, *Rev. Mex. Fis* **69** (2023) 061201, https://doi.org/10.31349/RevMexFis.69.061201.
31. P. Sahoo *et al.*, Nuclear Hulthén potentials for F and G Partial waves, *Research and Reviews: Journal of Physics* **10** (2021) 31, https://doi.org/10.37591/RRJoPHY.
32. A. Behera *et al.*, Study of nucleon-nucleon and alphanucleon elastic scattering by the Manning-Rosen potential, *Communications in Theoretical Physics* **72** (2020) 075301, https://doi.org/10.1088/1572-9494/ab8a1a.
33. G. N. Watson, A treatise on the theory of Bessel functions, **vol. 3** (The University Press, 1922).
34. R. N. Pérez, J. Amaro, and E. R. Arriola, The low-energy structure of the nucleon-nucleon interaction: statistical versus systematic uncertainties, *Journal of Physics G: Nuclear and Particle Physics* **43** (2016) 114001, https://doi.org/10.1088/0954-3899/43/11/114001.