Optical soliton and travelling wave solutions for the wick-type stochastic Fokas-Lenells equation

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In this study, we investigate the perturbed Fokas-Lenells equation with conformable fractional derivatives in the presence of white noise, employing two advanced methodologies. The analysis utilizes Hermite and inverse Hermite transformations within the framework of white noise theory to derive solutions to the model. We also construct traveling wave solutions, optical soliton solutions, and their respective stochastic counterparts.

Keywords: Optical solitons; travelling wave solutions; Fokas-Lenells equation; wick product; Hermite transform.

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1. Introduction

Physical systems use the extended nonlinear Schrödinger equations (NLSEs) with higher-order terms to explain more realistic circumstances. The Fokas-Lenells equation (FLE) is an integrable version of the NLSE that has recently received a great deal of interest [1–5]. The FLE describes a light pulse propagation applicable to a femtosecond or subpicosecond regime.

Randomness is present in all types of physical occurrences in the real world. The phenomenon's internal and external interferences are the source of this type of randomness. Due to the imprecision of experimental data, some random interference items should be incorporated into physical models in order to make the model closer to reality.

We analyze the stochastic form of the FLE with conformable fractional derivatives in this work. Because of the additional random terms, analyzing stochastic equations is more challenging than deterministic equations. Despite the fact that the stochastic equations have some difficulties obtaining explicit solutions due to the additional random terms and Wick product, extracting the Wick-type explicit solutions of the stochastic Wick-type nonlinear partial differential equations is crucial for their analysis. There are few studies on the solutions of stochastic equations, some of which are presented in [6–17]. In this study, we consider the Wick-type stochastic perturbed FLE namely we take into consideration this equation in a white noise environment. We use two powerful analytical methods, white noise theory and Hermite transformations to obtain various solutions for the model. The dimensionless form of FLE in the presence of dual dispersion, including perturbation is structured as:

$$iu_t + a(t)u_{xx} + b(t)u_{xt} + |u|^2\left(c(t)u + ih(t)u_x\right) = i\left[r(t)u_x + s(t)\left(|u|^2 u\right)_x + q(t)\left(|u|^2\right)_x u\right], \quad (1)$$

where the first term represents the temporal evolution of the pulses in birefringent fibers and $u(x,t)$ is the complex valued function that describes the profile of optical pulse. In (1), the coefficients $a(t)$, $b(t), r(t)$, $s(t)$, and $q(t)$ are variable coeffiecients of group velocity dispersion (GVD), spatio-temporal dispersion (STD), inter modal dispersion (IMD), self steepening perturbation term and nonlinear dispersion (ND) respectively.

Using conformable derivatives, Eq. (1) can be written as:

$$iD_t^{\beta}u + a(t)D_x^{2\beta}u + b(t)D_{x,t}^{\beta}u + |u|^2\left(c(t)u + ih(t)D_x^{\beta}u\right) = i\Bigg[r(t)D_x^{\beta}u + s(t)D_x^{\beta}\left(|u|^2 u\right) + q(t)uD_x^{\beta}\left(|u|^2\right)_x\Bigg], \quad (2)$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R}\times\mathbb{R}_+$.

The definition of conformable fractional derivative for the function $f : (0,\infty) \to \mathbb{R}$, with order β is given by [18]:

$$D_{\zeta}^{\beta} f(\zeta) = \lim_{\epsilon\to 0} \frac{f\left(\zeta + \epsilon\zeta^{1-\beta}\right) - f(\zeta)}{\epsilon}, \quad (3)$$

where $0 < \beta \leq 1$ and $\zeta > 0$.

Some properties of these derivatives can be presented as follows:

$$i.)\ D_{\zeta}^{\beta}\zeta^{\kappa} = \kappa\zeta^{\kappa-\beta}, \forall\kappa\in\mathbb{R},$$

$$ii.)\ D_{\zeta}^{\beta}(fg) = f_{\zeta}D^{\beta}g + g_{\zeta}D^{\beta}f,$$

$$iii.)\ D_{\zeta}^{\beta}(f\circ g) = \zeta^{1-\beta}g'(\zeta)f'(g(\zeta)),$$

$$iv.)\ D_{\zeta}^{\beta}\left(\tfrac{f}{g}\right) = \tfrac{g_{\zeta}D^{\beta}f - f_{\zeta}D^{\beta}g}{g^2}.$$

In this work, we study the Wick-type stochastic perturbed FLE with conformable fractional derivative as

$$iD_t^{\beta}U + A(t)\diamond D_x^{2\beta}U + B(t)\diamond D_{x,t}^{\beta}U + |U|^{\diamond 2}\left[C(t)\diamond U + iH(t)\diamond D_x^{\beta}U\right]$$
$$= i\left[R(t)\diamond D_x^{\beta}U + S(t)\diamond D_x^{\beta}(|U|^{\diamond 2}U) + Q(t)\diamond U\diamond D_x^{\beta}(|U|^{\diamond 2})\right], \quad (4)$$

with the wick product "$\diamond$" in the Kondratiev distribution space $(S)_{-1}$ which provides a method for renormalizing some infinite quantities in quantum field theory. In Eq. (4), $A(t)$, $B(t)$, $C(t)$, $H(t)$, $R(t)$, $S(t)$ and $Q(t)$ are Gaussian white noise functions on the Kondratiev distribution space. For additional information on stochastic Kondratiev spaces and the Wick product, see [6].

2. Mathematical analysis

By applying the Hermite transformation into Eq. (4), Wick products are transformed into ordinary products as

$$iD_t^{\beta}\widetilde{U}(x,t,\zeta) + \widetilde{A}(t,\zeta)D_x^{2\beta}\widetilde{U}(x,t,\zeta) + \widetilde{B}(t,\zeta)D_{x,t}^{\beta}\widetilde{U}(x,t,\zeta)$$
$$+ \left|\widetilde{U}(x,t,\zeta)\right|^2\left[\widetilde{C}(t,\zeta)\widetilde{U}(x,t,\zeta) + i\widetilde{H}(t,\zeta)D_x^{\beta}\widetilde{U}(x,t,\zeta)\right]$$
$$= i\left[\widetilde{R}(t,\zeta)D_x^{\beta}\widetilde{U}(x,t,\zeta) + \widetilde{S}(t,\zeta)D_x^{\beta}(\left|\widetilde{U}(x,t,\zeta)\right|^2\widetilde{U}(x,t,\zeta)) + \widetilde{Q}(t,\zeta)\widetilde{U}(x,t,\zeta)\,D_x^{\beta}(\left|\widetilde{U}(x,t,\zeta)\right|^2)\right], \quad (5)$$

where $\zeta = (\zeta_1, \zeta_2, ...) \in (\mathbb{C})$ is a vector parameter. For the sake of simplicity we will take $\widetilde{A}(t,\zeta) = a(t,\zeta)$, $\widetilde{B}(t,\zeta) = b(t,\zeta)$, $\widetilde{C}(t,\zeta) = c(t,\zeta)$, $\widetilde{H}(t,\zeta) = h(t,\zeta)$, $\widetilde{R}(t,\zeta) = r(t,\zeta)$, $\widetilde{S}(t,\zeta) = s(t,\zeta)$, $\widetilde{Q}(t,\zeta) = q(t,\zeta)$ and $\widetilde{U}(x,t,\zeta) = u(x,t,\zeta)$.

Let us consider the solution form as follows

$$u(x,t,\zeta) = P(\psi).e^{i\phi}, \quad (6)$$

where $\psi = \psi(x,t,\zeta)$ and $\phi = \phi(x,t,\zeta)$ are defined as:

$$\psi(x,t,\zeta) = \kappa\left(\frac{x^{\beta}}{\beta}\right) + v\int_0^t \frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau, \quad (7)$$

$$\phi(x,t,\zeta) = \ell\left(\frac{x^{\beta}}{\beta}\right) + w\int_0^t \frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau, \quad (8)$$

respectively. In Eqs. (7) and (8), κ, v, ℓ and w are free constants and nonzero functions θ and Γ are determined later. Applying transformation (6), we obtain the real and imaginary parts of Eq. (5) as

$$\left[a(t,\zeta)\kappa^2 + b(t,\zeta)\kappa v\theta(t,\zeta)\right]\frac{d^2P}{d\psi^2} - \left[w\Gamma(t,\zeta) + a(t,\zeta)\ell^2 + b(t,\zeta)\ell w\Gamma(t,\zeta) - r(t,\zeta)\ell\right]P$$
$$+ \left[c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\right]P^3 = 0, \quad (9)$$

and

$$\theta(t,\zeta)v + 2a(t,\zeta)\kappa\ell - r(t,\zeta)\kappa + b(t,\zeta)\left[\Gamma(t,\zeta)\kappa w + \theta(t,\zeta)\ell v\right] + \left[h(t,\zeta) + 3s(t,\zeta) - 2q(t,\zeta)\right]\kappa P^2 = 0, \quad (10)$$

respectively. The imaginary part implies

$$\Gamma(t,\zeta) = \frac{\kappa\left(r(t,\zeta) - 2a(t,\zeta)\ell\right) - v\theta(t,\zeta)\left(a(t,\zeta) + b(t,\zeta)\ell\right)}{\kappa b(t,\zeta)w}, \quad (11)$$

and

$$h(t,\zeta) = 2q(t,\zeta) - 3s(t,\zeta). \quad (12)$$

Equations (11) and (12) are true for all of the solutions presented in this study.

2.1. Traveling wave solutions by extended (G'/G)-expansion technique

Using the extended G'/G-expansion method the solution to Eq. (5) given as

$$P(\psi) = \sigma_0(t,\zeta) + \sum_{j=1}^{n} \sigma_i(t,\zeta)\left(\frac{G'}{G}\right)^{j} + \eta_i(t,\zeta)\left(\frac{G'}{G}\right)^{-j}, \tag{13}$$

where the functions $\sigma_0, \sigma_j, \eta_j$ will be determined. $G = G(\psi)$ satisfies the following second order linear differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{14}$$

where λ and μ are constants. Using the homogeneous balance between the terms $d^2P/d\psi^2$ and P^3 appearing in Eq. (9), the solution form can be written as

$$P(\psi) = \sigma_0(t,\zeta) + \sigma_1(t,\zeta)\left(\frac{G'}{G}\right) + \eta_1(t,\zeta)\left(\frac{G'}{G}\right)^{-1}. \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (9), equating the coefficients of (G'/G) with the same powers we get the following solution sets.

Set 1

$$\begin{aligned}
&\lambda \neq 0,\ b(t,\zeta) = -\frac{1}{\ell},\ c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\left(\lambda^2 - 4\mu\right) \neq 0, \quad b(t,\zeta)\theta(t,\zeta)\kappa \neq 0,\\
&\sigma_0(t,\zeta) = \pm\frac{\lambda\sqrt{\ell\left(r(t,\zeta) - a(t,\zeta)\ell\right)}}{\sqrt{(\lambda^2 - 4\mu)\left[c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\right]}}, \quad \sigma_1(t,\zeta) = \frac{2\sigma_0(t,\zeta)}{\lambda}, \quad \sigma_0(t,\zeta) \neq 0,\\
&\theta(t,\zeta) = -\frac{-2\kappa^2\left(\lambda^2 - 4\mu\right)a(t,\zeta) - 4\ell\left[r(t,\zeta) - a(t,\zeta)\ell\right]}{2\kappa v\left(\lambda^2 - 4\mu\right)b(t,\zeta)}, \quad \eta_1(t,\zeta) = 0.
\end{aligned} \tag{16}$$

Set 2

$$\begin{aligned}
&\lambda \neq 0,\ \Gamma(t,\zeta)\ell \neq 0,\ b(t,\zeta) = -\frac{1}{\ell}, \quad \ell^4 + 4\kappa^2\ell^2\sigma - 32\kappa^4\sigma^2 \neq 0,\\
&\sigma_0(t,\zeta) = \pm\frac{\lambda\sqrt{\ell\left(r(t,\zeta) - a(t,\zeta)\ell\right)}}{\sqrt{(\lambda^2 - 4\mu)\left[c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\right]}}, \quad \eta_1(t,\zeta) = \frac{2\mu\sigma_0(t,\zeta)}{\lambda}, \quad \sigma_0(t,\zeta) \neq 0,\\
&\theta(t,\zeta) = -\frac{-\left(2\lambda^2 - 4\mu\right)\ell\left[r(t,\zeta) - a(t,\zeta)\ell\right] - 2a(t,\zeta)\left(\lambda^2 - 4\mu\right)\kappa^2\mu}{2\kappa v\mu b(t,\zeta)}, \quad \sigma_1(t,\zeta) = 0.
\end{aligned} \tag{17}$$

Remark 1: In set 1, the sign of σ_1 changes depending on the sign of σ_0, and similarly, in set 2, the sign of η_1 changes based on the sign of σ_0.

Using the above solution sets and solutions of Eq. (14), and substituting Eq. (15) into Eq. (9), the following solutions are found.

*For **set 1**:*

i. The case $\lambda^2 - 4\mu > 0$:

We get the hyperbolic function traveling wave solution as

$$\begin{aligned}
u_1(x,t,\zeta) = \pm&\frac{\sqrt{\ell\lambda^2\left(r(t,\zeta) - a(t,\zeta)\ell\right)}\sqrt{\lambda^2 - 4\mu}\left(E\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right) + F\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right)}{\lambda\sqrt{(\lambda^2 - 4\mu)\left(c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\right)}\left(E\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right) + F\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right)}\\
&\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\},
\end{aligned} \tag{18}$$

where it has the following necessary conditions:

$$\ell\left(r(t,\zeta) - a(t,\zeta)\ell\right) > 0, \tag{19}$$

$$\left(c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell\right) > 0. \tag{20}$$

ii. The case $\lambda^2 - 4\mu < 0$:

We obtain trigonometric function traveling wave solution as

$$u_2(x,t,\zeta) = \pm \frac{\sqrt{\ell\lambda^2 \left(r(t,\zeta) - a\left(t,\zeta\right)\ell\right)}\sqrt{4\mu-\lambda^2}\left(F\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right) - E\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\right)}{\lambda\sqrt{\left(\lambda^2-4\mu\right)\left(c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell\right)}\left(E\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right) + F\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\right)}$$
$$\times \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{21}$$

with conditions (18) and

$$\left(c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell\right) < 0. \tag{22}$$

In Eqs. (18) and (21), E and F are arbitrary constants and $\psi = \psi\left(x,t,\zeta\right)$ and $\Gamma(t,\zeta)$ are defined as:

$$\psi\left(x,t,\zeta\right) = \kappa\left(\frac{x^\beta}{\beta}\right) + v\int_0^t \frac{-2a\left(\tau,\zeta\right)\kappa^2\left(\lambda^2-4\mu\right) - 4\ell\left[r(\tau,\zeta) - a\left(\tau,\zeta\right)\ell\right]}{2b(\tau,\zeta)\kappa v\left(\lambda^2-4\mu\right)\tau^{1-\beta}}d\tau, \tag{23}$$

and

$$\Gamma\left(\tau,\zeta\right) = \frac{\begin{array}{c}\kappa\left(r\left(\tau,\zeta\right) - 2a(\tau,\zeta)\ell\right)\left[2b(\tau,\zeta)\kappa v\left(\lambda^2-4\mu\right)\right] + \\ v\left[2a\left(\tau,\zeta\right)\kappa^2\left(\lambda^2-4\mu\right) + 4\ell\left[r(\tau,\zeta) - a\left(\tau,\zeta\right)\ell\right]\right]\left(a(\tau,\zeta) + b(\tau,\zeta)\ell\right)\end{array}}{2b(\tau,\zeta)\kappa^2 vw\left(\lambda^2-4\mu\right)}. \tag{24}$$

When $E = 0$ and $F \neq 0$, the solutions (18) and (21) convert to

$$u_{1,1}(x,t,\zeta) = \pm \frac{\sqrt{\ell\left(r(t,\zeta) - a\left(t,\zeta\right)\ell\right)}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)}{\sqrt{c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell}}\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{25}$$

$$u_{2,1}(x,t,\zeta) = \pm \frac{\sqrt{\ell\left(r(t,\zeta) - a\left(t,\zeta\right)\ell\right)}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)}{\sqrt{-\left(c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell\right)}}\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{26}$$

respectively.

On the other hand, if we take $E \neq 0$ and $F = 0$, then solutions (18) and (21) become

$$u_{1,2}(x,t,\zeta) = \pm \frac{\sqrt{\ell\left(r(t,\zeta) - a\left(t,\zeta\right)\ell\right)}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)}{\sqrt{\left(c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell\right)}}\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{27}$$

and

$$u_{2,2}(x,t,\zeta) = \mp \frac{\sqrt{\ell\left(r(t,\zeta) - a\left(t,\zeta\right)\ell\right)}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)}{\sqrt{-\left(c(t,\zeta) - h\left(t,\zeta\right)\ell + s(t,\zeta)\ell\right)}}\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{28}$$

respectively.

*For **set 2:***

i. The case $\lambda^2 - 4\mu > 0$:

The Eq. (4) admits the following hyperbolic function traveling wave solution

$$u_3(x,t,\zeta)=\pm\frac{\sqrt{\ell\lambda^2\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\begin{array}{c}\left(-E\lambda\sqrt{\lambda^2-4\mu}+F\left(\lambda^2-4\mu\right)\right)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\\+\left(-F\lambda\sqrt{\lambda^2-4\mu}+E\left(\lambda^2-4\mu\right)\right)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\end{array}\right]}{\lambda\sqrt{\left(\lambda^2-4\mu\right)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left(\begin{array}{c}\left(F\lambda-E\sqrt{\lambda^2-4\mu}\right)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\\+\left(E\lambda-F\sqrt{\lambda^2-4\mu}\right)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\end{array}\right)}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{29}$$

with necessary conditions (19) and (20).

ii. The case $\lambda^2-4\mu<0$:

The Eq. (4) admits the following trigonometric function traveling wave solution

$$u_4(x,t,\zeta)=\pm\frac{\sqrt{\ell\lambda^2\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\begin{array}{c}\left(E\left(\lambda^2-4\mu\right)-F\lambda\sqrt{4\mu-\lambda^2}\right)\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\\+\left(F\left(\lambda^2-4\mu\right)+E\lambda\sqrt{4\mu-\lambda^2}\right)\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\end{array}\right]}{\lambda\sqrt{\left(\lambda^2-4\mu\right)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left(\begin{array}{c}\left(E\lambda-F\sqrt{4\mu-\lambda^2}\right)\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\\+\left(F\lambda+E\sqrt{4\mu-\lambda^2}\right)\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\end{array}\right)}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{30}$$

with conditions (19) and (22).

In Eqs. (29) and (30), Γ is defined in Eq. (24) and ψ is defined as follows

$$\psi\left(x,t,\zeta\right)=\kappa\left(\frac{x^\beta}{\beta}\right)+v\int_0^t\frac{-2a\left(\tau,\zeta\right)\kappa^2\mu\left(\lambda^2-4\mu\right)-\left(2\lambda^2-4\mu\right)\ell\left[r(\tau,\zeta)-a\left(\tau,\zeta\right)\ell\right]}{2b(\tau,\zeta)\kappa v\mu\tau^{1-\beta}}d\tau. \tag{31}$$

Specifically, taking $E=0$ and $F\neq0$, results in the solutions (29) and (30) becoming

$$u_{3,1}(x,t,\zeta)=\pm\frac{\sqrt{\ell\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\lambda^2-4\mu-\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}{\sqrt{\left(\lambda^2-4\mu\right)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left[\lambda-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{32}$$

$$u_{4,1}(x,t,\zeta)=\pm\frac{\sqrt{\ell\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\lambda^2-4\mu-\lambda\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}{\sqrt{\left(\lambda^2-4\mu\right)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left[\lambda-\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}. \tag{33}$$

Then if we take $F=0$ and $E\neq 0$, the solutions (29) and (30) become

$$u_{3,2}(x,t,\zeta)=\pm\frac{\sqrt{\ell\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\lambda^2-4\mu-\lambda\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}{\sqrt{(\lambda^2-4\mu)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left[\lambda-\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{34}$$

$$u_{4,2}(x,t,\zeta)=\pm\frac{\sqrt{\ell\left(r(t,\zeta)-a\left(t,\zeta\right)\ell\right)}\left[\lambda^2-4\mu-\lambda\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}{\sqrt{(\lambda^2-4\mu)\left[c(t,\zeta)-h\left(t,\zeta\right)\ell+s(t,\zeta)\ell\right]}\left[\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right]}$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau,\zeta\right)}{\tau^{1-\beta}}d\tau\right)\right\}. \tag{35}$$

Remark 2: In the case of $\lambda^2-4\mu=0$, we do not have any solution for Eq. (4).

2.1.1. *Wick-type stochastic traveling wave solutions*

The theorem 4.1.1, proved by Holden *et al*. in [6], has been used in this part. Wick-type stochastic solutions of Eq. (4) are obtained, by applying the inverse Hermite transform to the above solutions

i. The case $\lambda^2-4\mu>0$:

$$U_1(x,t)=\pm\frac{\sqrt{\ell\lambda^2\left(R(t)-A\left(t\right)\ell\right)}\diamond\sqrt{\lambda^2-4\mu}\left(E\cosh^\diamond\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)+F\sinh^\diamond\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right)}{\lambda\sqrt{(\lambda^2-4\mu)\left[C(t)-H\left(t\right)\ell+S(t)\ell\right]}\diamond\left(E\sinh^\diamond\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)+F\cosh^\diamond\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\right)}$$
$$\diamond\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{36}$$

which is restricted the conditions

$$\ell\left(R(t)-A\left(t\right)\ell\right)>0, \tag{37}$$
$$\left[C(t)-H\left(t\right)\ell+S(t)\ell\right]>0. \tag{38}$$

ii. The case $\lambda^2-4\mu<0$:

$$U_2(x,t)=\pm\frac{\sqrt{\ell\lambda^2\left(R(t)-A\left(t\right)\ell\right)}\diamond\sqrt{4\mu-\lambda^2}\left(F\cos^\diamond\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)-E\sin^\diamond\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\right)}{\lambda\sqrt{(\lambda^2-4\mu)\left[C(t)-H\left(t\right)\ell+S(t)\ell\right]}\diamond\left(E\cos^\diamond\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)+F\sin^\diamond\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\right)}$$
$$\diamond\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{39}$$

with conditions (37) and

$$\left[C(t)-H\left(t\right)\ell+S(t)\ell\right]<0. \tag{40}$$

In Eqs. (36) and (39), ψ and Γ are defined as follows:

$$\psi(x,t)=\kappa\left(\frac{x^{\beta}}{\beta}\right)+v\int_0^t\frac{-2A(\tau)\kappa^2\left(\lambda^2-4\mu\right)-4\ell\left[R(\tau)-A(\tau)\ell\right]}{2B(\tau)\kappa v\left(\lambda^2-4\mu\right)\tau^{1-\beta}}d\tau, \tag{41}$$

$$\Gamma(\tau)=\frac{\begin{array}{c}\kappa\left(R(\tau)-2A(\tau)\ell\right)\diamond\left[2B(\tau)\kappa v\left(\lambda^2-4\mu\right)\right]+\\ v\left[2A(\tau)\kappa^2\left(\lambda^2-4\mu\right)+4\ell\left[R(\tau)-A(\tau)\ell\right]\right]\diamond\left(A(\tau)+B(\tau)\ell\right)\end{array}}{2B^2(\tau)\kappa^2 vw\left(\lambda^2-4\mu\right)}. \tag{42}$$

When $E=0$ and $F\neq 0$, the solutions (36) and (39) reduce to

$$U_{1,1}(x,t)=\pm\frac{\sqrt{\ell\left(R(t)-A(t)\ell\right)}\diamond\tanh^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)}{\sqrt{C(t)-H(t)\ell+S(t)\ell}}\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{43}$$

$$U_{2,1}(x,t)=\pm\frac{\sqrt{\ell\lambda^2\left(R(t)-A(t)\ell\right)}\diamond\cot^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)}{\lambda\sqrt{-\left(C(t)-H(t)\ell+S(t)\ell\right)}}\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{44}$$

respectively. On the other hand, if we take $E\neq 0$ and $F=0$, then solutions (36) and (39) become

$$U_{1,2}(x,t)=\pm\frac{\sqrt{\ell\left(R(t)-A(t)\ell\right)}\diamond\coth^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)}{\sqrt{C(t)-H(t)\ell+S(t)\ell}}\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{45}$$

$$U_{2,2}(x,t)=\mp\frac{\sqrt{\ell\left(R(t)-A(t)\ell\right)}\diamond\tan^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)}{\sqrt{-\left(C(t)-H(t)\ell+S(t)\ell\right)}}\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{46}$$

respectively.

Similarly, for set 2, we acquire the following Wick-type stochastic hyperbolic and trigonometric solutions to Eq. (4),

$$U_3(x,t)=\pm\frac{\sqrt{\ell\lambda^2\left(R(t)-A(t)\ell\right)}\diamond\left[\begin{array}{c}\left(-E\lambda\sqrt{\lambda^2-4\mu}+F\left(\lambda^2-4\mu\right)\right)\cosh^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\\+\left(-F\lambda\sqrt{\lambda^2-4\mu}+E\left(\lambda^2-4\mu\right)\right)\sinh^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\end{array}\right]}{\lambda\sqrt{\left(\lambda^2-4\mu\right)\left[C(t)-H(t)\ell+S(t)\ell\right]}\diamond\left(\begin{array}{c}\left(F\lambda-E\sqrt{\lambda^2-4\mu}\right)\cosh^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\\+\left(E\lambda-F\sqrt{\lambda^2-4\mu}\right)\sinh^{\diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\psi\right)\end{array}\right)}$$
$$\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{47}$$

$$U_4(x,t)=\pm\frac{\sqrt{\ell\lambda^2\left(R(t)-A(t)\ell\right)}\diamond\left[\begin{array}{c}\left(E\left(\lambda^2-4\mu\right)-F\lambda\sqrt{4\mu-\lambda^2}\right)\cos^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\\+\left(F\left(\lambda^2-4\mu\right)+E\lambda\sqrt{4\mu-\lambda^2}\right)\sin^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\end{array}\right]}{\lambda\sqrt{\left(\lambda^2-4\mu\right)\left[C(t)-H(t)\ell+S(t)\ell\right]}\diamond\left(\begin{array}{c}\left(E\lambda-F\sqrt{4\mu-\lambda^2}\right)\cos^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\\+\left(F\lambda+E\sqrt{4\mu-\lambda^2}\right)\sin^{\diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\psi\right)\end{array}\right)}$$
$$\diamond\exp\left\{i\left(\ell\left(\frac{x^{\beta}}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{48}$$

where Γ is defined in (42) and ψ is defined as follows:

$$\psi(x,t)=\kappa\left(\frac{x^{\beta}}{\beta}\right)+v\int_0^t\frac{-2\kappa^2\mu\left(\lambda^2-4\mu\right)A(\tau)-\left(2\lambda^2-4\mu\right)\ell\left[R(\tau)-A(\tau)\ell\right]}{2\kappa v\mu B(\tau)\tau^{1-\beta}}d\tau. \tag{49}$$

Eq. (47) is restricted with conditions (37) and (38) for the case $\lambda^2 - 4\mu > 0$. Similarly Eq. (48) is restricted with conditions (37) and (40) for the case $\lambda^2 - 4\mu < 0$.

The following dark and singular optical soliton solutions result from the solutions (47) and (48) when $E = 0$ and $F \neq 0$:

$$U_{3,1}(x,t) = \Bigg\{ \pm \frac{\lambda\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \pm \frac{\mu\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \diamond \left[-\lambda + \sqrt{\lambda^2 - 4\mu}\tanh^{\diamond}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\psi\right)\right]\Bigg\} \diamond \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{50}$$

$$U_{4,1}(x,t) = \Bigg\{ \pm \frac{\lambda\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \pm \frac{\mu\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \diamond \left[-\lambda - \sqrt{4\mu - \lambda^2}\cot^{\diamond}\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\psi\right)\right]\Bigg\} \diamond \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}. \tag{51}$$

Then if we take $F = 0$ and $E \neq 0$, the solutions (47) and (48) become

$$U_{3,2}(x,t) = \Bigg\{ \pm \frac{\lambda\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \pm \frac{\mu\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \diamond \left[-\lambda + \sqrt{\lambda^2 - 4\mu}\coth^{\diamond}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\psi\right)\right]\Bigg\} \diamond \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{52}$$

and

$$U_{4,2}(x,t) = \Bigg\{ \pm \frac{\lambda\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \pm \frac{\mu\sqrt{\ell\left(R(t) - A\left(t\right)\ell\right)}}{\sqrt{\left(\lambda^2 - 4\mu\right)\left[C(t) - H\left(t\right)\ell + S(t)\ell\right]}} \diamond \left[-\lambda - \sqrt{4\mu - \lambda^2}\tan^{\diamond}\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\psi\right)\right]\Bigg\} \diamond \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma\left(\tau\right)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{53}$$

respectively.

2.2. Bright and dark optical soliton solutions by Jacobi elliptic function ansatz technique

We will use the Jacobi elliptic sine and cosine function to get explicit solutions of Eq. (4) in this part. We assume that

$$P\left(\psi\right) = \lambda\left(t,\zeta\right) sn^{\rho}\left(\psi,m\right), \tag{54}$$

where sn is the Jacobi elliptic sine function, $m \in (0,1)$ is the modulus of this function, while $\lambda(t,\zeta)$ and ρ are determined later.

Using the hypothesis (54) , Eq. (9) reduces to

$$\Big[a(t,\zeta)\kappa^2 + b(t,\zeta)\kappa v\theta\left(t,\zeta\right)\Big]\Big\{\left(\rho-1\right)\rho\lambda\left(t,\zeta\right)sn^{\rho-2}\left(\psi,m\right) - \rho\left[m^2\left(\rho-1\right)+m+\rho\right]\lambda\left(t,\zeta\right)sn^{\rho}\left(\psi,m\right) + m\rho\left(m\rho+1\right)\lambda\left(t,\zeta\right)sn^{\rho+2}\left(\psi,m\right)\Big\} - \left[w\Gamma\left(t,\zeta\right) + a(t,\zeta)\ell^2 + b(t,\zeta)\ell w\Gamma(t,\zeta) - r(t,\zeta)\ell\right]\lambda\left(t,\zeta\right)sn^{\rho}\left(\psi,m\right) + \left[c\left(t,\zeta\right) - h(t,\zeta)\ell + s\left(t,\zeta\right)\ell\right]\lambda^3\left(t,\zeta\right)sn^{3\rho}\left(\psi,m\right) = 0. \tag{55}$$

Matching the exponents of $sn^{\rho+2}\left(\psi,m\right)$ and $sn^{3\rho}\left(\psi,m\right)$ in the above equation, yields

$$3\rho = \rho + 2, \tag{56}$$

which gives

$$\rho = 1. \tag{57}$$

In (55), equating coefficients of $sn^{\rho+j}(\psi,m)$, for $j=-2,0$, to zero we get

$$\lambda(t,\zeta)=\sqrt{-\frac{m(m+1)\left[a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)\right]}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}}, \tag{58}$$

$$\theta(t,\zeta)=-\frac{a(t,\zeta)(m+1)\kappa^2+w\Gamma(x,t)\left[b(t,\zeta)\ell+1\right]+a(t,\zeta)\ell^2-r(t,\zeta)\ell}{b(t,\zeta)(m+1)\kappa v}, \tag{59}$$

which requires the constraint

$$\frac{a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}<0. \tag{60}$$

So, we have the Jacobi elliptic function solution

$$u_1(x,t,\zeta)=\sqrt{-\frac{m(m+1)\left[a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)\right]}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}}sn\left(\kappa\left(\frac{x^\beta}{\beta}\right)+v\int_0^t\frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau,m\right)$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{61}$$

where $\Gamma(t,\zeta)$ and $\theta(t,\zeta)$ are defined in (11) and (59) respectively.

From Eq. (61), we get the dark optical soliton solution given below when $m\to1$, as

$$u_1(x,t,\zeta)=\sqrt{-\frac{2\kappa\left[a(t,\zeta)\kappa+b(t,\zeta)v\theta(t,\zeta)\right]}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}}\tanh\left(\kappa\left(\frac{x^\beta}{\beta}\right)+v\int_0^t\frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)$$
$$\times\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}. \tag{62}$$

Now, we assume that

$$P(\psi)=\lambda(t,\zeta)\,cn^\rho(\psi,m). \tag{63}$$

Thus real part reduces to

$$\begin{aligned}&\left[a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)\right]\Big\{(\rho-1)\rho\left(1-m^2\right)\lambda(t,\zeta)\,cn^{\rho-2}(\psi,m)+\rho\left[m^2(2\rho-1)+m-\rho\right]\\&\quad\times\lambda(t,\zeta)\,cn^\rho(\psi,m)-m\rho(m\rho+1)\lambda(t,\zeta)\,cn^{\rho+2}(\psi,m)\Big\}\\&\quad-\left[w\Gamma(t,\zeta)+a(t,\zeta)\ell^2+b(t,\zeta)\ell w\Gamma(t,\zeta)-r(t,\zeta)\ell\right]\lambda(t,\zeta)\,cn^\rho(\psi,m)\\&\quad+\left[c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell\right]\lambda^3(t,\zeta)\,cn^{3\rho}(\psi,m)=0.\end{aligned} \tag{64}$$

If necessary operations are performed for coefficients and exponents in (64), we get the value in (57) and also

$$\lambda(t,\zeta)=\sqrt{\frac{m(m+1)\left[a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)\right]}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}}, \tag{65}$$

$$\theta(t,\zeta)=\frac{a(t,\zeta)\ell^2-a(t,\zeta)\left(m^2+m+1\right)\kappa^2+w\Gamma(x,t)\left[b(t,\zeta)\ell+1\right]+-r(t,\zeta)\ell}{b(t,\zeta)(m^2+m+1)\kappa v}, \tag{66}$$

which requires the constraint

$$\frac{a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}>0. \tag{67}$$

So the Jacobi elliptic function solution is obtained as

$$u_2(\psi)=\sqrt{\frac{m(m+1)\left[a(t,\zeta)\kappa^2+b(t,\zeta)\kappa v\theta(t,\zeta)\right]}{c(t,\zeta)-h(t,\zeta)\ell+s(t,\zeta)\ell}}cn\left(\kappa\left(\frac{x^\beta}{\beta}\right)+v\int_0^t\frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau,m\right)$$
$$\times\exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right)+w\int_0^t\frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{68}$$

and then the dark optical soliton solution is acquired as follows when the modulus $m \to 1$,

$$u_2(x,t,\zeta) = \sqrt{\frac{2\kappa\left[a(t,\zeta)\kappa + b(t,\zeta)v\theta(t,\zeta)\right]}{c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell}}\,\sec h\left(\kappa\left(\frac{x^\beta}{\beta}\right) + v\int_0^t \frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)$$
$$\times \exp\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}. \tag{69}$$

2.2.1. Wick-type stochastic bright and dark soliton solutions

Equations (62) and (69) turn into following dark and bright optical stochastic soliton solutions using inverse Hermit transformations:

$$U_{1^*}(x,t) = \sqrt{-\frac{2\kappa\left[a(t,\zeta)\kappa + b(t,\zeta)\diamond v\theta(t,\zeta)\right]}{c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell}} \diamond \tanh^\diamond\left(\kappa\left(\frac{x^\beta}{\beta}\right) + v\int_0^t \frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)$$
$$\diamond \exp^\diamond\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{70}$$

$$U_{2^*}(x,t) = \sqrt{\frac{2\kappa\left[a(t,\zeta)\kappa + b(t,\zeta)\diamond v\theta(t,\zeta)\right]}{c(t,\zeta) - h(t,\zeta)\ell + s(t,\zeta)\ell}} \diamond \sec h^\diamond\left(\kappa\left(\frac{x^\beta}{\beta}\right) + v\int_0^t \frac{\theta(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)$$
$$\diamond \exp^\diamond\left\{i\left(\ell\left(\frac{x^\beta}{\beta}\right) + w\int_0^t \frac{\Gamma(\tau,\zeta)}{\tau^{1-\beta}}d\tau\right)\right\}, \tag{71}$$

respectively.

3. Conclusion

The present article aims to analyze the Wick-type stochastic FLE with conformable fractional derivatives. It is performed conversion between Wick products and ordinary products via Hermit and inverse Hermit transformations. Traveling wave solutions, bright and dark optical soliton solutions, and stochastic solutions of the model are obtained using the extended-G'/G and Jacobi elliptic functions ansatz methods. The constraint conditions for the existence of solitons are also presented. The results in this paper will provide important assistance to the study of soliton dynamical properties in a white noise environment. Furthermore, we believe that the approaches adopted in this article will inspire future researchers who will conduct research to address other issues of a similar nature.

Data availibility statement

Not applicable.

Conflict of interest

Not applicable.

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