Cross section analysis of neutron-light nuclei systems using Modified Pöschl-Teller potential

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The regular and irregular/Jost solutions of the Schrödinger equation with the Modified Pöschl-Teller potential are presented by implementing the differential equation technique to the problem. In this work the said potential is parameterized for nuclear systems by exploiting Jost formalism to estimate bound state energies and the scattering phase shifts. The results are in line with previous theoretical and experimental observations. The total elastic scattering cross sections are being calculated using the phase parameters.

Keywords: Modified Pöschl-Teller potential; jost function; scattering phase shifts; scattering cross section; $(n-d)$ and $(n{-}\mathrm{He}^3)$ systems.

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1. Introduction

Studying the energy spectra and wave functions of a quantum system under various potentials is an intriguing area of nonrelativistic quantum scattering theory [1], as it provides essential information about the system's behavior and properties. The Pöschl-Teller potential [2] is used to describe the vibrational behavior of diatomic molecules. It is particularly useful for describing bound states and vibrational spectra. In our context, the Pöschl-Teller potential has been modified [3-6] to suit the nuclear domain. This adaptation likely involves changes to the potential parameters or functional form to more accurately describe nucleon-nucleus interactions. The Jost function [7-9] is a mathematical tool used in scattering theory to analyze how particles interact with a potential. In our work, it is employed to handle the complexities of the nucleon-nucleus interaction under the modified Pöschl-Teller potential [3-6]. The Pöschl-Teller potential is indeed one of the potentials for which the Schrödinger equation can be solved exactly. The Pöschl-Teller potential is particularly interesting because it has applications in various areas of physics. Several researchers have studied the Pöschl-Teller potential in recent years [10-21]. Perturbation calculations and approximation techniques are crucial tools in solving the radial Schrödinger equation, especially for cases where exact analytical solutions are challenging to obtain. These methods are part of a broader toolkit for addressing complex quantum mechanical problems. The N-U method [22], Supersymmetric quantum mechanics [16,23,24], the screened centrifugal barrier [25-31] provide various ways to handle the challenges posed by different types of potentials and boundary conditions, allowing researchers to gain insights into the physical systems under study. While the harmonic oscillator potential provides a useful starting point for understanding vibrational modes in diatomic molecules, anharmonic potentials are essential for a more accurate description of molecular vibrations, especially at higher energy states. These potentials, such as the Morse potential [32], address the limitations of the harmonic approximation and provide a more complete picture of molecular behavior. The Pöschl-Teller potential remains a significant model in quantum mechanics due to its mathematical properties and its applicability in both nonrelativistic and relativistic contexts. The use of various approximation techniques and exact methods allows researchers [3-6,17,33-46] to explore a wide range of physical phenomena associated with this potential. Dong *et al*. [14] used traditional methods to solve the Schrödinger equation with the Pöschl-Teller potential and Infield *et al.* [14] used the factorization method. Aktas and Sever [16] and Diaz *et al*. [34] employed the SUSY QM techniques to solve the modified Pöschl-Teller potential. The modified Pöschl-Teller potential has found significant applications in various physical and theoretical contexts, including clathrate quantum statistical mechanics [47] and strong-field ionization dynamics [48]. The study of the Pöschl-Teller potential and its modifications extends to various advanced analytical techniques and methodologies, including the calculation of matrix elements for powers of x-dependent operators [49] within the framework of the Pöschl-Teller potential and applying quantization conditions [50,51] to find discrete energy levels of the system. The study of s-wave $(\ell = 0)$ [37,52-54] solutions for the Schrödinger equation with various potentials, including the Pöschl-Teller potential, is a well-explored area in quantum mechanics. The references [13,55-57] provide valuable insights into how to effectively approximate solutions for quantum systems involving the centrifugal term. Based on the above description the paper is organized, which contains methodology, results and discussion followed by conclusion.

2. Jost function for the Modified Pöschl-Teller potential

The modified Pöschl-Teller potential [3-6] is provided as

$$V(s) = -\frac{V_0}{\cosh^2(\alpha s)}, \tag{1}$$

where V_0 is the strength and α is related with the range of the potential.

The Schrödinger equation for this potential has the following form:

$$\left[\frac{d^2}{ds^2} + \frac{V_0}{\cosh^2(\alpha s)} - \frac{\ell(\ell+1)}{s^2}\right]\phi_\ell(\chi, s) = 0, \tag{2}$$

where $\chi^2 = 2mE/\hbar^2$, define the centre of mass momenta. To find the analytical solution, the suitable approximation to the centrifugal term [25] is considered as $1/s^2 = \alpha^2/\sinh^2(\alpha s)$. By changing a new variable $z = \tanh^2(\alpha s)$, the Eq. (2) is then rearranged as

$$4\alpha^2 z(1-z)^2\frac{d^2\phi_\ell(z)}{dz^2} + 2\alpha^2(1-z)^2\frac{d\phi_\ell(z)}{dz} + \left(\chi^2 + V_0(1-z) - \alpha^2\ell(\ell+1)\left[\frac{1}{z} - 1\right]\right)\phi_\ell(z) = 0. \tag{3}$$

Using the following trial wave function

$$\phi_\ell(\chi, z) = \frac{1}{\alpha}z^\delta(1-z)^\beta R(\chi, z), \tag{4}$$

in the Eq. (3), one obtains

$$z(1-z)R''(\chi,z) + \left(\left[2\delta + \frac{1}{2}\right] - z\left[2\delta + 2\beta + \frac{3}{2}\right]\right)R'(\chi,z) + \left(\left[-2\delta\beta - \frac{\beta}{2} - \delta + \frac{V_0}{4\alpha^2} - \delta(\delta-1) - \frac{\delta}{2} - \beta(\beta-1) - \beta\right] + \left[\delta(\delta-1) + \frac{\delta}{2} - \frac{\ell(\ell+1)}{4}\right]\frac{1}{z} + \left[\beta^2 + \frac{\chi^2}{4\alpha^2}\right]\frac{1}{1-z}\right)R(\chi,z) = 0. \tag{5}$$

To come up with the basic form of hypergeometric differential equation [58-61]

$$z(1-z)R''(\chi,z) + \left[C - (A+B+1)z\right]R'(\chi,z) + ABR(\chi,z) = 0, \tag{6}$$

the third term of Eq. (5) must be free of the dependence "z". The requirements are as follows:

$$\left(\delta(\delta-1) + \frac{\delta}{2} - \frac{\ell(\ell+1)}{4}\right) = 0, \tag{7}$$

and

$$\left(\beta^2 + \frac{\chi^2}{4\alpha^2}\right) = 0. \tag{8}$$

The values of δ and β are obtained as $\delta = 1/4 \pm (1/2)\sqrt{(1/4) + \ell(\ell+1)}$ and $\beta = \pm(i\chi/2\alpha)$. The quantities δ and β have two outcomes. Choosing $\delta = 1/4 + (1/2)\sqrt{(1/4) + \ell(\ell+1)}$ and $\beta = -(i\chi/2\alpha)$, Eq. (5) yields

$$z(1-z)R''(\chi,z) + \left(\left[2\delta + \frac{1}{2}\right] - z\left[2\delta + 2\beta + \frac{3}{2}\right]\right)R'(\chi,z) - \left(2\delta\beta + \frac{\beta}{2} + \delta^2 - \frac{V_0}{4\alpha^2} + \frac{\delta}{2} + \beta^2\right)R(\chi,z) = 0. \tag{9}$$

Equations (6) and (9) in comparison yield

$$A = \delta + \beta + \frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + \frac{V_0}{\alpha^2}}, \tag{10}$$

$$B = \delta + \beta + \frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4} + \frac{V_0}{\alpha^2}}, \tag{11}$$

and

$$C = 2\delta + \frac{1}{2}. \tag{12}$$

Solution to the Eq. (9), in its conventional form is ${}_2F_1(A,B:C;z)$. Therefore the expression for the regular solution for Pöschl-Teller potential will be expressed from Eq. (4) using the values of A, B, C, δ and β as

$$\phi_\ell(\chi,z)=\frac{1}{\alpha}(\tanh^2(\alpha s))^\delta(1-\tanh^2(\alpha s))^\beta{}_2F_1(A,B:C;\tanh^2(\alpha s)) \tag{13}$$

To get the irregular solution from Eq. (13), one can apply the transformation formulae [58,59]

$$\begin{aligned}{}_2F_1(m,p;d;x)&=\frac{\Gamma(d)\Gamma(d-m-p)}{\Gamma(d-m)\Gamma(d-p)}{}_2F_1(m,p;m+p-d+1;1-x)\\&+(1-x)^{d-m-p}\frac{\Gamma(d)\Gamma(m+p-d)}{\Gamma(m)\Gamma(p)}{}_2F_1(d-m,d-p;d-p-m;1-x),\end{aligned} \tag{14}$$

in Eq. (13) followed by the transformation [58,59]

$${}_2F_1(m,p;d;x)=(1-x)^{d-m-p}{}_2F_1(d-m,d-p;d;x), \tag{15}$$

to have

$$\begin{aligned}\phi_\ell(\chi,s)=&\frac{1}{2i\chi}\left[\frac{2\Gamma\left(2\delta+\frac{1}{2}\right)\Gamma\left(1+\frac{i\chi}{\alpha}\right)}{\Gamma(C-A)\Gamma(C-B)}\left[\tanh^2(\alpha s)\right]^{-\delta+1/2}\times\left[1-\tanh^2(\alpha s)\right]^{-\frac{i\chi}{2\alpha}}{}_2F_1\left(1-\frac{i\chi}{\alpha}-A,1-\frac{i\xi}{\alpha}-B;\right.\right.\\&\left.1-\frac{i\chi}{\alpha};1-\tanh^2(\alpha s)\right)-\frac{2\Gamma\left(2\delta+\frac{1}{2}\right)\Gamma\left(1-\frac{i\chi}{\alpha}\right)}{\Gamma(A)\Gamma(B)}\left[\tanh^2(\alpha s)\right]^{-\delta+1/2}\left[1-\tanh^2(\alpha s)\right]^{\frac{i\chi}{\alpha}}\\&\left.\times{}_2F_1\left(1+\frac{i\chi}{\alpha}-C+A,1+\frac{i\chi}{\alpha}-C+B;1+\frac{i\chi}{\alpha};1-\tanh^2(\alpha s)\right)\right].\end{aligned} \tag{16}$$

According to [8,9], the relationship between regular and irregular solutions is stated as

$$\phi_\ell(\chi,s)=\frac{1}{2i\chi}\left[F_\ell^{(-)}(\chi)f_\ell^{(+)}(\chi,s)-F_\ell^{(+)}(\chi)f_\ell^{(-)}(\chi,s)\right], \tag{17}$$

where the Jost function [7-9] $F_\ell^{(+)}(\chi)=(F_\ell^{(-)}(\chi))^*$ and the Jost solution $f_\ell^{(+)}(\chi,s)=(f_\ell^{(-)}(\chi,s))^*$. Now, by comparing Eqs. (16) and (17), Jost solution and the Jost function represented as

$$f_\ell^{(+)}(\chi,s)=\left[\tanh^2(\alpha s)\right]^{-\delta+1/2}\left[1-\tanh^2(\alpha s)\right]^{-\frac{i\chi}{2\alpha}}{}_2F_1\left(1-\frac{i\chi}{\alpha}-A,1-\frac{i\xi}{\alpha}-B;1-\frac{i\chi}{\alpha};1-\tanh^2(\alpha s)\right), \tag{18}$$

and again using the transformation given in Eq. (15) to the ${}_2F_1(.)$ function Eq. (18) leads to

$$f_\ell^{(+)}(\chi,s)=\left[\tanh^2(\alpha s)\right]^{\delta}\times\left[1-\tanh^2(\alpha s)\right]^{-\frac{i\chi}{2\alpha}}{}_2F_1\left(A,B;1-\frac{i\chi}{\alpha};1-\tanh^2(\alpha s)\right), \tag{19}$$

with the Jost function $F_\ell^{(+)}(\chi)$ as

$$F_\ell^{(+)}(\chi)=\frac{2\,\Gamma\left(2\delta+\frac{1}{2}\right)\Gamma\left(1-\frac{i\chi}{\alpha}\right)}{\Gamma(A)\Gamma(B)}. \tag{20}$$

However, one can reach at the same regular, irregular/Jost solution and Jost function with the remaining two roots which are $\delta=1/4-(1/2)\sqrt{(1/4)+\ell(\ell+1)}$ and $\beta=i\chi/2\alpha$ utilizing the transformation formula [58,59] given in Eq. (15). The bound state energies are generated by the zeros of the Jost function in the upper half of the complex momentum (χ)-plane, *i.e.* for $\chi=i\kappa_B$. Where $\kappa_B=(2mE_B/\hbar^2)^{1/2}$, with E_B and m denoting the binding energy and reduced mass respectively. Jost function $F_\ell^{(+)}(\chi)$ attains zero value at the poles of the gamma functions then

$$\delta+\beta+\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{V_0}{\alpha^2}}=-m;\qquad m=0,1,2... \tag{21}$$

Consequently, using the value $m = 0$, and Eq. (20) coupled with Eq. (10) yields

$$\kappa_B = -\alpha + \alpha\sqrt{\frac{1}{4} + \frac{V_0}{\alpha^2}} - \alpha\sqrt{\frac{1}{4} + \ell(\ell+1)}. \tag{22}$$

Jost function is a complex quantity thus one can define the phase shift as

$$\tan\delta = -\left[\mathrm{Im}\,\mathrm{F}_\ell^{(+)}(\chi)\,/\mathrm{Re}\mathrm{F}_\ell^{(+)}(\chi)\right]. \tag{23}$$

3. Results and discussion

Analysis on elastic neutron-deuteron (n-d) and $(n{-}\mathrm{He}^3)$ scattering are presented by applying Jost function method. The model parameters are obtained by parameterizing the nuclear modified Pöschl-Teller potential which are given in Table I and II. For the bound states $(1/2)^+$ (n-d) and S-wave $(n{-}\mathrm{He}^3)$, the parameters of the concerned potentials have been fixed by fitting their binding energies using Eq. (22). For (n-d) system the binding energy has been found to be -7.61 MeV and for $(n{-}\mathrm{He}^3)$ system it is -28.09 MeV. However, in order to get the best fit of the phase shifts at various laboratory energies, free iteration of the parameters in the computational approach has been applied to the unbound states. For computation of scattering phase shifts, bound state energies and total elastic cross sections

TABLE I. Parameters for the neutron-deuteron (n-d) system.

States	$\alpha(fm^{-1})$	$V_0(\mathrm{fm}^{-2})$
$1/2^+$	0.853	2.965
$1/2^-$	0.418	0.021
$3/2^-$	1.017	4.360
$3/2^+$	0.639	5.730

TABLE II. Parameters for the $(n{-}\mathrm{He}^3)$ system.

States	$\alpha(\mathrm{fm}^{-1})$	$V_0(\mathrm{fm}^{-2})$
$^1S_0$	0.416	2.621
$^3S_1$	0.685	7.802
$^1P_1$	1.415	2.015
$^3P_0$	1.092	5.075
$^3D_1$	1.074	7.076

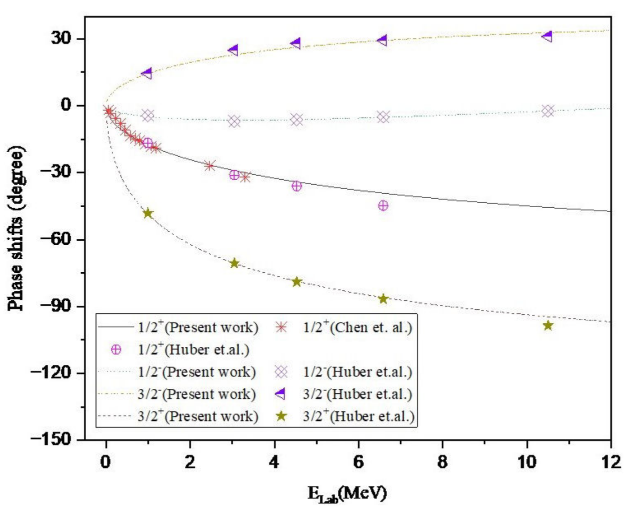

FIGURE 1. Scattering phase shifts for (n-d) system. The standard results are from Chen *et al.* [62] and Hüber *et al.* [63].

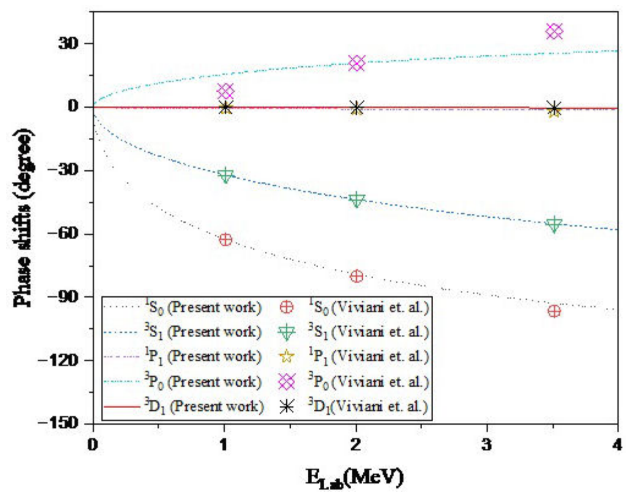

FIGURE 2. Scattering phase shifts for $(n - He^3)$ system. The standard data are from Viviani *et al.* [64].

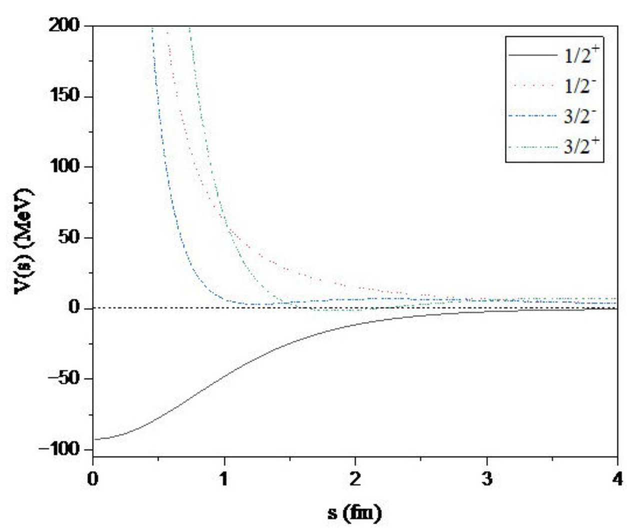

FIGURE 3. (n-d) potentials as a function of s.

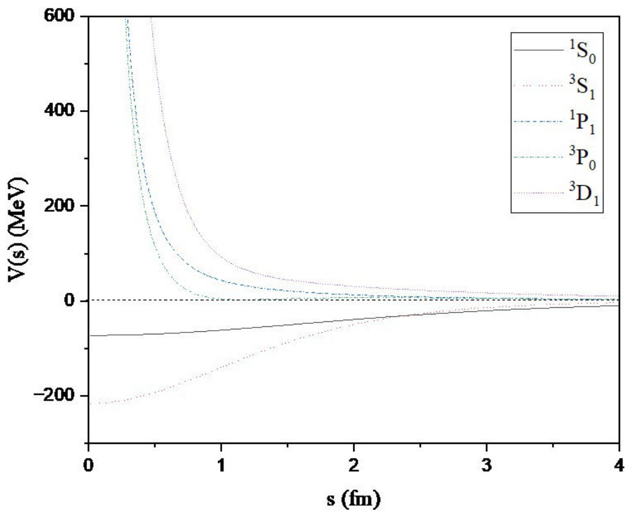

FIGURE 4. $(n{-}\mathrm{He}^3)$ potentials as a function of s.

$\hbar^2/2m$=31.1025 MeV fm^2 and $\hbar^2/2m = 27.6466$ MeV fm^2 has been used judiciously for (n-d) and $(n{-}\mathrm{He}^3)$ systems respectively.

Figure 1 depicts the (n-d) phase shifts as a function of laboratory energy up to 12 MeV along with the findings of Chen *et al*. [62] and Hüber *et al*. [63]. Figure 2 represents the $(n{-}\mathrm{He}^3)$ phase shifts with those of Viviani *et al*. [64] up to 4 MeV. It can be seen from Fig. 1 that up to a lab energy of 12 MeV, our parameters for various states of the (n-d) system reflect the right-phase parameters. The results are in close agreement with the results of Chen *et al*. [62] and Hüber *et al*. [63]. The $(n{-}\mathrm{He}^3)$ scattering phase parameters, depicted in Fig. 2, demonstrate excellent agreement with the standard data of Viviani *et al*. [64] except the state where the phase shifts are discernible slightly at laboratory energies 1 and 3.5 MeV. For the scattering of the Pöschl-Teller model of the (n-d) and $(n{-}\mathrm{He}^3)$ systems, the related potentials for various states are depicted in Figs. 3 and 4. The scattering cross section represents an invaluable tool for quantifying the intrinsic rate at which a particular event takes place throughout the scattering of a pair of particles. Numerous nucleon-nucleon model interactions have low-energy n-d parameter computations [65-71] available. The present text studies how much level the model's computations will be capable of producing accurate cross-section data in light of the minor disparities between the results of these phase shift analyses and those of our computation. Considering the systems under investigation, the total scattering cross sections in this article have been computed and compared with the standard data [67,72] that currently exist. The computed results are tabulated in Table III and IV along with standard data [67,72] for the (n-d) and $(n{-}\mathrm{He}^3)$ systems. The effects of S-, P-, and D-waves have been taken into account while calculating the cross section. The calculated outcomes for the (n-d) system are in good agreement with the results of Ref. [67]. However for $(n{-}\mathrm{He}^3)$ system the computed results differ slightly with experimental result [72]. This may be attributed to the fact that the obtained phase parameters for state differ slightly from those of Viviani *et al*. [64]. It is notable that the cross section data mainly agree with the results obtained by Seagrave *et al*. [72].

TABLE III. Total elastic cross-section for (n-d) system.

E_{Lab}(MeV)	σ^{nd}(b) (Present Work)	σ^{nd}(b) (Ref.[67])	E_{Lab}(MeV)	σ^{nd}(b) (Present Work)	σ^{nd}(b) (Ref.[67])
0.5	5.184	—-	8.0	1.244	1.120 ± 0.067
1.0	4.247	—	8.5	1.184	—-
1.5	3.593	—-	9.0	1.132	1.028 ± 0.062
2	3.133	—-	9.5	1.083	—-
2.5	2.754	2.375 ± 0.140	10.0	1.037	—-
3.0	2.459	2.149 ± 0.129	10.25	1.020	0.938 ± 0.056
3.5	2.241	1.985 ± 0.119	11.0	0.959	—-
4.0	2.054	1.863 ± 0.112	11.5	0.923	—-
4.5	1.900	1.732 ± 0.103	12.0	0.878	0.819 ± 0.049
5.0	1.768	1.608 ± 0.096	12.5	0.850	—-
5.5	1.650	—-	13.0	0.829	—-
6.0	1.550	1.448 ± 0.087	13.5	0.802	—-
6.5	1.463	—-	14.0	0.769	0.694 ± 0.042
7.0	1.379	1.254 ± 0.075	14.5	0.742	—-
7.5	1.308	—-	15.0	0.717	—-

TABLE IV. Total elastic cross sections for $(n{-}\mathrm{He}^3)$ system.

$E_{\rm Lab}$(MeV)	σ_e(mb) (Present Work)	σ_e(mb) (Ref.[72])
0.5	3.034	———
1.0	2.530	1.96 ± 0.06
1.5	2.212	—– –
2.0	1.968	2.52 ± 0.07
2.5	1.777	—– –
3.0	1.640	——-
3.5	1.495	2.34 ± 0.06
4.0	1.382	—– –
4.5	1.284	—– –
5.0	1.210	——-
5.5	1.131	——-
6.0	1.064	1.69 ± 0.07

4. Conclusion

Several authors have attempted to develop localized versions of nonlocal potentials or phase-equivalent local potentials to represent nonlocal interactions. One aspect of the current methodology is the generation of smooth potentials. The method used here is extended to real potentials by applying the Taylor series expansion to the wave functions. The computed phase parameters are then utilised to estimate cross-sections for the studied systems. Up to very high energies, the generated local potentials reproduce phase shifts that are in good agreement with standard data. It is well known that the folding models for alpha-nucleus scattering often use the non-local separable or phase equivalent local interactions of different forms. In the present work, we have investigated nucleon-nucleon and alpha-nucleon systems by constructing velocity-dependent interaction models and have obtained satisfactory results. The current procedure can easily be applied to electromagnetically distorted nonlocal potentials of higher rank. In our future work, we will address all partial wave cases, along with the spin dependence of the interaction.

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Conflict of interest

The authors declare that they have no conflict of interest in this work.

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