Sharing symmetries

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Making use of the fact that the solutions of the Schrödinger equation for the harmonic oscillator can be expressed in terms of the solutions of the Schrödinger equation for a free particle, we find the effect on the wavefunctions of the harmonic oscillator produced by the spatial translations and the Galilean transformations on the wavefunctions of a free particle. We find that these symmetry transformations applied to the ground state of the harmonic oscillator produce the coherent states.

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1. Introduction

In a recent paper [1] it has been shown that the problem of a free particle can be related to that of a harmonic oscillator, both in classical mechanics and in quantum mechanics, making use of the coordinate transformation

$$q = q' \sec \omega t', \qquad t = \frac{\tan \omega t'}{\omega}, \tag{1}$$

where ω is a constant (the angular frequency of the oscillator). Specifically, it was shown that $\Psi_{\rm osc}$ is a solution of the Schrödinger equation for the harmonic oscillator if and only if

$$\Psi_{\rm free}(q,t) = \cos^{1/2} \omega t' \times \exp\left(\frac{{\rm i}m\omega}{2\hbar} q'^2 \tan \omega t'\right) \Psi_{\rm osc}(q', t'), \tag{2}$$

is a solution of the Schrödinger equation for a free particle [with q' and t' expressed in terms of q and t according to Eq. (1)]. Since the Hamiltonian for a free particle is invariant under translations in space and in time as well as under Galilean transformations, from any given solution of the Schrödinger equation for a free particle we have simple ways of finding new solutions which are essentially equivalent to the original one, differing only by the choice of the coordinate system. Such trivial transformations, however, become non-trivial when translated to the wavefunctions of the harmonic oscillator.

The aim of this paper is find the induced effect of space translations and Galilean transformations on the solutions of the Schrödinger equation for a harmonic oscillator through the correspondence (2). We show that, despite the difference between these transformations on the wavefunctions for a free particle, their effects on the wavefunctions of the harmonic oscillator are almost identical. In particular, when applied to the wavefunction of the ground state of the oscillator we obtain the standard coherent states.

2. Taking advantage of the symmetries for a free particle

In this section we derive the effects on the wavefunctions of the harmonic oscillator induced by the spatial translations and the Galilean transformations applied to the wavefunctions of a free particle. These two examples are treated separately.

2.1. The induced effect of a spatial translation

Under a spatial translation, which can be defined by $Q = q - a$, $P = p$, where a is a constant, the wavefunction of a particle transforms into

$$\widetilde{\Psi}(q,t) = {\rm e}^{-{\rm i}\chi(t,a)/\hbar}\, \Psi(q - a, t), \tag{3}$$

where χ is a real-valued function of t and a. If the Hamiltonian is invariant under these translations, the function χ can be chosen in such a way that $\widetilde{\Psi}$ is a solution of the corresponding Schrödinger equation if Ψ is a solution of this equation. In the case of the Hamiltonian for a free particle, χ can be taken equal to zero (see, *e.g.*, Ref. [2]).

Thus, if $\Psi_{\rm free}$ is a solution of the Schrödinger equation for a free particle,

$$\widetilde{\Psi}_{\rm free}(q,t) = \Psi_{\rm free}(q - a, t) \tag{4}$$

is also a solution of this equation and, according to Eq. (2), the corresponding wavefunction of the harmonic oscillator is

$$\begin{aligned}\Psi_{\rm osc}(q', t') &= \sec^{1/2} \omega t' \times \exp\left(-\frac{{\rm i}m\omega}{2\hbar} q'^2 \tan \omega t'\right) \widetilde{\Psi}_{\rm free}(q,t), \\ &= \sec^{1/2} \omega t' \times \exp\left(-\frac{{\rm i}m\omega}{2\hbar} q'^2 \tan \omega t'\right) \Psi_{\rm free}(q - a, t), \end{aligned} \tag{5}$$

[with q and t related to q' and t' by means of Eqs. (1)].

According to Eqs. (1), when q is replaced by $q-a$, q' is replaced by $q'-a\cos\omega t'$ and, making use again of Eq. (2), we have

$$\Psi_{\rm free}(q-a,t)=\cos^{1/2}\omega t'\exp\left(\frac{{\rm i}m\omega}{2\hbar}(q'-a\cos\omega t')^2\tan\omega t'\right)\Psi_{\rm osc}(q'-a\cos\omega t',t'). \tag{6}$$

Substituting this last equation into (5) we obtain a relation between wavefunctions of the harmonic oscillator

$$\widetilde{\Psi}_{\rm osc}(q',t')=\exp\left(\frac{{\rm i}m\omega}{2\hbar}(-2aq'\sin\omega t'+a^2\sin\omega t'\cos\omega t')\right)\Psi_{\rm osc}(q'-a\cos\omega t',t') \tag{7}$$

[compare with Eq. (4)]. Thus, if $\Psi_{\rm osc}$ is any solution of the Schrödinger equation for the harmonic oscillator, $\widetilde{\Psi}_{\rm osc}$ is another solution of this equation.

For instance, taking $\Psi_{\rm osc}$ as the (normalized) wavefunction of the ground state,

$$\Psi_{\rm osc}(q',t')=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}q'^2-\frac{\rm i}{2}\omega t'\right), \tag{8}$$

from Eq. (7) we obtain the solution

$$\widetilde{\Psi}_{\rm osc}(q',t')=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{{\rm i}m\omega}{2\hbar}(-2aq'\sin\omega t'+a^2\sin\omega t'\cos\omega t')\right)\exp\left(-\frac{m\omega}{2\hbar}(q'-a\cos\omega t')^2-\frac{\rm i}{2}\omega t'\right) \tag{9}$$

which reduces to

$$\widetilde{\Psi}_{\rm osc}(q',t')=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{m\omega}{2\hbar}(-q'^2+2aq'{\rm e}^{-{\rm i}\omega t'}-a^2\cos\omega t'{\rm e}^{-{\rm i}\omega t'})-\frac{\rm i}{2}\omega t'\right). \tag{10}$$

These wavefunctions are the standard coherent states of the harmonic oscillator (see, *e.g.*, Ref. [3]). In fact, from Eq. (9) we immediately obtain

$$|\widetilde{\Psi}_{\rm osc}(q',t')|^2=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega}{\hbar}(q'-a\cos\omega t')^2\right),$$

which is a gaussian packet centered at the point $q'=a\cos\omega t'$.

In fact, since the first factor on the right-hand side of Eq. (7) has modulus equal to 1, if $\Psi_{\rm osc}$ is any of the stationary states of the harmonic oscillator (not only the ground state), the probability density corresponding wavefunction $\Psi_{\rm osc}(q',t')$ will oscillate rigidly, preserving its form, with the classical frequency ω (alternatively, one can verify that the wavefunction (10) is an eigenfunction of the destruction operator, $(1/\sqrt{2m\omega\hbar})({\rm i}\hat{p}'+m\omega\hat{q}')$, with eigenvalue $\sqrt{m\omega/2\hbar}\,a\,{\rm e}^{-{\rm i}\omega t'}$).

As is well known, the infinitesimal generator of the translations (4) is, up to a constant factor, the linear momentum operator, which trivially commutes with the standard Hamiltonian of a free particle. Making use of (7) we can readily find the corresponding conserved operator for the harmonic oscillator. By expanding the right-hand side of (7) as a series in the parameter a one finds that the infinitesimal generator of the (unitary) transformations (7) can be taken as the hermitean operator $\hat{A}\equiv\cos\omega t'\hat{p}'+m\omega\sin\omega t'\hat{q}'$. In fact, one can verify that ${\rm i}\hbar\partial\hat{A}/\partial t+[\hat{A},\hat{H}_{\rm osc}]=0$, where $\hat{H}_{\rm osc}$ is the standard Hamiltonian operator of the harmonic oscillator. The "conservation" of $\hat{A}$ means that the expectation value of $\hat{A}$ in any solution of the Schrödinger equation is constant.

2.2. The induced effect of a Galilean transformation

Under a Galilean transformation, defined by $Q=q-Vt$, $P=p-mV$, where V is a constant, the wavefunction, $\Psi(q,t)$, of a particle transforms into

$$\widetilde{\Psi}(q,t)=\exp\left[{\rm i}\big(mVq-\tfrac{1}{2}mV^2t-\chi(t,V)\big)/\hbar\right]\Psi(q-Vt,t), \tag{11}$$

where $\chi(t,V)$ is a function of t and V only. If the Hamiltonian is invariant under the Galilean transformations, χ can be chosen in such a way that $\widetilde{\Psi}$ is a solution of the corresponding Schrödinger equation if Ψ is a solution of this equation (see, *e.g.*, Ref. [2]). The standard Hamiltonian for a free particle is invariant under the Galilean transformations and χ can be taken equal to zero. Hence, if $\Psi_{\rm free}$ is a solution of the Schrödinger equation for a free particle,

$$\widetilde{\Psi}_{\rm free}(q,t)=\exp\left[{\rm i}\big(mVq-\tfrac{1}{2}mV^2t\big)/\hbar\right]\Psi_{\rm free}(q-Vt,t) \tag{12}$$

is another solution of this equation.

Following the same steps as in section 2.1, with the aid of Eq. (2) we find the wavefunction of the harmonic oscillator corresponding to $\widetilde{\Psi}_{\rm free}$

$$\begin{aligned}\Psi_{\rm osc}(q',t') &= \sec^{1/2}\omega t' \exp\left(-\frac{{\rm i}m\omega}{2\hbar}q'^2\tan\omega t'\right)\widetilde{\Psi}_{\rm free}(q,t)\\ &= \sec^{1/2}\omega t' \exp\left(-\frac{{\rm i}m\omega}{2\hbar}q'^2\tan\omega t'\right)\exp\left[{\rm i}\big(mVq-\tfrac{1}{2}mV^2t\big)/\hbar\right]\Psi_{\rm free}(q-Vt,t)\\ &= \sec^{1/2}\omega t' \exp\left(-\frac{{\rm i}m\omega}{2\hbar}q'^2\tan\omega t'\right)\exp\left(\frac{{\rm i}m}{2\hbar}\big(2Vq'\sec\omega t'-(V^2/\omega)\tan\omega t'\big)\right)\Psi_{\rm free}(q-Vt,t), \qquad (13)\end{aligned}$$

but [see Eqs. (6) and (1)]

$$\begin{aligned}\Psi_{\rm free}(q-Vt,t) &= \cos^{1/2}\omega t' \exp\left(\frac{{\rm i}m\omega}{2\hbar}\big(q'-Vt\cos\omega t'\big)^2\tan\omega t'\right)\Psi_{\rm osc}(q'-Vt\cos\omega t',t')\\ &= \cos^{1/2}\omega t' \exp\left(\frac{{\rm i}m\omega}{2\hbar}\big(q'-(V/\omega)\sin\omega t'\big)^2\tan\omega t'\right)\Psi_{\rm osc}(q'-(V/\omega)\sin\omega t',t'). \qquad (14)\end{aligned}$$

Combining the previous equations one finds the relation

$$\widetilde{\Psi}_{\rm osc}(q',t') = \exp\left(\frac{{\rm i}m\omega}{2\hbar}\big(2(V/\omega)q'\cos\omega t'-(V/\omega)^2\sin\omega t'\cos\omega t'\big)\right)\Psi_{\rm osc}(q'-(V/\omega)\sin\omega t',t') \qquad (15)$$

[*cf.* Eq. (12)]. Comparing (7) and (15) we see that the application of (15) to the ground state (8) also leads to the coherent states.

Thus, despite the differences between the spatial translations and the Galilean transformations, under the correspondence (2), they induce the transformations (7) and (15) on the wavefunctions of the harmonic oscillator that only differ by a translation in the time t' by $\pi/2\omega$ (with the parameter a replaced by V/ω).

3. Concluding remarks

As shown above, by simple substitutions, some obvious symmetries of the Schrödinger equation for a free particle lead to non-trivial symmetries of the Schrödinger equation for the harmonic oscillator. It should be clear that one can apply the transformations (7) and (15) to any solution of the Schrödinger equation for the harmonic oscillator, including non-stationary states.

The transformations (7) and (15) can also be obtained in the following way: according to Eqs. (1), the one-parameter group of translations $q \mapsto q-a$, $t \mapsto t$, amounts to $q' \mapsto q' - a\cos\omega t'$, $t' \mapsto t'$. These transformations are variational symmetries of the standard Lagrangian for the harmonic oscillator (see, *e.g.*, Ref. [4]) and belong to the class of coordinate transformations considered in Ref. [2] (that do not mix coordinates and momenta). Following the procedure presented in Ref. [2] a somewhat lengthy computation shows that the wavefunctions of the harmonic oscillator transform as in (7). Similarly, the one-parameter group of Galilean transformations $q \mapsto q - Vt$, $t \mapsto t$, which amounts to $q' \mapsto q' - (V/\omega)\sin\omega t'$, $t' \mapsto t'$, has the same characteristics as the previous group and leads to (15).

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