# Relating problems in two dimensions. A particle in a uniform magnetic field and a free particle

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We give a coordinate transformation in the extended configuration space that maps the trajectories of a free particle in two dimensions into the trajectories of a charged particle in a uniform magnetic field. We show that this coordinate transformation also relates the solutions of the Schrödinger equations for these two problems.

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## 1. Introduction

In a recent paper [1] it has been shown that by means of an appropriate coordinate transformation mixing coordinates and time one can relate the problem of a free particle in one dimension with that of a one-dimensional harmonic oscillator, in such a way that the standard Lagrangian of one of these problems is transformed into the standard Lagrangian of the other. With the same coordinate transformation, the solutions of the (time-dependent) Schrödinger equation for the one-dimensional harmonic oscillator are expressed in terms of the solutions of the (time-dependent) Schrödinger equation for a free particle in one dimension. Among other things, this relation between wavefunctions allows us to find in a trivial manner the relation between the corresponding propagators. Taking advantage of this result, we can establish similar relations between a free particle in two or three dimensions with the *isotropic* harmonic oscillator in two or three dimensions, respectively (both within the framework of classical mechanics and within the framework of quantum mechanics).

The aim of this paper is to show that we can find similar results relating a free particle in two dimensions with a charged particle in a uniform magnetic field. Specifically, we consider a coordinate transformation in the extended configuration space of a particle in two dimensions, which maps the trajectories of a free particle into the trajectories of a charged particle in a uniform magnetic field and we find that the same coordinate transformation is useful relating the solutions of the Schrödinger equations for these two problems.

In Sec. 2 we introduce the coordinate transformation in the extended configuration space to be employed and we show that this transformation maps the standard Lagrangian of a free particle into the standard Lagrangian of a particle in a magnetic field. In Sec. 3 we show that the same coordinate transformation maps the Schrödinger equation for a free particle in two dimensions into, essentially, the Schrödinger equation for a charged particle in a uniform magnetic field. In Sec. 4 we show that by starting from a Lagrangian for a free particle different from the standard one, we obtain a Lagrangian for the particle in the magnetic field which also differs from the usual one.

## 2. Relation in the classical mechanics framework

Throughout this paper we shall consider the coordinate transformation

$$\begin{aligned} x' &= x - y\tan\omega t, \\ y' &= y + x\tan\omega t, \\ t' &= \frac{1}{\omega}\tan\omega t, \end{aligned} \tag{1}$$

where $\omega$ is a constant. As we can readily verify, the mapping (1) takes the orbits of a free particle in two dimensions into the orbits of a charged particle in a uniform magnetic field. Indeed, according to Eqs. (1), the equations

$$x' = A + Bt', \quad y' = C + Dt',$$

where $A, B, C$ and $D$ are arbitrary constants, which represent a motion with constant velocity (a free particle), correspond (in the unprimed variables) to

$$\begin{aligned} x - y\tan\omega t &= A + B\frac{1}{\omega}\tan\omega t, \\ y + x\tan\omega t &= C + D\frac{1}{\omega}\tan\omega t. \end{aligned}$$

Solving these equations for $x$ and $y$ as functions of $t$ we obtain

$$x = \frac{A + D/\omega}{2} + \frac{A - D/\omega}{2}\cos 2\omega t + \frac{C + B/\omega}{2}\sin 2\omega t$$

and

$$y = \frac{C - B/\omega}{2} - \frac{A - D/\omega}{2}\sin 2\omega t + \frac{C + B/\omega}{2}\cos 2\omega t,$$

which are circles traversed with constant angular velocity, $2\omega$, and represent the motion of a charged particle in a uniform magnetic field. The angular velocity $2\omega$ is the cyclotron frequency, $eB_0/mc$, where $e$ is the electric charge of the particle, $m$ is its mass and $B_0$ is the magnitude of the magnetic field (in cgs units).

Another more convenient manner of exhibiting this relationship consists in showing that Eqs. (1) map the standard Lagrangian of one of these problems into the standard Lagrangian of the other. As a first step, from Eqs. (1) we find

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \dot{x}\cos^2\omega t - \dot{y}\sin\omega t\cos\omega t - \omega y$$

and

$$\frac{\mathrm{d}y'}{\mathrm{d}t'} = \dot{y}\cos^2\omega t + \dot{x}\sin\omega t\cos\omega t + \omega x,$$

where, following the standard usage, $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t$ and $\dot{y} \equiv \mathrm{d}y/\mathrm{d}t$. Hence, under the transformation (1), the standard Lagrangian for a free particle in two dimensions,

$$L_{\text{free}} = \frac{m}{2}\left[\left(\frac{\mathrm{d}x'}{\mathrm{d}t'}\right)^2 + \left(\frac{\mathrm{d}y'}{\mathrm{d}t'}\right)^2\right], \tag{2}$$

is transformed into (see, *e.g.*, Ref. [2])

$$\begin{aligned} L_{\text{free}}\frac{\mathrm{d}t'}{\mathrm{d}t} &= \frac{m}{2}\Big[\left(\dot{x}\cos^2\omega t - \dot{y}\sin\omega t\cos\omega t - \omega y\right)^2 \\ &\quad + \left(\dot{y}\cos^2\omega t + \dot{x}\sin\omega t\cos\omega t + \omega x\right)^2\Big]\sec^2\omega t \\ &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + m\omega\left(x\dot{y} - y\dot{x}\right) \\ &\quad + \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial t}, \end{aligned} \tag{3}$$

where

$$F \equiv \frac{m\omega}{2}(x^2 + y^2)\tan\omega t. \tag{4}$$

Apart from the terms containing $F$ (that do not contribute to the equations of motion), the right-hand side of (3) is just the standard Lagrangian for a particle, with electric charge $e$, in a uniform magnetic field of magnitude $B_0$ perpendicular to the $xy$-plane if one identifies $m\omega$ with $eB_0/2c$. (As we shall see, the function $F$ is relevant in the quantum-mechanical version.)

Thus, the standard Lagrangian for a charged particle in a uniform magnetic field is the standard Lagrangian of a free particle expressed in certain coordinates of the extended configuration space. Interestingly enough, something similar holds regarding the quantum mechanical version of these problems.

## 3. Relation in the quantum mechanics framework

The Schrödinger equation for a free particle in two dimensions is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi_{\text{free}}}{\partial x'^2} + \frac{\partial^2\Psi_{\text{free}}}{\partial y'^2}\right) = \mathrm{i}\hbar\frac{\partial\Psi_{\text{free}}}{\partial t'}. \tag{5}$$

With the aid of the chain rule and Eqs. (1) one finds that

$$\begin{aligned} \frac{\partial}{\partial t'} = \cos^2\omega t\Big[&\frac{\partial}{\partial t} + \omega(y - x\tan\omega t)\frac{\partial}{\partial x} \\ &- \omega(x + y\tan\omega t)\frac{\partial}{\partial y}\Big] \end{aligned} \tag{6}$$

and, in a similar way,

$$\frac{\partial}{\partial x'} = \cos\omega t\left(\cos\omega t\frac{\partial}{\partial x} - \sin\omega t\frac{\partial}{\partial y}\right) \tag{7}$$

and

$$\frac{\partial}{\partial y'} = \cos\omega t\left(\sin\omega t\frac{\partial}{\partial x} + \cos\omega t\frac{\partial}{\partial y}\right). \tag{8}$$

Substituting (6)–(8) into Eq. (5) and cancelling a common factor $\cos^2\omega t$ one finds that this equation amounts to

$$\begin{aligned} &-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi_{\text{free}}}{\partial x^2} + \frac{\partial^2\Psi_{\text{free}}}{\partial y^2}\right) \\ &= \mathrm{i}\hbar\Big[\frac{\partial\Psi_{\text{free}}}{\partial t} + \omega\left(y\frac{\partial\Psi_{\text{free}}}{\partial x} - x\frac{\partial\Psi_{\text{free}}}{\partial y}\right) \\ &\quad - \omega\tan\omega t\left(x\frac{\partial\Psi_{\text{free}}}{\partial x} + y\frac{\partial\Psi_{\text{free}}}{\partial y}\right)\Big], \end{aligned} \tag{9}$$

where $\Psi_{\text{free}}$ is now considered as a function of $(x, y, t)$.

Expressing $\Psi_{\text{free}}$ in the form

$$\Psi_{\text{free}} = \cos\omega t\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m\omega}{2}(x^2 + y^2)\tan\omega t\right)\Psi_{\text{mag}} \tag{10}$$

[see Eq. (4)], from Eq. (9) we find that $\Psi_{\text{mag}}$ satisfies the Schrödinger equation for a charged particle in a uniform magnetic field:

$$\begin{aligned} \frac{1}{2m}\Big[&\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} + m\omega y\right)^2 \\ &+ \left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y} - m\omega x\right)^2\Big]\Psi_{\text{mag}} = \mathrm{i}\hbar\frac{\partial\Psi_{\text{mag}}}{\partial t}, \end{aligned}$$

with $\omega$ related to the magnitude of the magnetic field by $\omega = eB_0/(2mc)$.

Furthermore, making use of (10) and the formula for the change of variables in a double integral

$$\begin{aligned} \int_{\mathbb{R}^2}|\Psi_{\text{free}}|^2\mathrm{d}x'\,\mathrm{d}y' &= \int_{\mathbb{R}^2}\cos^2\omega t\,|\Psi_{\text{mag}}|^2\sec^2\omega t\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2}|\Psi_{\text{mag}}|^2\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which means that $\Psi_{\rm free}$ is normalized if and only if $\Psi_{\rm mag}$ is normalized.

### 3.1. An example

The wavefunction

$$\Psi_{\rm free}(x',y',t') = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{1+{\rm i}\omega t'} \times \exp\left[-\frac{m\omega\left(x'^2+y'^2\right)}{2\hbar(1+{\rm i}\omega t')}\right], \qquad (11)$$

is a solution of the Schrödinger equation for a free particle (see, *e.g.*, Refs. [1,3]). Hence, according to Eqs. (10) and (1),

$$\Psi_{\rm mag}(x,y,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \times \exp\left[-\frac{m\omega}{2\hbar}(x^2+y^2) - {\rm i}\omega t\right], \qquad (12)$$

is a solution of the Schrödinger equation for a charged particle in a uniform magnetic field, which happens to be a stationary state. Even though the wavefunctions (11) and (12) have the same form at $t = t' = 0$, in the case of the free particle this wave packet spreads with the time, while in the case of the interaction with the magnetic field the probability density does not change with the time, as it should be for a stationary state.

## 4. Non-standard Lagrangians

It should be clear that the equations of motion for a free particle can be obtained making use of the non-standard Lagrangian

$$L_{\rm ns} = \frac{m}{2}\left[\left(\frac{{\rm d}x'}{{\rm d}t'}\right)^2 - \left(\frac{{\rm d}y'}{{\rm d}t'}\right)^2\right], \qquad (13)$$

instead of (2). Following the same steps as above one readily finds that

$$\begin{aligned} L_{\rm ns}\frac{{\rm d}t'}{{\rm d}t} &= \frac{m}{2}\Bigg[\left(\dot{x}\cos^2\omega t - \dot{y}\sin\omega t\cos\omega t - \omega y\right)^2 \\ &\quad - \left(\dot{y}\cos^2\omega t + \dot{x}\sin\omega t\cos\omega t + \omega x\right)^2\Bigg]\sec^2\omega t \\ &= \frac{m}{2}\left[(\dot{x}^2-\dot{y}^2)\cos 2\omega t - 2\dot{x}\dot{y}\sin 2\omega t\right] \\ &\quad + \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\dot{y} + \frac{\partial G}{\partial t}, \end{aligned} \qquad (14)$$

where

$$G \equiv \frac{m\omega}{2}(y^2-x^2)\tan\omega t - m\omega xy.$$

In this manner, we obtain a non-standard Lagrangian for the charged particle in a uniform magnetic field: $\frac{m}{2}\left[(\dot{x}^2-\dot{y}^2)\cos 2\omega t - 2\dot{x}\dot{y}\sin 2\omega t\right]$, which was previously obtained in Ref. [4]. It may be remarked that this Lagrangian does not depend on the coordinates, but depends on the time even though the field is static.

## 5. Concluding remarks

The Lagrangian (13) illustrates two simple points (which, however, do not seem to be widely known): on the one hand, a Lagrangian need not possess the symmetries of the equations of motion derived from it (*e.g.*, the Lagrangian (13) is not invariant under rotations on the $x'y'$-plane, while the equations of motion of a free particle are invariant under these rotations) and, on the other hand, two Lagrangians leading to the same equations of motion need not differ by a total derivative with respect to the time of a function of the coordinates and the time.

In the framework of classical mechanics, if two mechanical systems are related by a change of coordinates as in the examples mentioned here, it is as if they were one and the same problem expressed in terms of different coordinates and, therefore, their variational symmetries must be the same. This gives a necessary condition for two problems to be related by a change of coordinates in the extended configuration space: their variational symmetry groups have to be isomorphic to each other. The question remains whether this condition is also sufficient.

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