Gravitation, Mathematical Physics and Field Theory Revista Mexicana de Física **71** 060701 1–8 NOVEMBER-DECEMBER 2025

On the Lagrange's equilateral homographic flat motion of three bodies interacting with the Newton gravitational force

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Received 30 January 2025; accepted 6 June 2025

We present the equilateral three-body motion with three different masses according to Newton gravitational force, which was discovered by Lagrange, tracing conic trajectories. We will extend to several bodies the generalization of the equilateral triangle solution discovered by Lagrange. The flat n-body problem of several different masses can be solved in closed and elementary form if we assume that the polygon formed by several celestial bodies always remains similar to itself. The Lagrange proof was simplified by C. Carathéodory and we extend without problem this proof to several bodies. The bodies move on a fixed plane with two independent coordinates: one rotation around the center of mass, and one radial expansion. At any time the position vector of each body is the same multiple of the acceleration vector of the body. Bodies move tracing similar conics with the pole of each conic at the center of mass. For the three-body Lagrange's case, a rigid triangle function of the masses discovered by Simó, is described with very simple geometry. Which we should place in a particular position imposed by the Lagrange's solution. We present a set of mathematical properties which are not well known.

Keywords: Homographic motion; central configurations; conics; few body problem; Lagrange equilateral; three-body problem.

DOI: https://doi.org/10.31349/RevMexFis.71.060701

1. Introduction

We want to review the flat homographic motion of several bodies interacting by the Newton gravitational force. By homographic we mean the angles formed by the lines connecting the bodies are constant. In the case of three bodies, Euler discovered a collineal solution [1] that has been reviewed recently [2], which will not be considered in this paper. We turn our attention into the flat solution found by Lagrange [3] (although the masses could be different) he proved they move in a fixed plane, forming an equilateral triangle; any body with its position vector relative to the center of mass being the same multiple of the acceleration vector of that body (Laplace called this a central configuration). The Lagrange proof is not in most of the textbooks of Mechanics. However, A. Sommerfeld includes that theorem in his text of Mechanics [4] since he uses one simplified proof of this Lagrange's Theorem published by C. Carathéodory in 1933 [5].

This is an exercise affordable to Physics, Astronomy and Mathematics students, and will be useful for some researchers in the field of Central Configurations that do not know this simple proof. The proof is very similar to that included in the Sommerfeld's text Mechanics, "an introductory course attended by students usually in their third or fourth years", in Germany in the last century.

We note the proof considered by Carathéodory and Sommerfeld for three bodies is valid for a larger number of bodies until it is demonstrated the bodies move in a fixed plane that rotates around the center of mass in the same plane. This is presented in the next section. After this fact is proved, the geometric configuration of the bodies become fastly growing in complexity from an equilateral triangle for three bodies to a complex polygon for four or five bodies, with angles depending on the values of the masses, as we discussed elsewhere [6, 7].

2. Proof that a flat homographic motion remains in a fixed plane

The hypothesis of flat homographic motion assumes that the several positions of the body $\mathbf{r}_k$ in cartesian coordinates obey the equation

$$\mathbf{r}_k = R\,\mathcal{G}\,\mathbf{c}_k, \tag{1}$$

where R is a time dependent dilatation, $\mathcal{G}$ is a time dependent rotation around the center of mass, from the frame of the plane containing the bodies to the inertial frame and k is a label to number the bodies. Vectors $\mathbf{c}_k$ are constant vectors with third component equal to zero

$$\mathbf{c}_k = \begin{pmatrix} A_k \\ B_k \\ 0 \end{pmatrix}, \tag{2}$$

where A_k, B_k are the cartesian coordinates of vector $\mathbf{c}_k$ in the rotating plane, before the time dependent dilatation by R. The center of mass of the three bodies is in the same plane, and with no loss of generality one assumes this point does not move and it is at the origin of a fixed system of coordinates of the plane containing the bodies, with its third axis of coordinates, orthogonal to such plane.

Rotation $\mathcal{G}$ is around the center of mass. To be a rotation this matrix has the property that its inverse matrix is its transposed matrix

$$\mathcal{G}^{\mathrm{T}}\mathcal{G} = \mathcal{E}, \tag{3}$$

where $\mathcal{E}$ is the unit matrix.

The time derivative of this equation gives

$$\mathcal{G}^{\mathrm{T}}\dot{\mathcal{G}} + \dot{\mathcal{G}}^{\mathrm{T}}\mathcal{G} = \mathcal{O}, \tag{4}$$

where $\mathcal{O}$ is the zero matrix. Hence, it follows that matrix $\mathcal{G}^{\mathrm{T}}\dot{\mathcal{G}}$ is an anti-symmetric matrix which defines the angular velocity

$$\mathcal{G}^{\mathrm{T}}\dot{\mathcal{G}} = -\dot{\mathcal{G}}^{\mathrm{T}}\mathcal{G} = \omega\times\ . \tag{5}$$

We have the velocities of the bodies

$$\dot{\mathbf{r}}_k = (\dot{R}\,\mathcal{G} + R\,\dot{\mathcal{G}})\,\mathbf{c}_k = \mathcal{G}(\dot{R}\mathcal{E} + R\,\omega\times)\mathbf{c}_k, \tag{6}$$

where a dot on a letter denotes the time derivative and where

$$\dot{\mathcal{G}} = \mathcal{G}\,\omega\times \tag{7}$$

is the velocity of the rotation matrix $\mathcal{G}$ in terms of the angular velocity ω, computed in the referential of the fixed plane:

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad \omega\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_3 & \omega_1 & 0 \end{pmatrix}. \tag{8}$$

Therefore the acceleration is

$$\ddot{\mathbf{r}}_k = \mathcal{G}\left[\ddot{R}\mathcal{E} + 2\dot{R}\omega\times + R\dot{\omega}\times + R(\omega\times)^2\right]\mathbf{c}_k, \tag{9}$$

where we introduce the unit matrix $\mathcal{E}$, since the other terms inside the square brackets are matrices. This acceleration, to satisfy the Newton equations of motion, should be equal to

$$\sum_{i\neq k}\frac{m_i(\mathbf{r}_i - \mathbf{r}_k)}{|\mathbf{r}_i - \mathbf{r}_k|^3} = \frac{1}{R^2}\mathcal{G}\sum_{i\neq k}\frac{m_i(\mathbf{c}_i - \mathbf{c}_k)}{|\mathbf{c}_i - \mathbf{c}_k|^3}$$
$$= \frac{1}{R^2}\mathcal{G}\begin{pmatrix} L_k \\ M_k \\ 0 \end{pmatrix}, \tag{10}$$

where L_k, M_k are the constant components of the vector forming the sum on the left, which is a function of vectors $\mathbf{c}_k$ in the plane containing the bodies.

Equating equations (9) and (10), and multiplying both members of that equation with $R^2\mathcal{G}^{-1}$, we come to

$$\left[R^2\ddot{R}\mathcal{E} + 2R^2\dot{R}\omega\times + R^3\dot{\omega}\times + R^3(\omega\times)^2\right]\mathbf{c}_k$$
$$= \begin{pmatrix} L_k \\ M_k \\ 0 \end{pmatrix}. \tag{11}$$

From this equation, when it is valid for three or more non collinear bodies, Carathéodory proves $\omega_1 = \omega_2 = 0$ as follows.

We write the third component of this equation for body k in terms of the components of vectors ω and $\mathbf{c}_k$, one has

$$A_k\left[-R\dot{\omega}_2 - 2\dot{R}\omega_2 + R\omega_3\omega_1\right]$$
$$+ B_k\left[R\dot{\omega}_1 + 2\dot{R}\omega_1 + R\omega_3\omega_2\right] = 0. \tag{12}$$

Because the bodies are not collineal, the vectors $(A_1, A_2, A_3, ...)$ and $(B_1, B_2, B_3, ...)$ are linearly independent; therefore, the quantities inside the square brackets of this equation are zero. It results in

$$-R\dot{\omega}_2 - 2\dot{R}\omega_2 + R\omega_3\omega_1 = 0,$$
$$R\dot{\omega}_1 + 2\dot{R}\omega_1 + R\omega_3\omega_2 = 0. \tag{13}$$

Cancelling ω_3 of both equations (13) we have the integrable result

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[R^4(\omega_1+\omega_2)^2\right] = 0 \ \rightarrow\ R^4(\omega_1+\omega_2)^2 = C^2, \tag{14}$$

where C is a non negative number, constant of integration. Next we will prove $C = 0$. But just now we only are certain that the complex number $\omega_1 + \mathrm{i}\omega_2$ obeys

$$R^2(\omega_1 + \mathrm{i}\omega_2) = \mathrm{C}\exp(\mathrm{i}\xi), \tag{15}$$

with ξ some real number. This number disappears if C is equal to zero.

Next we consider the components 1 and 2 of the equation (11) to have

$$\{R^2\ddot{R} - R^3(\omega_2^2 + \omega_3^2)\}A_k$$
$$- \{2\omega_3R^2\dot{R} + R^3(-\omega_1\omega_2 + \dot{\omega}_3)\}B_k = L_k, \tag{16}$$

and

$$\{2\omega_3R^2\dot{R} + R^3(\omega_1\omega_2 + \dot{\omega}_3)\}A_k$$
$$+ \{R^2\ddot{R} + -R^3(\omega_1^2 + \omega_3^2)\}B_k = M_k. \tag{17}$$

Since vectors of components A_k and B_k are linearly independent we find 4 particular values A_i, B_i, A_j, B_j, such that the determinant is not zero

$$\begin{vmatrix} A_i & A_j \\ B_i & B_j \end{vmatrix} \neq 0.$$

Substitution of these 4 values in Eq. (16) or Eq. (17), gives a system of two linear equations with constant coefficients, that implies the two brackets in each equation are constant. It follows that the 4 brackets should be constant. Then, from the differences of the first and the fourth, and the second and the third, we obtain the equations

$$R^3(\omega_1^2 - \omega_2^2) = \beta, \quad 2R^3\omega_1\omega_2 = \gamma, \tag{18}$$

with β and γ two unknown constants. These are the real part and imaginary part of the constant complex number

$$\beta + \mathrm{i}\gamma = \mathrm{R}^3(\omega_1 + \mathrm{i}\omega_2)^2. \tag{19}$$

If we assume R variable, the only possibility for satisfying both (15) and (19) is to have $\omega_1 = \omega_2 = 0$. If we assume R constant, then (19) implies ω_1 and ω_2 are constants, and (13) gives $\omega_3 = 0$ if one ω_1 or ω_2 is different from zero. Such case is impossible since the axis of rotation could coincide with the plane of the bodies. In both cases we conclude $\omega_1 = \omega_2 = 0$, and (13) is satisfied identically without implying that ω_3 is zero.

The rotation $\mathcal{G}$ is of the form

$$\mathcal{G} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

with

$$\omega_3 = \dot{\psi}. \tag{21}$$

The brackets in Eqs. (16) and (17) become simplified, now we find instead of four just two different, but one of them is zero. To prove this, consider the vector of angular momentum which is a constant of motion.

The angular momentum of the bodies is the constant of motion which is expressed in terms of our Eqs. (1) and (6)

$$\sum_k m_k \mathbf{r}_k \times \dot{\mathbf{r}}_k = \sum_k m_k R\,(\mathcal{G}\,\mathbf{c}_k) \times \left[\mathcal{G}(\dot{R} + R\,\omega\times)\mathbf{c}_k\right]. \tag{22}$$

The right hand side of this equation is simplified using first, that the product $\times$ of two rotated vectors is the rotation of the product $\times$ of the vectors

$$\sum_k m_k \mathbf{r}_k \times \dot{\mathbf{r}}_k = \sum_k m_k R\,(\mathcal{G}\,\mathbf{c}_k) \times \left\{\dot{R}(\mathcal{G}\mathbf{c}_k) + R\left[(\mathcal{G}\omega)\times(\mathcal{G}\mathbf{c}_k)\right]\right\}. \tag{23}$$

Furthermore the $\times$ vector of two equal vectors is zero and the triple $\times$ product simplifies as vectors $\mathcal{G}\omega$ and $\mathcal{G}\mathbf{c}_k$ are orthogonal and the square of vector $\mathcal{G}\mathbf{c}_k$ is the square of vector $\mathbf{c}_k$. Finally ω is an eigenvector of $\mathcal{G}$ with eigenvalue 1 (since it is parallel to the axis of rotation)

$$\sum_k m_k \mathbf{r}_k \times \dot{\mathbf{r}}_k = R^2\mathcal{G}\omega \sum_k m_k(\mathcal{G}\mathbf{c}_k)\cdot(\mathcal{G}\mathbf{c}_k) = R^2\omega\sum_k m_k \mathbf{c}_k\cdot\mathbf{c}_k. \tag{24}$$

Conservation of angular momentum for the homographic motion simplifies to

$$R^2\omega_3 = R^2\dot{\psi} = J, \tag{25}$$

where J is a constant.

Taking into account that $\omega_1 = \omega_2 = 0$, the time derivative of this constant, equal to zero, is a factor of a bracket in Eqs. (16) and (17). The other bracket is another constant that we denote by the number $-\nu$

$$-\nu = R^2\ddot{R} - R^3\omega_3^2 = R^2\ddot{R} - \frac{J^2}{R}. \tag{26}$$

Equations (16) and (17) are simplified to

$$-\nu\begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} L_k \\ M_k \end{pmatrix}, \tag{27}$$

which will be discussed in the next section.

3. The central configurations of the homographic motion in the fixed plane

Last Eq. (27) was solved by Lagrange and Laplace to obtain a remarkable simple solution for three bodies. The complexity of the equation is explicit if we return to the original expression in terms of the vector notation in Eqs. (2) and (10)

$$-\nu\mathbf{c}_k = \sum_{i\neq k}\frac{m_i(\mathbf{c}_i - \mathbf{c}_k)}{|\mathbf{c}_i - \mathbf{c}_k|^3}, \tag{28}$$

or coming back to our first hypothesis (1)

$$-\frac{\nu}{R^3}\mathbf{r}_k = \sum_{i\neq k}\frac{m_i(\mathbf{r}_i - \mathbf{r}_k)}{|\mathbf{r}_i - \mathbf{r}_k|^3}, \tag{29}$$

namely, at each time, the position is proportional to the acceleration of each particle. This is called a central configuration.

For three bodies Eq. (29) is written as

$$-\frac{\nu}{R^3}\mathbf{r}_k = \frac{m_i(\mathbf{r}_i - \mathbf{r}_k)}{|\mathbf{r}_i - \mathbf{r}_k|^3} + \frac{m_j(\mathbf{r}_j - \mathbf{r}_k)}{|\mathbf{r}_j - \mathbf{r}_k|^3}, \tag{30}$$

with i, j and k different.The $\times$ product with vector $\mathbf{r}_k$ leads to

$$\frac{m_i\mathbf{r}_i\times\mathbf{r}_k}{|\mathbf{r}_i - \mathbf{r}_k|^3} + \frac{m_j\mathbf{r}_j\times\mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|^3} = \mathbf{0}. \tag{31}$$

Since the center of mass is at the origin of coordinates we have

$$m_i\mathbf{r}_i + m_j\mathbf{r}_j + m_k\mathbf{r}_k = \mathbf{0}. \tag{32}$$

The $\times$ product with $\mathbf{r}_k$ gives

$$m_i\mathbf{r}_i\times\mathbf{r}_k + m_j\mathbf{r}_j\times\mathbf{r}_k = \mathbf{0}. \tag{33}$$

Substitution of (33) in (31) produces

$$m_i\mathbf{r}_i\times\mathbf{r}_k\left(\frac{1}{|\mathbf{r}_i - \mathbf{r}_k|^3} - \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|^3}\right) = \mathbf{0}. \tag{34}$$

Therefore, for three bodies, they are at the vertexes of an equilateral triangle

$$|\mathbf{r}_i - \mathbf{r}_k| = |\mathbf{r}_j - \mathbf{r}_k|, \quad |\mathbf{c}_i - \mathbf{c}_k| = |\mathbf{c}_j - \mathbf{c}_k|. \tag{35}$$

In the case when the number of bodies is larger than three, the shape of the constant polygon formed by the vectors $\mathbf{c}_k$

has not a simple shape. The shape is now strongly dependent on the relative value of the masses, actually only a finite number of shapes are allowed for 4 bodies [8] and, for finite 5 different masses perhaps one will have the same property [9]. Nevertheless. if the central configuration holds, then all the bodies describe conic sections, with a common focus at the center of mass, and with the same eccentricity.

To prove this, we assume for a while that a system of coordinates has been defined such that the angle ψ of the rotation $\mathcal{G}$ around the fixed direction of our plane is the angle between the inertial system of the plane and the principal direction of inertia of the several bodies. We shall make explicit in the next section that the coordinate system exists, and verify that the Newton equation of motion allows constants of motion that are compatible with restrictions imposed by the central configuration, Eq. (28).

Substitution of the angular velocity

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}, \tag{36}$$

in the expression (5) for the velocity of body k, we compute the kinetic energy

$$\begin{aligned} K &= \frac{1}{2}\sum_k m_k |\dot{\mathbf{r}}_k|^2 \\ &= \frac{1}{2}\left(\sum_k m_k |\mathbf{c}_k|^2\right)\left(\dot{R}^2 + R^2\dot{\psi}^2\right). \end{aligned} \tag{37}$$

The potential energy becomes

$$\begin{aligned} V &= -\sum_j \sum_{j\neq k} \sum_k \frac{m_j m_k}{|\mathbf{r}_j - \mathbf{r}_k|} \\ &= -\frac{1}{R}\left(\sum_j \sum_{j\neq k} \sum_k \frac{m_j m_k}{|\mathbf{c}_j - \mathbf{c}_k|}\right). \end{aligned} \tag{38}$$

We will write the constants in the kinetic energy and the potential energy with the notation

$$M^* = \left(\sum_k m_k |\mathbf{c}_k|^2\right), \quad C = \left(\sum_j \sum_{j\neq k} \sum_k \frac{m_j m_k}{|\mathbf{c}_j - \mathbf{c}_k|}\right).$$

The Lagrangian $L = K - V$ of several bodies for the homographic motion in the plane becomes

$$L(R, \dot{R}, \dot{\psi}) = \frac{1}{2}M^*\left(\dot{R}^2 + R^2\dot{\psi}^2\right) + \frac{C}{R}, \tag{39}$$

that is formally identical (except for the meaning of the constants in it) to the Lagrangian for the two body relative motion of two bodies moving in the Newton gravitational problem, where R is the relative distance and angle ψ is the real anomaly. For the 2-body problem, R and ψ are the polar coordinates of the relative position in the plane orthogonal to the angular momentum vector. This Lagrangian gives the constants of motion J in Eq. (25) and the energy

$$E = \frac{1}{2}M^*\left(\dot{R}^2 + R^2\dot{\psi}^2\right) - \frac{C}{R}. \tag{40}$$

Using these constants we relate the constants by

$$\nu = \frac{C}{M^*}, \tag{41}$$

which was obtained by replacing $\dot{\psi}$ from (25) in the equation of the energy (40). Taking the derivative with respect to time, and identifying the resulting equation with (26) .

The orbit is the conic

$$R = \frac{p}{1 - \epsilon\cos(\psi - \psi_0)}, \tag{42}$$

where ψ_0 is a constant angle, p is the *latus rectum*

$$p = \frac{J^2}{\nu}$$

and ϵ is the eccentricity

$$\epsilon = \sqrt{1 + \frac{2EJ^2}{M^* k^2}}.$$

A particular example, when the bodies move on ellipses is illustrated in Fig. 1.

4. A system of coordinates for three bodies in the plane

We review in this section a system of coordinates for studying the Newton equations of motion of three bodies, interacting with the gravitational force in the plane [10]. With no loss of generality, we assume the origin of coordinates is at the center of mass of the bodies. This implies the 6 Cartesian coordinates in the inertial frame of the plane obey the two constraints

$$\begin{aligned} m_1 x_1 + m_2 x_2 + m_3 x_3 &= 0, \\ \text{and} \quad m_1 y_1 + m_2 y_2 + m_3 y_3 &= 0. \end{aligned} \tag{43}$$

These two conditions lead to only four independent coordinates.

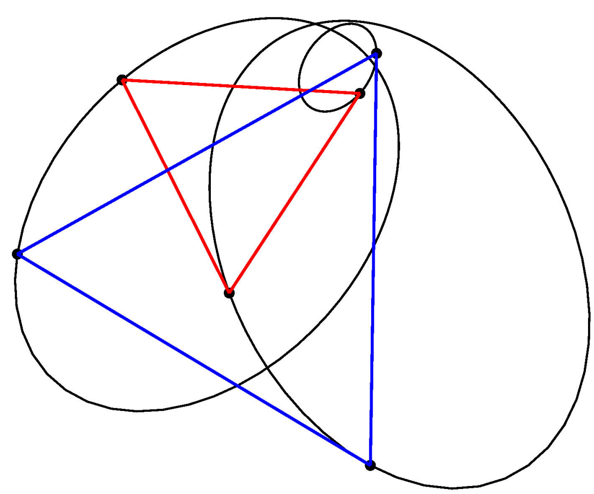

FIGURE 1. Three bodies tracing ellipses in a Lagrange's three-body solution. At two different times, the simultaneous positions forming equilateral triangles is highlighted.

We start considering three bodies in the plane. The Cartesian coordinates are written in terms of 4 new coordinates: ψ, R_1, R_2, σ

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}. \tag{44}$$

The angle ψ, that we used before, is the angle that diagonalizes the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \mu R_1^2 & 0 \\ 0 & \mu R_2^2 \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \tag{45}$$

where μ is a reference mass, that for 3 bodies we define as

$$\mu = \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}}, \tag{46}$$

and coordinates R_1 and R_2 are defined by Eq. (45).

Matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}, \tag{47}$$

is a constant matrix that satisfies

$$\begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} A \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} A^{\mathrm{T}} \begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \tag{48}$$

for any value of the fourth coordinate σ.

We will see that for the homographic motion of the Lagrange's equilateral triangle we choose $\sigma = 0$ that makes the $\sigma-$matrix to become the 2×2 unit matrix. The σ coordinate is hence the angle measured with respect to the Lagrange equilateral configuration.

The components of the matrix (47) obey until now 5 conditions: two of them are inherited from the conditions of the center of mass (43) that we write as

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{49}$$

Three conditions follow from the Eq. (48)

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \tag{50}$$

since it is a symmetric matrix with three independent equations. Now we combine these two equations in the form

$$\frac{1}{\mu} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ r & r & r \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & r \\ a_2 & b_2 & r \\ a_3 & b_3 & r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{51}$$

where

$$r = \sqrt{\frac{\mu}{m}}, \qquad m = m_1 + m_2 + m_3. \tag{52}$$

From Eq. (51) we use the left inverse of a matrix is equal to its right inverse and we arrive to

$$\begin{pmatrix} a_1 & b_1 & r \\ a_2 & b_2 & r \\ a_3 & b_3 & r \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ r & r & r \end{pmatrix} = \begin{pmatrix} \mu/m_1 & 0 & 0 \\ 0 & \mu/m_2 & 0 \\ 0 & 0 & \mu/m_3. \end{pmatrix}, \tag{53}$$

that gives the independent equations

$$a_j^2 + b_j^2 = \frac{\mu}{m_j} - \frac{\mu}{m}, \qquad a_i a_j + b_i b_j = -\frac{\mu}{m}. \tag{54}$$

These lead to very important geometric properties. The positions of the three coordinates in matrix (47) form a triangle which square of sides is equal to

$$(a_i - a_j)^2 + (b_i - b_j)^2 = \frac{\mu}{m_i} + \frac{\mu}{m_j}. \tag{55}$$

The center of mass of this triangle is at the orthocenter:

$$a_i(a_j - a_k) + b_i(b_j - b_k) = 0, \tag{56}$$

with i, j, k different.

This constant triangle, function of the masses, appears in a paper presented by C. Simó [11] studying the bifurcation of the three body motion using the ratio of kinetic energy and angular momentum. We will refer to this triangle as Simó's triangle.

A new independent condition is necessary to fix the position of this triangle to determine its 6 components for the Lagrange's case of equilateral motion. The additional condition to hold all the properties stated before is

$$a_1 m_1^2 b_1 + a_2 m_2^2 b_2 + a_3 m_3^2 b_3 = 0. \tag{57}$$

Then we have 3 vectors orthogonal to vector (b_1, b_2, b_3), namely (m_1, m_2, m_3), $(a_1 m_1, a_2 m_2, a_3 m_3)$ and $(a_1 m_1^2, a_2 m_2^2, a_3 m_3^2)$. Therefore these three vectors are linearly dependent, and we have the properties

$$m_i^2 a_i = X m_i a_i + Y \mu m_i) \quad \rightarrow \quad a_i = \frac{Y\mu}{m_i - X}, \tag{58}$$

with X and Y two real numbers to be determined, X with dimension of mass, and Y without dimensions. The X number is determined from the condition

$$a_1 m_1 + a_2 m_2 + a_3 m_3 = 0 \quad \rightarrow$$
$$\frac{m_1}{m_1 - X} + \frac{m_2}{m_2 - X} + \frac{m_3}{m_3 - X} = 0. \tag{59}$$

This last is a quadratic equation written as

$$\left(\frac{X}{\mu}\right)^2 - 2\left(\frac{X}{\mu}\right)\alpha + 3 = 0,$$
$$\alpha = \mu\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right). \tag{60}$$

With the two solutions

$$X_a = \mu(\alpha + \sqrt{\alpha^2 - 3}), \quad X_b = \mu(\alpha - \sqrt{\alpha^2 - 3}). \tag{61}$$

Nevertheless the vectors of components a_i and b_i obey symmetric equations therefore b_i satisfies properties like (58), that explains the subindex added to the two X's

$$a_i = \frac{Y_a\,\mu}{m_i - X_a}, \quad b_i = \frac{Y_b\,\mu}{m_i - X_b}. \tag{62}$$

Numbers Y_a, Y_b are normalization factors to satisfy equations in Eq. (50)

$$\sum_i m_i a_i^2 = \sum_i m_i b_i^2 = \mu. \tag{63}$$

Further properties of these quantities are in Ref. [12], for example

$$m1 > X_a > m_2 > X_b > m_3, \tag{64}$$

where we assume choice of indexes to have $m_1 > m_2 > m_3$.

Coming back to our Eqs. (1) and (2), we relate to our coordinates in this section by transforming R_1 and R_2 into polar coordinates

$$R_1 = R\cos\theta, \quad R_2 = R\sin\theta. \tag{65}$$

We recover the Lagrange's equilateral triangle case with angles σ and θ constants with the particular values

$$\sigma = 0, \quad \cos^2\theta = \frac{\alpha + \sqrt{\alpha^2 - 3}}{2\alpha},$$
$$\sin^2\theta = \frac{\alpha - \sqrt{\alpha^2 - 3}}{2\alpha}. \tag{66}$$

To prove this constant value for θ, we compute the characteristic equation for the matrix (45), in the case the three bodies are placed at the vertices of an equilateral triangle. The answer to this exercise is again the quadratic Eq. (60) with solutions (61), for a proper choice of the length of the triangle. We rewrite Eq. (1) in the form

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = R \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$
$$\times \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}, \tag{67}$$

which coincides with values (66) of our coordinates if

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} = \begin{pmatrix} \sqrt{X_a/)2\mu)} & 0 \\ 0 & \sqrt{X_b/(2\mu)} \end{pmatrix}$$
$$\times \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}. \tag{68}$$

We prove these are the coordinates for an equilateral triangle:

$$(A_i - A_j)^2 + (B_i - B_j)^2 = \frac{X_a}{2\mu}(a_i - a_j)^2 + \frac{X_b}{2\mu}(b_i - b_j)^2 =$$

(because Eqs. (68))

$$\frac{1}{2\mu}\left[(m_i a_i - m_j a_j)(a_i - a_j) + (m_i b_i - m_j b_j)(b_i - b_j)\right] =$$

$$\frac{1}{2\mu}\left[m_i(a_i^2 + b_i^2) + m_j(a_j^2 + b_j^2) - (m_i + m_j)(a_i a_j + b_i b_j)\right] =$$

(because Eqs. (54))

$$\frac{1}{2\mu}\left[m_i\left(\frac{\mu}{m_i} - \frac{\mu}{m}\right) + m_j\left(\frac{\mu}{m_j} - \frac{\mu}{m}\right)\right.$$
$$\left. - (m_i + m_j)\left(-\frac{\mu}{m}\right)\right] = 1. \tag{69}$$

This ends the proof.

5. The geometry of the Simó's triangle and its Lagrange's fixed position

In this section we will express some equations in compact form, using matrix notation. If M is the matrix

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{70}$$

then Eq. (50) is

$$\frac{1}{\mu} A M A^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{71}$$

Although the properties of the Simó's triangle presented in the previous section are remarkable, we want in this section to include several geometric properties that are expressed by means of very symmetric simple expressions.

Associated to the Lagrange' s constraint presented in Eq. (57) we find a generalized equation similar to the previous one

$$\frac{1}{\mu} A M^2 A^{\mathrm{T}} = \begin{pmatrix} X_a & 0 \\ 0 & X_b \end{pmatrix}. \tag{72}$$

Other simple property is obtained computing the area of the Simó's triangle by using the Heron's formula for the area in terms of the distances (55). It results in that the value of the area is equal to 1/2. This is related to our choice of the reference mass μ.

For finite positive masses, the angles of this triangle are acute. The center of mass is at the orthocenter where the altitudes cross. Each altitude separates the triangle in two right triangles. It follows the angle between one altitude and one side is the complement of one angle at the vertex. If we denote by α, β, γ, the angle between sides and altitudes, we verify the repetition of the angles in two different vertices (see Fig. 2). Each angle of the triangle is the sum of two of these angles. These angles have a trigonometric tangent that is a simple function of the masses as follows

$$\tan(\beta+\gamma) = \frac{1}{\tan(\alpha)} = \frac{m_1}{\mu}, \tag{73}$$

$$\tan(\gamma+\alpha) = \frac{1}{\tan(\beta)} = \frac{m_2}{\mu}, \tag{74}$$

$$\tan(\alpha+\beta) = \frac{1}{\tan(\gamma)} = \frac{m_3}{\mu}. \tag{75}$$

The angles between two altitudes is the supplementary angle and we have

$$\tan(\alpha+\pi/2) = -\frac{m_1}{\mu}, \tag{76}$$

$$\tan(\beta+\pi/2) = -\frac{m_2}{\mu}, \tag{77}$$

$$\tan(\gamma+\pi/2) = -\frac{m_3}{\mu}. \tag{78}$$

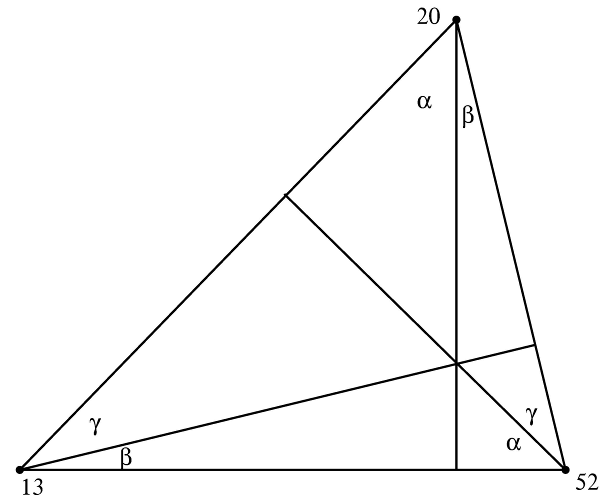

FIGURE 2. The angles α, β and γ between the altitudes and the sides in the Simó's triangle, for masses $m_1 = 52$, $m_2 = 20$, $m_3 = 13$.

It follows a theorem, valid for the Simó's triangle (any triangle with three acute angles) the purely geometric property

$$\tan(\alpha)\tan(\beta)\tan(\gamma) = \tan(\alpha)+\tan(\beta)+\tan(\gamma), \tag{79}$$

which we deduce from our definition of μ.

The Simó's triangle has been fixed in the particular position determined by coordinates that appear as entries of the matrix (47). We give polar coordinates for those positions

$$a_i = \rho_i \cos(\sigma_i), \quad b_i = \rho_i \sin(\sigma_i). \tag{80}$$

The first equation in (54) gives the radial coordinates ρ_i in terms of the masses

$$\rho_i = \sqrt{a_i^2 + b_i^2} = \sqrt{\frac{\mu}{m_j} - \frac{\mu}{m}}. \tag{81}$$

We used these polar coordinates in Ref. [12], calling the σ_i collision angles. At this reference we present the property

$$(\tan(\sigma_1)+\tan(\sigma_2)+\tan(\sigma_3)) \times \left(\frac{1}{\tan(\sigma_1)} + \frac{1}{\tan(\sigma_2)} + \frac{1}{\tan(\sigma_3)}\right) = 9. \tag{82}$$

The difference of these angles is equal to the angle between two altitudes

$$\sigma_2 - \sigma_1 = \gamma + \pi/2, \quad \sigma_3 - \sigma_2 = \alpha + \pi/2, \quad \sigma_1 - \sigma_3 = \beta + \pi/2. \tag{83}$$

From these equations we prove

$$\tan(\sigma_1) = \frac{\mu}{m_2 - m_3}\left(\frac{m_3}{m_2} + \frac{m_2}{m_3} - \frac{m_2 + m_3}{m_1}\right) + \frac{m_2 + m_3}{m_2 - m_3}\sqrt{\frac{\mu^2}{m_1^2} + \frac{\mu^2}{m_2^2} + \frac{\mu^2}{m_3^2} - 1}, \tag{84}$$

$$\tan(\sigma_2) = \frac{\mu}{m_3 - m_1}\left(\frac{m_3}{m_1} + \frac{m_1}{m_3} - \frac{m_1 + m_3}{m_2}\right) + \frac{m_1 + m_3}{m_3 - m_1}\sqrt{\frac{\mu^2}{m_1^2} + \frac{\mu^2}{m_2^2} + \frac{\mu^2}{m_3^2} - 1}, \tag{85}$$

and

$$\tan(\sigma_3) = \frac{\mu}{m_1 - m_2}\left(\frac{m_2}{m_1} + \frac{m_1}{m_2} - \frac{m_1 + m_2}{m_3}\right) + \frac{m_1 + m_2}{m_1 - m_2}\sqrt{\frac{\mu^2}{m_1^2} + \frac{\mu^2}{m_2^2} + \frac{\mu^2}{m_3^2} - 1}. \tag{86}$$

6. Historical notes and conclusions

The first theorem presented in this paper (see the abstract) was proved in the old book by A. Wintner, *The analytical foundations of Celestial Mechanics* [13] sec. 374.

According to Wintner the proof in his book was obtained originally by P. Pizzetti [15]. The Carathéodory proof presented in this paper was not included in Wintner's book. This proof is simpler than the original proofs by Lagrange and Pizzetti, which could be qualified with Sommerfeld [4] as rather involved. This fact is illustrated again when we find in another text of Mechanics written by Prof. K. R. Symon [16], where at the last section of the last chapter he considers the linear stability of the Lagrange's three body solution. Prof. Symon assumes like Laplace that the solution is in a confined fixed plane. According to Wintner [13], p. 431, this is the important part of the Lagrange theorem, that was disregarded by Laplace.

The presentation (5) of the angular velocity in terms of the rotation matrix, as in this paper, is not as popular in texts of mechanics, as it should be. My opinion is that this is the only clear definition of angular velocity in the 3-D case; notwithstanding it uses matrices. The Wintner book, cited above [13], is a good reference to this presentation of the angular velocity, nevertheless it should be noted that he uses the obsolete term *reciprocal matrix* instead of inverse matrix. We have given in the European Journal of Physics [17] other older references to such approach dated 1938.

The Simó's triangle is a rigid triangle used to determine our coordinates. The σ angle is an internal rotation of this triangle for a general configuration. For the Lagrange equilateral solution this angle is zero. For the general configuration, the σ angle is defined with respect to the Lagrange position. At the end of section 4 we prove that in the referential of the principal inertia moments the equilateral triangle is transformed into the Simó's triangle by performing two expansions along the principal inertia directions with magnitudes $\sqrt{X_a/2\mu}$ and $\sqrt{X_b/2\mu}$.

Our Sec. 5 includes simple relations between the trigonometric tangents of the angles of the Simó's triangle and the masses.

The position of the Simó's triangle with respect to the principal inertia referential, in the Lagrange case, has been determined by redundant expressions that are hidden in the literature. Besides Eq. (62), we remark Eqs. (72) and (82), and the set of Eqs. (84-86), all of them different characterizations of that position. These last three are probably original.

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