

Coherent states of a free particle from the coherent states of the harmonic oscillator

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We construct the coherent states of a free particle by implementing a coordinate transformation in the extended configuration space, which establishes a correspondence between the solutions of the Schrödinger equation for a harmonic oscillator and those for a free particle. Using this framework, we derive analytical expressions for the coherent states of a free particle, avoiding the complexities of non-normalizable fiducial states, integrals of motion, or group-theoretic approaches. Our method provides a systematic way to characterize these states at any instant while ensuring that they satisfy the Robertson-Schrödinger uncertainty relation.

Keywords: Coherent states; free particle; uncertainty relations.

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1. Introduction

Recent studies [1,2] have shown that the classical and quantum dynamics of a free particle can be related to those of a harmonic oscillator through an appropriate coordinate transformation within the extended configuration space, given by

$$q = q' \sec \omega t', \quad t = \frac{\tan \omega t'}{\omega}, \quad (1)$$

where ω is a constant representing the angular frequency of the oscillator. More precisely, it was demonstrated that $\Psi_{\text{OSC}}(q', t')$ satisfies the Schrödinger equation for the harmonic oscillator if and only if

$$\Psi_{\text{free}}(q, t) = \cos^{1/2} \omega t' \times \exp\left(\frac{im\omega}{2\hbar} q'^2 \tan \omega t'\right) \Psi_{\text{OSC}}(q', t'), \quad (2)$$

satisfies the Schrödinger equation for a free particle, where q' and t' are expressed in terms of q and t according to the coordinate transformation defined by Eq. (1). This coordinate transformation has also been employed in Ref. [1] to determine the propagator of the harmonic oscillator in terms of the propagator of a free particle.

Building on this approach, the aim of this paper is to make use of the coordinate transformation (1), and the correspondence of the wave functions, given by Eq. (2), to construct the coherent states of a free particle based on the well-known coherent states of the harmonic oscillator. As we will demonstrate below, this approach allows us to characterize the coherent states in detail at any given instant, without requiring complex techniques such as non-normalizable fiducial states [3], integrals of motion [4,5], or group-theoretic methods [6,7].

The structure of the paper is organized as follows: In Sec. 2, we provide a brief overview of the fundamental concepts of coherent states for the harmonic oscillator. Then, we derive the coherent states of a free particle under different initial conditions. Finally, in Sec. 3, we present our concluding remarks.

2. The coherent states for a free particle

The coherent states of a harmonic oscillator were first introduced by Schrödinger [8], as quantum states whose expectation values for the position and momentum follow the corresponding classical solutions. These states exhibit several remarkable properties, many of which were first systematically studied by Glauber [9], who emphasized their importance in the quantum mechanical treatment of optical coherence and formally introduced the term coherent state. As a result, coherent states have found numerous applications across multiple fields, including semiclassical descriptions of quantum systems [10], condensed matter physics and radiation theory [11], quantum computing [12], noncommutative geometry [13], and even quantum gravity [14], among others. However, despite their physical significance, the coherent states of a free particle have not been extensively studied due to the difficulties associated with deriving their explicit analytical expressions. In Refs. [15,16], the coherent states of a free particle, represented as Gaussian wave packets, were constructed using the invariant operator method according to the Lewis-Riesenfeld theory [17]. This approach is based on determining a set of solutions to the Schrödinger equation by using non-Hermitian linear integrals of motion, subject to specific constraints on the parameters. While this method is not limited to the free particle and can be applied to other

quadratic systems, constructing an invariant operator can be highly non trivial in certain cases. The difficulty arises from the need to employ quantum canonical transformations via unitary operators, a formalism that often becomes intricate due to the emergence of nonlinearities.

In the present paper, we propose an alternative and simpler method for deriving the coherent states of a free particle, relying solely on the wave function correspondence given by Eq. (2). To achieve this, we begin by considering the most general coherent state of the harmonic oscillator in the position representation [18,19],

$$\Psi_{\text{Osc}}(q', t') = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-i\frac{\langle\hat{q}'\rangle\langle\hat{p}'\rangle}{2\hbar}\right) \times \exp\left[-\frac{m\omega}{2\hbar}(q' - \langle\hat{q}'\rangle)^2 + \frac{i}{\hbar}\langle\hat{p}'\rangle q' - \frac{i}{2}\omega t'\right], \quad (3)$$

where $\langle\hat{q}'\rangle$ and $\langle\hat{p}'\rangle$ denote the expectation values of the position and momentum operators respectively. This expression differs from a standard Gaussian wave packet by a phase factor, which is essential to ensure the temporal stability of the coherent states [18,20]. Let us consider the case in which the expectation values of position and momentum are given by $\langle\hat{q}'\rangle = q_0 \cos \omega t'$ and $\langle\hat{p}'\rangle = -m\omega q_0 \sin \omega t'$, where q_0 is a real constant. Then, the coherent state (3), takes the form

$$\Psi_{\text{Osc}}(q', t') = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}(q' - q_0 \cos \omega t')^2 - i\left(\frac{m\omega}{\hbar}q_0 q' \sin \omega t' - \frac{m\omega}{4\hbar}q_0^2 \sin 2\omega t' + \frac{1}{2}\omega t'\right)\right]. \quad (4)$$

This wave function describes an oscillating wave packet that oscillates at the classical frequency while maintaining its shape. Furthermore, by applying Stirling's formula, it can be shown that the energy of this state is approximately equal to that of a classical oscillator with the same amplitude [21]. Substituting this expression into the right-hand side of Eq. (2), we obtain the resulting wave function, in terms of q and t ,

$$\Psi_{\text{free}}(q, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{1+i\omega t}} \times \exp\left[-\frac{m\omega}{2\hbar} \frac{1}{1+i\omega t}(q - q_0)^2\right]. \quad (5)$$

This state represents the coherent state of a free particle, with the wave packet initially centered at q_0 and having zero average momentum [15,21]. Calculating the time-dependent standard deviations $\sigma_q(t)$, $\sigma_p(t)$ and the covariance $\sigma_{qp}(t)$ for the state $\Psi_{\text{free}}(q, t)$, given in Eq. (5), we obtain

$$\begin{aligned} \sigma_q(t) &= \sqrt{\langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}(1 + \omega^2 t^2)^{1/2}}, \\ \sigma_p(t) &= \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}}, \\ \sigma_{qp}(t) &= \langle(\hat{q} - \langle\hat{q}\rangle)(\hat{p} - \langle\hat{p}\rangle) + (\hat{p} - \langle\hat{p}\rangle)(\hat{q} - \langle\hat{q}\rangle)\rangle = \frac{\hbar\omega t}{2}. \end{aligned} \quad (6)$$

This result implies that the coherent state $\Psi_{\text{free}}(q, t)$ minimizes the Robertson-Schrödinger uncertainty relation [22],

$$\sigma_q^2(t)\sigma_p^2(t) - \sigma_{qp}^2(t) = \frac{\hbar}{4}, \quad (7)$$

for all instants of time, ensuring the stability of the coherent state.

Now, let us consider the general expectation values of position and momentum for a coherent state of the harmonic oscillator, given by:

$$\begin{aligned} \langle\hat{q}'\rangle &= q_0 \cos \omega t' + \frac{p_0}{m} \sin \omega t', \\ \langle\hat{p}'\rangle &= -m\omega q_0 \sin \omega t' + p_0 \cos \omega t', \end{aligned} \quad (8)$$

where q_0 and p_0 are real constants. Substituting these expressions into Eq. (3) and applying the correspondence relation (2), we express the coherent state of the free particle in terms of the coordinates q and t as:

$$\begin{aligned} \Psi_{\text{free}}(q, t) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{1+i\omega t}} \\ &\times \exp\left[-\frac{m\omega}{2\hbar} \frac{1}{1+i\omega t}(q - q_0)^2 + \frac{i}{\hbar} \frac{1}{1+i\omega t} p_0 \left(q - q_0 - \frac{p_0}{2m} t\right) + \frac{i}{2\hbar} q_0 p_0\right]. \end{aligned} \quad (9)$$

This wave function satisfies the Schrödinger equation for a free particle and coincides with the solution obtained in [16], where coherent states were constructed using a Hermitian linear invariant operator according to the Lewis-Riesenfeld approach. It is noteworthy that the coherent state of the free particle derived in Eq. (9), satisfies the same standard deviations and covariance relations given in Eq. (6). For this state, the expectation values of the position and momentum operators follow the classical trajectory, given by $\langle\hat{q}\rangle = q_0 + p_0/mt$ and $\langle\hat{p}\rangle = p_0$, respectively. Consequently, this implies that the state in Eq. (9) minimizes the Robertson-Schrödinger uncertainty relation for any instant of time.

3. Conclusions

In this paper, we have obtained the coherent states for a free particle. These states were constructed by using a coordinate transformation in the extended configuration space, which establishes a correspondence between solutions of the Schrödinger equation for the harmonic oscillator and those for a free particle. It is worthwhile to mention that the method developed in this work can also be applied to generalized coherent states in various quantum systems, including nonlinear systems and those associated with Lie groups. Our intention is to make use of this correspondence between quantum systems to investigate these features in more general scenarios. This investigation will be addressed in future work.

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