A non-Newtonian approach to electromagnetic curves in optical fiber

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The investigation within this article delves into the non-Newtonian geometric attributes exhibited by a linearly polarized light wave along an optical fiber within the framework of the 3D multiplicative Riemann manifold, employing multiplicative derivative and integral. While conducting this research, the unique arguments of multiplicative analysis (angle, norm, distance, etc.) are used. Within this context, the optical fiber is presumed as a one-dimensional entity embedded in the 3D Riemannian space, establishing a connection between the linearly polarized light wave's evolution and the geometric phase. Consequently, a novel form of the multiplicative geometric phase model is formulated, integrating the principles of multiplicative calculus. Additionally, the concept of multiplicative magnetic curves generated by the electric field Q is introduced. Notably, this study stands out due to its unique utilization of multiplicative derivatives and integrals in the computational processes. The article culminates by presenting illustrative examples consistent with the outlined theoretical framework, accompanied by visual representations. The distinctiveness of this research lies in its departure from conventional methodologies, incorporating multiplicative calculus into the calculations. Remarkably, multiplicative computing demonstrates its applicability across diverse domains, including physics, engineering, mathematical biology, fluid mechanics, and signal processing. The pervasive use of multiplicative derivatives and integrals signifies their profound significance as a novel mathematical approach, contributing substantially to problem-solving methodologies across various scientific disciplines.

Keywords: Non-Newtonian calculus; optical fiber; electromagnetic theory; multiplicative Riemann manifolds; berry phase.

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1. Introduction

The conventional understanding of arithmetic revolves around the conceptualization of measurement and computation, employing fundamental arithmetic operations such as addition, subtraction, multiplication, and division. These operations are used to establish connections between numbers and model mathematical problems. The foundational operations of addition and multiplication, along with the inherent ordering relations, shape the structure of this ordered field, which can be articulated in a manner distinct from conventional definitions. Traditional analysis, rooted in Newtonian principles, is anchored in infinitesimal modifications of the addition process. In contrast, branching from classical analysis, alternative methodologies have emerged, relying on diverse arithmetic operations. For example, in 1887, Volterra V. proposed a Volterra-type analysis, or multiplicative analysis, which based measurements on multiplication operations, as opposed to traditional calculus [1]. This paradigm shift forms the basis of distinct analytical methodologies. In the perspective of multiplicative analysis, the tasks performed by addition and subtraction in Newtonian analysis were replaced by multiplication and division. After the introduction of multiplicative analysis, the work by Grossman M. and Katz R. between 1972 and 1983 had a great impact on this field [2, 3]. This extensive research endeavor led to the conceptualization and formalization of non-Newtonian analysis, an innovative paradigm encompassing fundamental definitions and concepts in mathematics. On the other hand, bigeometric analysis extends this proportional perspective to both the alterations in functional values and the changes in arguments. Here, ratios govern the measurement of both functions and their arguments, providing a comprehensive proportional viewpoint. Anageometric calculation, in contrast, employs a linear approach for measuring the changes in functional values, akin to traditional calculus using linear differences. Meanwhile, arguments within anageometric calculus are quantified by ratios, aligning more with a proportional measurement scale. These various non-Newtonian calculi offer diverse mathematical methodologies by blending proportional and linear measurement concepts, thereby expanding the spectrum of mathematical operations and problem-solving techniques. Although multiplicative analysis did not attract much attention at first, its advantages, especially in measuring metric concepts, made it popular. The main reason for this is that in multiplicative analysis, measurement operations are measured proportionally. This alternative framework particularly excels in addressing problems characterized by exponential or proportional changes. Its inherent capacity to handle "scale changes" during computations often yields more precise results for specific problem sets. The introduction of multiplicative calculus presents an innovative departure from conventional mathematical approaches, offering enhanced solutions tailored to particular problems. Central to this paradigm shift are the concepts of multiplicative derivative and integral, which form the cornerstone of this alternative approach. Their applications span across diverse domains, including probability theory, statistics, financial mathematics, physics, engineering, and economics [4–8]. This versatile methodology extends superior problem-solving capabilities within these domains, embracing a proportional perspective that complements and enhances conventional mathematical analyses. Recently, multiplicative analysis has entered almost every field of mathematics and has become a favorite of researchers. For example, the most significant results of multiplicative analysis are given in [9, 10], some developments in multiplicative numerical analysis in [11–13], properties of multiplicative differential equations in [14–16]. Bashirov A. et al. presented us with the preliminary information that will form the basis for multiplicative analysis in [17]. Yılmaz E. et al. established the multiplicative Dirac system structure and introduced it to the literature in [18–20]. Georgiev S. has presented the most important and concrete studies in this field with the books he published on multiplicative analysis in 2022 [21–23]. In these works, Georgiev S.G. not only took the operators as multiplicative, but also equipped all mathematical systems with multiplicative arguments and presented us with a full-fledged multiplicative space.

The adaptation of multiplicative analysis to the field of geometry began with the work of Georgiev S.G. in this field [22, 23]. In addition, Nurkan S. K. et al. examined vector analysis with multiplicative arguments and presented them to researchers [24]. Afterwards, Aydın M. E. et al. studied multiplicative rectifier curves in depth [25, 26]. Has A. and Yılmaz B. introduced magnetic curves to the literature by using multiplicative arguments on multiplicative Riemann manifolds [27]. In more recent research endeavors, Has A. and Yılmaz B. introduced non-Newtonian (multiplicative) conics to the academic discourse, further extending the application scope of multiplicative analysis [28]. These pioneering contributions collectively amplify the application landscape of multiplicative analysis within geometry, paving the way for innovative approaches and enriched perspectives within this field.

The profound interconnections between geometry, a fundamental sub-discipline of mathematics, and physics are integral to our understanding of various natural phenomena. Within the realm of differential geometry, the investigation of space curves stands out as an intriguing field, showcasing numerous significant applications across diverse branches of physics. Notably, the correlation between these two domains was extensively explored in numerous studies. For instance, Rytov's law, elucidating the rotation of the polarization plane along an optical fiber, is expounded upon with introductions and geometric interpretations found in [29]. Furthermore, investigations into the rotation of polarization concerning geometric effects in low birefringence single-mode optical fibers are conducted by Kugler M., Ross J. N., and others [30, 31]. Additionally, several authors have contributed varied characterizations of the geometric phase, offering comprehensive insights into this phenomenon [32–38].

This paper delves into deriving the equations of geometric phase for a linearly polarized light wave within an optical fiber, considering three distinct states of the multiplicative polarization vector. The utilization of multiplicative calculus facilitates the formulation of these equations. Furthermore, our study encompasses the derivation of multiplicative Rytov curves in this context. We introduce the concept of multiplicative electromagnetic curves generated along the polarization plane of an optical fiber by the electric field, employing the principles of multiplicative derivative within the 3D multiplicative Riemann manifold. Additionally, we complement our theoretical findings with visual representations using the Geogebra program. A key distinguishing aspect of our study lies in the meticulous consideration of conditions within the multiplicative space during our calculations.

2. Preliminaries

In this section, basic information about multiplicative analysis will be presented and then multiplicative differential geometry will be introduced.

2.1. Background on multiplicative calculus

The definitions and theorems that will be presented in this section are taken from the works of Georgiev S.G. [21–23].

Since the multiplicative space has an exponential structure, the sets of multiplicative real numbers are we have

$$\mathbb{R}_* = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+, \ \mathbb{R}_*^+ = \{e^x : x \in \mathbb{R}^+\} = (1, \infty) \text{and} \quad \mathbb{R}_*^- = \{e^x : x \in \mathbb{R}^-\} = (0, 1). \tag{1}$$

The basic multiplicative operations for all $m, n \in \mathbb{R}_*$, is

$$\begin{aligned} m +_* n &= e^{\log m + \log n} = mn, \\ m -_* n &= e^{\log m - \log n} = m/n, \\ m \cdot_* n &= e^{\log m \log n} = m^{\log n}, \end{aligned} \tag{2}$$

$$m /_* n = e^{\log m / \log n} = m^{\frac{1}{\log n}}, \quad n \neq 1. \tag{3}$$

According to the multiplicative addition operation, the multiplicative neutral and unit elements are $0_* = 1$ and $1_* = e$, respectively.

The inverse elements of multiplicative addition and multiplicative multiplication operations for all $m \in \mathbb{R}_*$ are as follows, respectively:

$$-_* m = 1/m, \quad m^{-1_*} = e^{\frac{1}{\log m}}. \tag{4}$$

Absolute value function in multiplicative space, we have

$$|m|_* = \begin{cases} m, & m \geq 0_* \\ -_* m, & m < 0_* \end{cases}. \tag{5}$$

With the help of multiplicative arguments, the multiplicative power function can be given as for all $m \in \mathbb{R}_*$ and $k \in \mathbb{N}$

$$m^{k_*} = e^{(\log m)^k}, \quad m^{\frac{1}{2}_*} = \sqrt[*]{m} = e^{\sqrt{\log m}}. \tag{6}$$

A vector whose components are elements of the space $\mathbb{R}_*$ is called a multiplicative vector and satisfies the following properties $\overrightarrow{\mathbf{r}} = (r_1, r_2, ..., r_n)$, $\overrightarrow{\mathbf{s}} = (s_1, s_2, ..., s_n) \in \mathbb{R}_*^n$ multiplicative vectors and $\lambda \in \mathbb{R}_*$, as follows

$$\overrightarrow{\mathbf{r}} +_* \overrightarrow{\mathbf{s}} = (r_1 +_* s_1, ..., r_n +_* s_n) = (r_1 s_1, .., r_n s_n), \quad (7)$$

$$\lambda \cdot_* \overrightarrow{\mathbf{r}} = (\lambda \cdot_* r_1, ..., \lambda \cdot_* r_n) = (r_1^{\log \lambda}, ..., r_n^{\log \lambda}) = e^{\log \overrightarrow{\mathbf{r}} \log \lambda}, \quad (8)$$

where $\log \overrightarrow{\mathbf{r}} = (\log r_1, \log r_2, ..., \log r_n)$. Let $\overrightarrow{\mathbf{r}} = (r_1, r_2, ..., r_n)$ and $\overrightarrow{\mathbf{s}} = (s_1, s_2, ..., s_n) \in \mathbb{R}_*^n$ be two multiplicative vectors in the multiplicative vector space $\mathbb{R}_*^n$. Thus, the multiplicative inner product of two multiplicative vectors is follows

$$\langle \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \rangle_* = r_1 \cdot_* s_1 +_* ... +_* r_n \cdot_* s_n = e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{s}} \rangle}. \quad (9)$$

If the multiplicative vectors $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$ are multiplicative orthogonal to each other, they are denoted by $\overrightarrow{\mathbf{r}} \perp_* \overrightarrow{\mathbf{s}}$ and this relation is as follows

$$\langle \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \rangle_* = 0_*. \quad (10)$$

We gave a visual of multiplicative orthogonal vectors in Fig. 1.

The multiplicative norm of the multiplicative vector $\overrightarrow{\mathbf{r}} \in \mathbb{R}_*^n$ is given by the multiplicative inner product The multiplicative norm of a vector $\mathbf{u} \in \mathbb{R}_*^n$ is defined as follows

$$\|\overrightarrow{\mathbf{r}}\|_* = e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{r}} \rangle^{\frac{1}{2}}}. \quad (11)$$

Let $\overrightarrow{\mathbf{r}} = (r_1, r_2, r_3)$ and $\overrightarrow{\mathbf{s}} = (s_1, s_2, s_3)$ be 3D multiplicative vectors, and the multiplicative cross products of $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$, we have

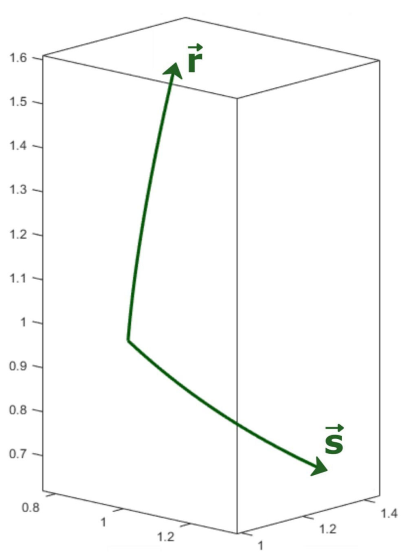

FIGURE 1. Multiplicative orthogonal vectors.

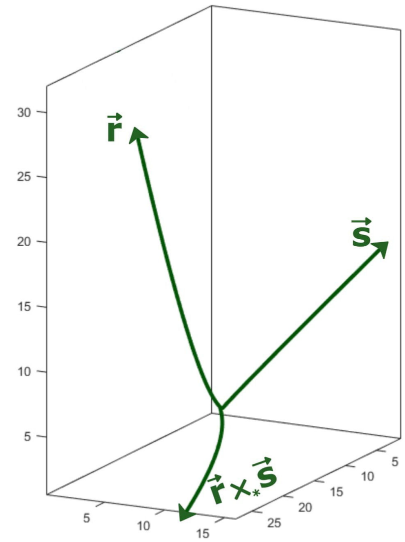

FIGURE 2. Multiplicative orthogonal system.

$$\overrightarrow{\mathbf{r}} \times_* \overrightarrow{\mathbf{s}} = (e^{\log r_2 \log s_3 - \log r_3 \log s_2}, e^{\log r_3 \log s_1 - \log r_1 \log s_3}, e^{\log r_1 \log s_2 - \log r_2 \log s_1}). \quad (12)$$

Multiplicative cross product preserves the properties of traditional cross product with its arguments. For example, cross-products of multiplicative vectors $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$ are multiplicative orthogonal to both $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$. We give this visually in Fig. 2.

The multiplicative angle between the multiplicative unit direction vectors $\overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \in \mathbb{R}_*^n$ is given by

$$\phi = \arccos_*(e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{s}} \rangle}). \quad (13)$$

Multiplicative trigonometric functions with the help of multiplicative angles

$$\sin_* \phi = e^{\sin \log \phi}, \quad \cos_* \phi = e^{\cos \log \phi}, \quad (14)$$

$$\tan_* \phi = e^{\tan \log \phi}, \quad \cot_* \phi = e^{\cot \log \phi}. \quad (15)$$

Multiplicative trigonometric functions provide the same algebraic properties as traditional trigonometric functions, but with their arguments. For example, there is the equality $\sin_*^{2_*} \theta +_* \cos_*^{2_*} \theta = 1_*$. For other relations, see [21].

The multiplicative derivative of the multiplicative function $f(t) \subset \mathbb{R}_*$ for $t \in I \subset \mathbb{R}_*$ is as follows

$$f^*(t) = \lim_{h \to 0_\star} \left(\left(f\left(t +_\star h\right) -_\star f\left(t\right) \right) /_\star h \right) = \lim_{h \to 1} \exp\left[\frac{\log f\left(th\right) - \log f\left(t\right)}{\log\left(h\right)} \right] = \lim_{h \to 1} \exp\left[\frac{th f'\left(th\right)}{f\left(th\right)} \right] = e^{t \frac{f'(t)}{f(t)}}. \quad (16)$$

Multiplicative differentiation realizes many properties provided in classical differentiation, such as linearity, Leibniz rule, chain rules, etc., based on multiplicative arguments. For examples $(f(x) \cdot_* g(x))^* = f^*(x) \cdot_* g(x) +_* g^*(x) \cdot_* f(x)$. It can also be stated as $f^*(x) = d_* f /_* d_* x$. For other relations, see [21].

The multiplicative integral of the multiplicative function $f(t) \subset \mathbb{R}_*$ is as follows for $t \in I \subset \mathbb{R}_*$

$$\int_* f(x) \cdot_* d_* x = e^{\int \frac{1}{x} \log f(x) dx}, \quad x \in \mathbb{R}_*. \tag{17}$$

2.2. Multiplicative differential geometry

Curves in three-dimensional multiplicative space are geometric objects that can be mathematically expressed by a parametric equation and move within a three-dimensional environment. These multiplicative curves are typically defined as vector functions representing the x_1, x_2 and x_3 coordinates of a multiplicative curve's points with respect to a parameter (usually denoted as s). For instance, consider a multiplicative curve represented as $\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s))$. This multiplicative curve describes the position of each point along the multiplicative curve within a certain interval of the parameter s, where $x_1(s)$, $x_2(s)$, and $x_3(s)$ functions define the x_1, x_2 and x_3 coordinates of the multiplicative curve. Let $a < b$ denote real numbers. A multiplicative function $\mathbf{x} : [a, b] \to \mathbb{R}^3_*$ is called a multiplicative vector-valued function. A multiplicative vector valued function can be described in coordinates as

$$\mathbf{x}(s) = x_1(s) \cdot_* i +_* x_2(s) \cdot_* j +_* x_3(s) \cdot_* k \tag{18}$$

where $i = (e, 1, 1)$, $j = (1, e, 1)$ and $j = (1, 1, e)$. Also in short, $\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s))$. $\mathbf{x}(s) : [a, b] \to \mathbb{R}^3_*$, $\mathbf{x}(s) = [x_1(s), x_2(s), x_3(s)]$ is called smooth C^∞ if the coordinate functions $x_1(s)$, $x_2(s)$ and $x_3(s)$ are infinitely many times multiplicative differentiable on the open interval (a, b) and multiplicative continuous on (a, b). For $s \in (a, b)$, its multiplicative derivative $\mathbf{x}(s) : [a, b] \to \mathbb{R}^3_*$ is given by $\mathbf{x}^*(s) = (x^*_1(s), x^*_2(s), x^*_3(s))$, i.e., by the multiplicative derivatives of the coordinate functions. We call $\mathbf{x}(s)$ multiplicative naturally parametrized curve if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^k_* and $\|\mathbf{x}^*(s)\|_* = e$, for each $s \in I$ [22]. Given $s_0 \in I$, the multiplicative arc length of a multiplicative regular parametrized curve $\mathbf{x}(s)$ from the point s_0, is by definition

$$h(s) = \int_{*s_0}^{s} \|\mathbf{x}^*(t)\|_* \cdot_* d_* t. \tag{19}$$

Also multiplicative smooth curve is said to be multiplicative regular if $\mathbf{x}^*(s) \neq 0_*$ for all $s \in [a, b]$, [22].

The multiplicative Frenet vectors of the multiplicative curve $\mathbf{x}(s)$ are as follows [22]

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{x}^*(s), \\ \mathbf{N}(s) &= \mathbf{x}^{**}(s) /_* \|\mathbf{x}^{**}(s)\|_*, \\ \mathbf{B}(s) &= \mathbf{T}(s) \times_* \mathbf{N}(s), \end{aligned} \tag{20}$$

where vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the multiplicative tangent, normal and binormal vectors of the curve $\mathbf{x}$, respectively and these vectors are multiplicative orthogonal to each other as pairs, thus forming a multiplicative orthogonal system.

The multiplicative derivative equivalents of the multiplicative Frenet vectors are as follows [22]

$$\begin{aligned} \mathbf{T}^* &= e^{\log \kappa \log \mathbf{N}}, \\ \mathbf{N}^* &= e^{-\log \kappa \log \mathbf{T} + \log \tau \log \mathbf{B}}, \\ \mathbf{B}^* &= e^{-\log \tau \log \mathbf{N}}, \end{aligned} \tag{21}$$

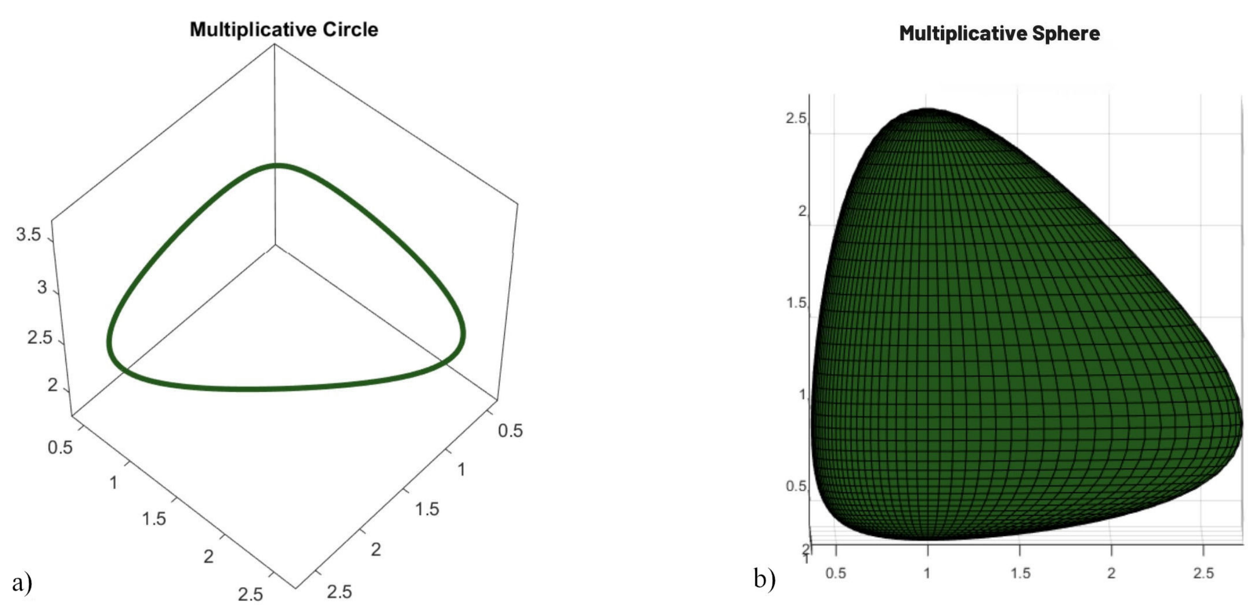

FIGURE 3. A multiplicative circle and sphere with centered at multiplicative origin $O(0_*, 0_*, 0_*)$ and radius 1_*.

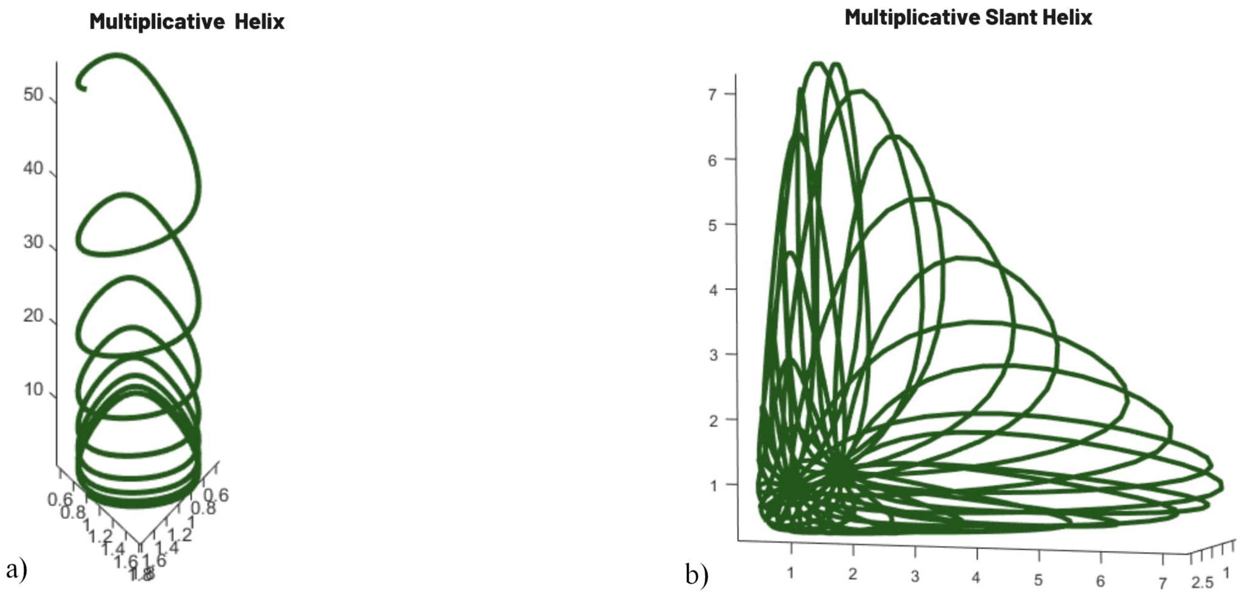

FIGURE 4. A multiplicative helix and slant helix.

or with multiplicative arguments

$$\begin{aligned} \mathbf{T}^* &= \kappa \cdot_* \mathbf{N}, \\ \mathbf{N}^* &= -_* \kappa \cdot_* \mathbf{T} +_* \tau \cdot_* \mathbf{B}, \\ \mathbf{B}^* &= -_* \tau \cdot_* \mathbf{N}, \end{aligned} \tag{22}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of $\mathbf{x}$ and they are given by [22]

$$\kappa(s) = \|\mathbf{x}^{**}(s)\|_* = e^{(\langle \log \mathbf{x}^{**}, \log \mathbf{x}^{**} \rangle)^{\frac{1}{2}}}, \tag{23}$$

$$\tau(s) = \langle \mathbf{N}^*(s), \mathbf{B}(s) \rangle_* = e^{\langle \log \mathbf{N}^*(s), \log \mathbf{B}(s) \rangle}. \tag{24}$$

Has A. and Yılmaz B. studied some geometric structures with multiplicative arguments [26]. We gave the multiplicative unit circle whose center $O(0_*, 0_*, 0_*)$ is the multiplicative origin in Fig. 3.

They also obtained some characterizations of multiplicative helices and multiplicative slant helices as follows, respectively; [26]

$$\tau /_* \kappa = c, \quad c \in \mathbb{R}_* \tag{25}$$

and

$$\left(\kappa^{2*} /_* (\kappa^{2*} +_* \tau^{2*})\right) \cdot_* (\tau /_* \kappa)^* = c, \quad c \in \mathbb{R}_*. \tag{26}$$

Additionally, Fig. 4 shows the helix with parameterization

$$\mathbf{x}(s) = \left((e^3 /_* e^5) \cdot_* \cos_* s, (e^3 /_* e^5) \cdot_* \sin_* s, (e^4 /_* e^5) \cdot_* s\right),$$

and the slant helix with parameterization

$$\mathbf{x}(s) = \left(\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s)\right),$$

where

$$\begin{aligned} \mathbf{x}_1(s) &= (e^9 /_* e^{400}) \cdot_* \sin_* 25s +_* (e^{25} /_* e^{144}) \cdot_* \sin_* 9s, \\ \mathbf{x}_2(s) &= -_*(e^9 /_* e^{400}) \cdot_* \cos_* 25s +_* (e^{25} /_* e^{144}) \cdot_* \cos_* 9s, \\ \mathbf{x}_3(s) &= (e^{15} /_* e^{136}) \cdot_* \sin_* 17s. \end{aligned}$$

3. Multiplicative geometric phase with multiplicative derivative

This section introduces the representation of an optical fiber within a multiplicative space curve employing the principles of multiplicative calculus. We begin by considering $\mathbf{x}(s)$ within the multiplicative Euclidean space. Given that the optical fiber is a one-dimensional entity embedded within a $3D$ multiplicative Riemann manifold, we establish a correlation between the evolution of a linearly polarized light wave and a multiplicative geometric phase. Simultaneously, the orientation of the linearly polarized light wave is determined by the direction of the electric field $\mathbf{Q}(s)$. Consequently, the direction $\mathbf{Q}(s)$ along an optical fiber is formulated within the context of the multiplicative Frenet frame $\{\mathbf{T} = (t_1, t_2, t_3), \mathbf{N} = (n_1, n_2, n_3), \mathbf{B} = (b_1, b_2, b_3)\}$ as follows:

$$\mathbf{Q}^*(s) = e^{(\log\lambda_1 \log t_1 + \log\lambda_2 \log n_1 + \log\lambda_3 \log b_1, \log\lambda_1 \log t_2 + \log\lambda_2 \log n_2 + \log\lambda_3 \log b_2, \log\lambda_1 \log t_3 + \log\lambda_2 \log n_3 + \log\lambda_3 \log b_3)}$$

or equally

$$\mathbf{Q}^*(s) = \lambda_1 \cdot_* \mathbf{T} +_* \lambda_2 \cdot_* \mathbf{N} +_* \lambda_3 \cdot_* \mathbf{B}, \qquad (27)$$

where λ_1, λ_2 and λ_3 are multiplicative differentiable functions.

The polarization state of light is examined in the following for three distinct cases.

Case 1. In this initial scenario, it is assumed that the $\mathbf{Q}$ resides within multiplicative plane that is multiplicative orthogonal to $\mathbf{T}$. Consequently, we can express this as

$$\langle \mathbf{Q}, \mathbf{T} \rangle_* = e^{\langle \log \mathbf{Q}, \log \mathbf{T} \rangle} = 0_*. \qquad (28)$$

Take the above equation be multiplicative differentiated with respect to s, as follows

$$\langle \mathbf{Q}^*, \mathbf{T} \rangle_* +_* \langle \mathbf{Q}, \mathbf{T}^* \rangle_* = 0_*,$$

and

$$\langle \mathbf{Q}^*, \mathbf{T} \rangle_* = -_* \kappa \cdot_* \langle \mathbf{Q}^*, \mathbf{N} \rangle_*.$$

From Eq. (27) and multiplicative Frenet formulae, we have

$$\lambda_1 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{T} \rangle} +_* \lambda_2 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{N} \rangle}$$
$$+_* \lambda_3 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{B} \rangle} +_* \kappa \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{N} \rangle} = 0_*.$$

Take upon making the required adjustment, we acquire the initial coefficient λ_1 in Eq. (27) as

$$\lambda_1 = -_* \kappa \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_*. \qquad (29)$$

We assume there is no loss mechanism in the optical fiber due to absorption; we have $\langle \mathbf{Q}, \mathbf{Q} \rangle_* = c$, where $c \in \mathbb{R}_*$. Then, by taking multiplicative derivatives of the equation $\langle \mathbf{Q}, \mathbf{Q} \rangle_* = c$, we obtain the following

$$e^2 \cdot_* \langle \mathbf{Q}^*, \mathbf{Q} \rangle_* = 0_*$$

and

$$\langle \mathbf{Q}^*, \mathbf{Q} \rangle_* = 0_*.$$

Upon this using Eq. (27), we can write

$$\lambda_1 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{T} \rangle} +_* \lambda_2 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{N} \rangle}$$
$$+_* \lambda_3 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{B} \rangle} = 0_*.$$

Thereupon, consider Eq. (28) in the above equation, we get

$$\lambda_2 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{N} \rangle} +_* \lambda_3 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{B} \rangle} = 0,$$

and so

$$\lambda_2 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{N} \rangle} = -_* \lambda_3 \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{B} \rangle}$$
$$\text{or} \quad \lambda_2 \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* = -_* \lambda_3 \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*.$$

From Eq. (28), since $\langle \mathbf{Q}, \mathbf{N} \rangle_* \neq 0_*$ and $\langle \mathbf{Q}, \mathbf{B} \rangle_* \neq 0_*$, λ_2 and λ_3 are multiplicative proportional to each other and we can give

$$\lambda_2 = e^{\log \lambda \langle \log \mathbf{Q}, \log \mathbf{B} \rangle} \quad \text{and} \quad \lambda_3 = e^{-\log \lambda \langle \log \mathbf{Q}, \log \mathbf{N} \rangle}$$

or

$$\lambda_2 = \lambda \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \quad \text{and} \quad \lambda_3 = -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_*. \qquad (30)$$

So let's use Eq. (29) and (30) in Eq. (27), then we give

$$d \cdot_* \mathbf{Q} /_* d \cdot_* s = e^{-\log \kappa \langle \log \mathbf{Q}, \log \mathbf{N} \rangle \log \mathbf{T} + \log \lambda \langle \log \mathbf{Q}, \log \mathbf{B} \rangle \log \mathbf{N} - \log \lambda \langle \log \mathbf{Q}, \log \mathbf{N} \rangle \log \mathbf{B}}, \qquad (31)$$

or with multiplicative arguments

$$d \cdot_* \mathbf{Q} /_* d \cdot_* s = -_* \kappa \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{T} +_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{N} -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{B}. \qquad (32)$$

We know that, is

$$d \cdot_* \mathbf{Q} /_* d \cdot_* s = e^{-\log \kappa \langle \log \mathbf{Q}, \log \mathbf{N} \rangle \log \mathbf{T} + \log \lambda (\langle \log \mathbf{Q}, \log \mathbf{B} \rangle \log \mathbf{N} - \langle \log \mathbf{Q}, \log \mathbf{N} \rangle \log \mathbf{B})}. \qquad (33)$$

By leveraging the properties of the multiplicative vector-product and applying the multiplicative Frenet formulae, we attain the following result

$$d \cdot_* \mathbf{Q} /_* d \cdot_* s = -_* \kappa \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{T} +_* \lambda \cdot_* (\mathbf{Q} \times_* \mathbf{T}). \qquad (34)$$

The rotation around $\mathbf{T}$ is represented by the second term on the right side of the aforementioned equation. Assuming $\mathbf{T}$ is multiplicative parallel transported (*i.e.*, $\lambda = 0_*$), as follow

$$d \cdot_* \mathbf{Q} /_* d \cdot_* s = -_* \kappa \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{T}.$$

Also, the multiplicative polarization vector is written as follows

$$\mathbf{Q} = \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{N} +_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{B}. \qquad (35)$$

Whenupon multiplicative differentiating Eq. (35) with respect to s and comparing the result with Eq. (32), it is evident that

$$\mathbf{Q}^* = -_*\kappa \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* \cdot_* \mathbf{T} +_* ((d_*/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_*) \cdot_* \mathbf{N}$$
$$+_* ((d_*/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_* +_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_*) \cdot_* \mathbf{B},$$

and so, we establish the following system

$$(d_*/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* = \tau \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_*,$$
$$(d_*/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_* = -_*\tau \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_*.$$

Furthermore, considering that $\langle \mathbf{Q}, \mathbf{Q}\rangle_* = c$ where $c \in \mathbb{R}_*$, employing multiplicative spherical coordinates, we can express the following relationship

$$\mathbf{Q} = e^{\log e^{\sin \log \theta} \log \mathbf{N} + \log e^{\cos \log \theta} \log \mathbf{B}}, \tag{36}$$

or with multiplicative arguments

$$\mathbf{Q} = \sin_* \theta \cdot_* \mathbf{N} +_* \cos_* \theta \cdot_* \mathbf{B}, \tag{37}$$

where θ multiplicative angle. Thereupon computing the multiplicative differentiating of Eq. (37) and using multiplicative Frenet formulae, the resultant expression is given as follows

$$\mathbf{Q}^* = -_* \sin_* \theta \cdot_* \kappa \cdot_* \mathbf{T} +_* ((d_*\theta/_*d_*s) \cdot_* \cos_* \theta -_* \cos_* \theta \cdot_* \tau) \cdot_* \mathbf{N} +_* (\sin_* \theta \cdot_* \tau -_* (d_*\theta/_*d_*s) \cdot_* \sin_* \theta) \cdot_* \mathbf{B}, \tag{38}$$

and then

$$\mathbf{Q}^* = -_* \kappa \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* \cdot_* \mathbf{T} +_* ((d_*\theta/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B}\rangle_*) \cdot_* \mathbf{N} +_* (\tau \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* -_* (d_*\theta/_*d_*s) \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_*) \cdot_* \mathbf{B}\,.$$

As another impression, we give

$$\mathbf{Q}^* = e^{-\log \kappa \langle \log \mathbf{Q}, \log \mathbf{N}\rangle \log \mathbf{T} + \log((d_*\theta/_*d_*s - \tau)(\langle \mathbf{Q}, \mathbf{B}\rangle \log \mathbf{N} - \langle \mathbf{Q}, \mathbf{N}\rangle) \log \mathbf{B}}.$$

By employing the characteristics of the multiplicative vector-product in the equation above, we derive the following result

$$\mathbf{Q}^* = -_*\kappa \cdot_* \langle \mathbf{Q}, \mathbf{N}\rangle_* \cdot_* \mathbf{T} +_* ((d_*\theta/_*d_*s) -_* \tau) \cdot_* (\mathbf{Q} \times_* \mathbf{T}). \tag{39}$$

Optical fiber does not exhibit a preference for rotation of the field. Typically observed in the presence of optical activity, the coefficient of the second term on the right side of the aforementioned equation should be taken equal to 0_*, we obtain

$$(d_*\theta/_*d_*s) -_* \tau = 0_*.$$

Taking both sides of the above equation are integrated using the multiplicative integral, the result yields

$$\int_* (d_*\theta/_*d_*s) d_*s = e^{\int \frac{1}{s} \log e^{s \frac{d \log \theta}{ds}} ds} = e^{\int \frac{d \log \theta}{ds} ds} = \theta, \tag{40}$$

and

$$\theta = \int_* \tau \cdot_* d_*s.$$

Hence, we derive the multiplicative $\mathbf{Q}_t$-Rytov curve, which represents the traced curve of the polarization vector in the optical context, subject to the condition $\langle \mathbf{Q}, \mathbf{T}\rangle_* = 0_*$, where $\mathbf{T}$ denotes the direction

$$\mathbf{Q}_t = \mathbf{x}(s) +_* \mathbf{Q}(s),$$

and

$$\mathbf{Q}(s) = \sin_* \left(\int_* \tau \cdot_* d_*s \right) \cdot_* \mathbf{N}(s) +_* \cos_* \left(\int_* \tau \cdot_* d_*s \right) \cdot_* \mathbf{B}(s)\,.$$

Case 2. For the second case, the following assumption can be made for the fundamental fiber optical.

$$\langle \mathbf{Q}, \mathbf{N} \rangle_* = e^{\langle \log \mathbf{Q}, \log \mathbf{N} \rangle} = 0_* .$$

Consider Eq. (27) and taking multiplicative differentiating aforementioned equation with respect to s and using multiplicative Frenet formulae, we obtain

$$\langle \lambda_1 \cdot_* \mathbf{T} +_* \lambda_2 \cdot_* \mathbf{N} +_* \lambda_3 \cdot_* \mathbf{B}, \mathbf{N} \rangle_* +_* \langle \mathbf{Q}, (-_* \kappa \cdot_* \mathbf{T} +_* \tau \cdot_* \mathbf{B}) \rangle_* = 0_* ,$$

or equivalently

$$\lambda_1 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{N} \rangle} +_* \lambda_2 \cdot_* e^{\langle \log \mathbf{N}, \log \mathbf{N} \rangle} +_* \lambda_3 \cdot_* e^{\langle \log \mathbf{B}, \log \mathbf{N} \rangle} -_* \kappa \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{T} \rangle} +_* \tau \cdot_* e^{\langle \log \mathbf{Q}, \log \mathbf{B} \rangle} = 0_* .$$

Whereupon making the necessary adjustments, the second coefficient λ_2 in Eq. (27) can be obtained as

$$\lambda_2 = \kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* . \tag{41}$$

Taking the multiplicative differentiating $\langle \mathbf{Q}, \mathbf{Q} \rangle_* = c$, $c \in \mathbb{R}_*$, we get

$$e^2 \cdot_* \langle \mathbf{Q}^*, \mathbf{Q} \rangle_* = 0_* ,$$

and

$$\langle \mathbf{Q}^*, \mathbf{Q} \rangle_* = 0_* .$$

Consider with Eq. (27) and above equation, as follow

$$\lambda_1 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{Q} \rangle} +_* \lambda_2 \cdot_* e^{\langle \log \mathbf{N}, \log \mathbf{Q} \rangle} +_* \lambda_3 \cdot_* e^{\langle \log \mathbf{B}, \log \mathbf{Q} \rangle} = 0_* ,$$

and finally

$$\lambda_1 \cdot_* e^{\langle \log \mathbf{T}, \log \mathbf{Q} \rangle} = -_* \lambda_3 \cdot_* e^{\langle \log \mathbf{B}, \log \mathbf{Q} \rangle}$$

or simplify

$$\lambda_1 \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* = -_* \lambda_3 \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* = 0_* .$$

When both $\langle \mathbf{Q}, \mathbf{T} \rangle_* \neq 0_*$ and $\langle \mathbf{Q}, \mathbf{B} \rangle_* \neq 0_*$, it follows that λ_1 and λ_2 are proportional to each other, resulting in, we get

$$\lambda_1 = \lambda \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \quad \text{and} \quad \lambda_3 = -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* . \tag{42}$$

Let's incorporate Eq. (41) and (42) in Eq. (27), as follow

$$\mathbf{Q}^* = \lambda \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{T} +_* (\kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*) \cdot_* \mathbf{N} -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* \cdot_* \mathbf{B}. \tag{43}$$

Once necessary adjustments are applied, the following expression as

$$\mathbf{Q}^* = (\kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*) \cdot_* \mathbf{N} -_* \lambda \cdot_* (\mathbf{Q} \times_* \mathbf{N}).$$

Just as in the *Case 1*, the latter part of the equation above corresponds to rotation around $\mathbf{N}$. Assuming the parallel transport of $\mathbf{N}$, we obtain

$$\mathbf{Q}^* = (\kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*) \cdot_* \mathbf{N}. \tag{44}$$

Furthermore, considering these aspects, we can express the polarization vector as follows

$$\mathbf{Q} = \langle \mathbf{Q}, \mathbf{T} \rangle_* \cdot_* \mathbf{T} +_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{B}. \tag{45}$$

Thereupon computing the multiplicative differentiating of Eq. (45), the following as

$$\mathbf{Q}^* = (d_* /_* d_* s) \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* \cdot_* \mathbf{T} +_* (\kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*) \cdot_* \mathbf{N} +_* (d_* /_* d_* s) \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{B}.$$

Thus, upon comparing the aforementioned equation with Eq.(44), we establish the following system

$$(d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*=0_*,$$
$$(d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{B}\rangle_*=0_*.$$

Meanwhile, the representation of the polarization vector in accordance with multiplicative spherical coordinates is expressed as

$$\mathbf{Q}=e^{\log e^{\cos\log\theta}\log\mathbf{T}+e^{\sin\log\theta}\log\mathbf{B}},\tag{46}$$

or with multiplicative arguments

$$\mathbf{Q}=\cos_*\theta\cdot_*\mathbf{T}+_*\sin_*\theta\cdot_*\mathbf{B}.\tag{47}$$

If we take the multiplicative derivative of this equation, the resulting expression becomes

$$\mathbf{Q}^*=(\kappa\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*-_*\tau\cdot_*\langle\mathbf{Q},\mathbf{B}\rangle_*)\cdot_*\mathbf{N}+_*(d_*\theta/_*d_*s)\cdot_*(\mathbf{Q}\times_*\mathbf{N}).\tag{48}$$

Since Eq. (48) , we obtain

$$d_*\theta/_*d_*s=0_*,$$

and so

$$\theta=c,\quad c\in\mathbb{R}_*.\tag{49}$$

Hence, we can infer that $\mathbf{Q}$ induces the parallel transport along $\mathbf{N}$. Consequently, is characterized $\mathbf{Q}_n$-Rytov multiplicative curve in the second case. The orientation of the polarized light state evolves in accordance with $\mathbf{N}$. As a result, we derive the $\mathbf{Q}_n$-Rytov multiplicative curve and the polarization vector within the optical context, subject to the conditions $\langle\mathbf{Q},\mathbf{B}\rangle_*=0_*$ respectively, as

$$\mathbf{Q}_n(s)=\mathbf{x}(s)+_*\mathbf{Q}(s),\tag{50}$$

and

$$\mathbf{Q}=\cos_*\theta\cdot_*\mathbf{T}+_*\sin_*\theta\cdot_*\mathbf{B},\tag{51}$$

where θ multiplicative constant angle.

Case 3. In the last scenario, assuming that $\mathbf{Q}$ lies within a multiplicative plane and is multiplicative orthogonal to $\mathbf{B}$, the expression can be formulated as follows

$$\langle\mathbf{Q},\mathbf{B}\rangle_*=e^{\langle\log\mathbf{Q},\log\mathbf{B}\rangle}=0_*.$$

Multiplicative differentiating the above equation with respect to the s and employing the multiplicative Frenet elements, we obtain

$$\langle\lambda_1\cdot_*\mathbf{T}+_*\lambda_2\cdot_*\mathbf{N}+_*\lambda_3\cdot_*\mathbf{B},\mathbf{B}\rangle_*$$
$$-_*\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*=0_*.$$

By making the necessary adjustments, λ_3 in Eq. (27) is obtained as follows

$$\lambda_3=\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*.\tag{52}$$

Let $\langle\mathbf{Q},\mathbf{Q}\rangle_*=c$, $c\in\mathbb{R}_*$ take its multiplicative derivative and do the necessary adjustment, we can see that

$$\lambda_1\cdot_*e^{\langle\log\mathbf{T},\log\mathbf{Q}\rangle}+_*\lambda_2\cdot_*e^{\langle\log\mathbf{N},\log\mathbf{Q}\rangle}$$
$$+_*\lambda_3\cdot_*e^{\langle\log\mathbf{B},\log\mathbf{Q}\rangle}=0_*,$$

and finally

$$\lambda_1\cdot_*e^{\langle\log\mathbf{T},\log\mathbf{Q}\rangle}=-_*\lambda_2\cdot_*e^{\langle\log\mathbf{N},\log\mathbf{Q}\rangle},$$

or equivalently

$$\lambda_1\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*=-_*\lambda_2\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*.$$

For $\langle\mathbf{Q},\mathbf{T}\rangle_*\neq0_*$ and $\langle\mathbf{Q},\mathbf{N}\rangle_*\neq0_*$, λ_1 and λ_2 are multiplicative proportional to each other giving as

$$\lambda_1=\lambda\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\quad\text{and}\quad\lambda_2=-_*\lambda\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*.\tag{53}$$

By using Eqs. (52) and (53) within Eq. (27), in this case we get

$$\mathbf{Q}^*=\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{B}+_*\lambda\cdot_*(\mathbf{Q}\times_*\mathbf{B}).\tag{54}$$

The second term on the right side of the previous equation represents a rotation around $\mathbf{B}$. Assuming that $\mathbf{B}$ undergoes parallel transport, we can express this as

$$\mathbf{Q}^*=\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{B}.\tag{55}$$

In the scenario where $\langle\mathbf{Q},\mathbf{B}\rangle_*=0_*$, the polarization vector can also be expressed as

$$\mathbf{Q}=\langle\mathbf{Q},\mathbf{T}\rangle_*\cdot_*\mathbf{T}+_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{N}.$$

Whereupon taking the multiplicative derivative of the aforementioned equation, we can express it as follows

$$\mathbf{Q}^*=((d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*-_*\kappa\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*)\cdot_*\mathbf{T}$$
$$+_*(\kappa\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*-_*(d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*)\cdot_*\mathbf{N}$$
$$+_*\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{B}.$$

Upon comparison with Eq. (55), the ensuing system of equations emerges

$$(d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*=\kappa\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*,$$
$$(d_*/_*d_*s)\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*=-_*\kappa\cdot_*\langle\mathbf{Q},\mathbf{T}\rangle_*.$$

Utilizing multiplicative spherical coordinates, the representation of the polarization vector can be articulated as

$$\mathbf{Q}=e^{\log e^{\cos\log\theta}\log\mathbf{T}+e^{\sin\log\theta}\log\mathbf{N}}, \tag{56}$$

or with multiplicative arguments

$$\mathbf{Q}=\cos_*\theta\cdot_*\mathbf{T}+_*\sin_*\theta\cdot_*\mathbf{N}, \tag{57}$$

where θ constant multiplicative angle. Compute the multiplicative derivative of Eq. (57), the resulting expression is

$$\mathbf{Q}^*=\tau\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{B}-_*(\kappa+_*(d_*\theta/_*d_*s))\cdot_*(\mathbf{Q}\times_*\mathbf{B}).$$

Since optical fiber, is have $\kappa+_*(d_*\theta/_*d_*s)=0_*$. If we perform integration on both sides using multiplicative integral, it becomes apparent that

$$\int_*(d_*\theta/_*d_*s)d_*s=e^{\int\frac{1}{s}\log e^{s\frac{d\log\theta}{ds}}ds}$$
$$=e^{\int\frac{d\log\theta}{ds}ds}=\theta, \tag{58}$$

and

$$\theta=-_*\int_*\kappa\cdot_*d_*s.$$

Similarly to the aforementioned theory, in the last scenario where $\langle\mathbf{Q},\mathbf{B}\rangle_*=0_*$, we derive the $\mathbf{Q}_b-$Rytov multiplicative curve and the polarization vector within the optical context, respectively, as follows

$$\mathbf{Q}_b=\mathbf{x}(s)+_*\mathbf{Q}(s), \tag{59}$$

and

$$\mathbf{Q}=\cos_*\left(-_*\int_*\kappa\cdot_*d_*s\right)\cdot_*\mathbf{T}$$
$$-_*\sin_*\left(-_*\int_*\kappa\cdot_*d_*s\right)\cdot_*\mathbf{N}. \tag{60}$$

4. An approximation with multiplicative properties to the electromagnetic curves generated by the electric field Q along the polarization plane on the optical fiber

If F represents an optical fiber describing the multiplicative curve $\mathbf{x}(s)$ in a $3D$ multiplicative Riemann manifold, then the multiplicative electromagnetic curve within the optical fiber, taking into account the multiplicative derivative, is defined as

$$\phi(\mathbf{Q})=V\times_*\mathbf{Q}=\mathbf{Q}^* \tag{61}$$

where V is a multiplicative Killing vector field. Also influenced by the multiplicative electromagnetic field, resulting in the generation of a force known as the multiplicative Lorentz force. The multiplicative Lorentz force, also referred to as the electromagnetic force, emerges from the combined effects of the electric and magnetic forces acting on a point charge due to multiplicative electromagnetic fields. This force significantly impacts the motion of particles, leading to diverse multiplicative trajectories along the optical fiber, recognized as multiplicative electromagnetic curves [27]. The subsequent section will delineate three distinct categories of multiplicative electromagnetic curves based on the three cases of $\mathbf{Q}$ established in the preceding section.

4.1. Multiplicative electromagnetic curves for $\mathbf{Q}\perp_*\mathbf{T}$

We have previously shown in Eq. (62) that is

$$d\cdot_*\mathbf{Q}/_*d\cdot_*s=-_*\kappa\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{T}+_*\lambda\cdot_*\langle\mathbf{Q},\mathbf{B}\rangle_*\cdot_*\mathbf{N}$$
$$-_*\lambda\cdot_*\langle\mathbf{Q},\mathbf{N}\rangle_*\cdot_*\mathbf{B}. \tag{62}$$

By employing the equation for the multiplicative Lorentz force, it can be expressed as

$$e^{\langle\log\phi(\mathbf{Q}),\log\mathbf{T}\rangle}=e^{-\langle\log\phi(\mathbf{T}),\log\mathbf{Q}\rangle},$$
$$e^{\langle\log\phi(\mathbf{Q}),\log\mathbf{N}\rangle}=e^{-\langle\log\phi(\mathbf{N}),\log\mathbf{Q}\rangle},$$
$$e^{\langle\log\phi(\mathbf{Q}),\log\mathbf{B}\rangle}=e^{-\langle\log\phi(\mathbf{B}),\log\mathbf{Q}\rangle}.$$

Then, applying the multiplicative Lorentz force ϕ on the multiplicative Frenet frame, it is determined that

$$\phi(\mathbf{T})=\kappa\cdot_*\mathbf{N},$$
$$\phi(\mathbf{N})=-_*\kappa\cdot_*\mathbf{T}-_*\lambda\cdot_*\mathbf{B},$$
$$\phi(\mathbf{B})=-\lambda\cdot_*\mathbf{N}.$$

Also, the multiplicative Killing magnetic vector field is spanned by the multiplicative Frenet trihedron.

$$V=e^{\log c_1\log\mathbf{T}+\log c_2\log\mathbf{N}+\log c_3\log\mathbf{B}},$$

or with multiplicative arguments

$$V=c_1\cdot_*\mathbf{T}+_*c_2\cdot_*\mathbf{N}+_*c_3\cdot_*\mathbf{B}.$$

From the definition of the multiplicative Lorentz force, we obtain

$$\phi(\mathbf{T})=V\times_*\mathbf{T},$$
$$\phi(\mathbf{N})=V\times_*\mathbf{N},$$
$$\phi(\mathbf{B})=V\times_*\mathbf{B}.$$

Therefore, given the suppositions and utilizing the frame along with the aforementioned equations, the multiplicative magnetic vector field of the $\mathbf{Q}_tM-$trajectories is obtained as

$$V=-_*\lambda\cdot_*\mathbf{T}+_*\kappa\cdot_*\mathbf{B}.$$

Under the supposition that $\mathbf{T}$ is orthogonal to $\mathbf{Q}$, the multiplicative Lorentz force equations, according to the multiplicative Frenet frame, are as follows

$$\phi(\mathbf{T}) = \kappa \cdot_* \mathbf{N},$$
$$\phi(\mathbf{N}) = -_*\kappa \cdot_* \mathbf{T},$$
$$\phi(\mathbf{B}) = 0_*.$$

Subsequently, the multiplicative magnetic field is obtained as follows

$$V = \kappa \cdot_* \mathbf{B}.$$

4.2. Multiplicative electromagnetic curves for $\mathbf{Q} \perp_* \mathbf{N}$

This second section, is obtained the equation $\mathbf{Q}$ for the case $\langle \mathbf{Q}, \mathbf{N} \rangle_* = 0_*$, from Eq. (43) as

$$\mathbf{Q}^* = \lambda \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_* \cdot_* \mathbf{T} +_* (\kappa \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* -_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{B} \rangle_*) \cdot_* \mathbf{N} -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* \cdot_* \mathbf{B}.$$

We use the multiplicative Lorentz force ϕ on multiplicative Frenet trihedron, as follows

$$\phi(\mathbf{T}) = \kappa \cdot_* \mathbf{N} -_* \lambda \cdot_* \mathbf{B},$$
$$\phi(\mathbf{N}) = -_*\kappa \cdot_* \mathbf{T} +_* \tau \cdot_* \mathbf{B},$$
$$\phi(\mathbf{B}) = \lambda \cdot_* \mathbf{T} -_* \tau \cdot_* \mathbf{N}.$$

The representation of the multiplicative magnetic vector field according to the multiplicative Frenet vectors can be expressed as

$$V = c_1 \cdot_* \mathbf{T} +_* c_2 \cdot_* \mathbf{N} +_* c_3 \cdot_* \mathbf{B}.$$

Upon performing the requisite calculations, we determine the magnetic vector field of the $\mathbf{Q}_n M$-trajectories

$$V = \tau \cdot_* \mathbf{T} +_* \lambda \cdot_* \mathbf{N} +_* \kappa \cdot_* \mathbf{B}.$$

If the assumption is made that $\mathbf{N}$ is multiplicative orthogonal to $\mathbf{Q}$, the multiplicative Lorentz force equations according to the multiplicative Frenet frame can be derived as follows

$$\phi(\mathbf{T}) = \kappa \cdot_* \mathbf{N},$$
$$\phi(\mathbf{N}) = -_*\kappa \cdot_* \mathbf{T} +_* \tau \cdot_* \mathbf{B},$$
$$\phi(\mathbf{B}) = -_*\tau \cdot_* \mathbf{N},$$

and multiplicative magnetic field as

$$V = \tau \cdot_* \mathbf{T} +_* \kappa \cdot_* \mathbf{B}.$$

4.3. Multiplicative electromagnetic curves for $\mathbf{Q} \perp_* \mathbf{B}$

This last section, is obtained the equation $\mathbf{Q}$ for the case $\langle \mathbf{Q}, \mathbf{N} \rangle_* = 0_*$, from Eq. (54) as

$$\mathbf{Q}^* = \lambda \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{T} -_* \lambda \cdot_* \langle \mathbf{Q}, \mathbf{T} \rangle_* \cdot_* \mathbf{N} +_* \tau \cdot_* \langle \mathbf{Q}, \mathbf{N} \rangle_* \cdot_* \mathbf{B}.$$

Utilizing the multiplicative Lorentz force on the multiplicative Frenet trihedron, as follows

$$\phi(\mathbf{T}) = -_*\lambda \cdot_* \mathbf{N},$$
$$\phi(\mathbf{N}) = \lambda \cdot_* \mathbf{T} +_* \tau \cdot_* \mathbf{B},$$
$$\phi(\mathbf{B}) = -_*\tau \cdot_* \mathbf{N}.$$

Thereupon performing the requisite calculations, we determine the magnetic vector field of the $\mathbf{Q}_b M$-trajectories

$$V = \tau \cdot_* \mathbf{T} -_* \lambda \cdot_* \mathbf{B}.$$

If we assume that $\mathbf{B}$ is orthogonal to $\mathbf{Q}$, then we derive the multiplicative Lorentz force equations and the multiplicative magnetic field, we obtain

$$\phi(\mathbf{T}) = 0_*,$$
$$\phi(\mathbf{N}) = \tau \cdot_* \mathbf{B},$$
$$\phi(\mathbf{B}) = -_*\tau \cdot_* \mathbf{N},$$

and

$$V = \tau \cdot_* \mathbf{T}.$$

Example 1. Let's establish the connection between a linear polarized light wave along an optical fiber and the multiplicative curve $\mathbf{x}(s)$ using the multiplicative Frenet trihedron $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. The multiplicative curve $\mathbf{x}(s)$ with an arbitrary multiplicative parameter s is expressed by the equation

$$\mathbf{x}(s) = ((e^3/_*e^5) \cdot_* \cos_* s, (e^3/_*e^5) \cdot_* \sin_* s, (e^4/_*e^5) \cdot_* s).$$

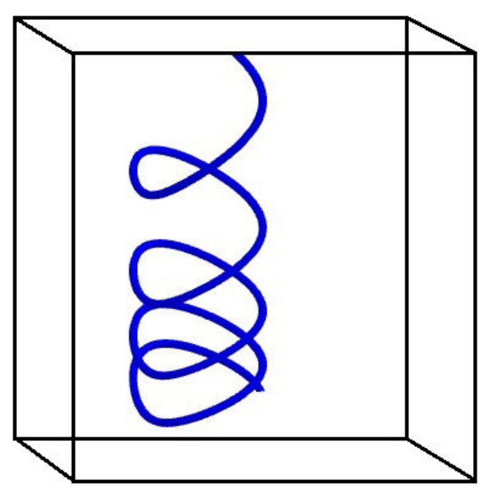

FIGURE 5. Multiplicative curve $\mathbf{x}$.

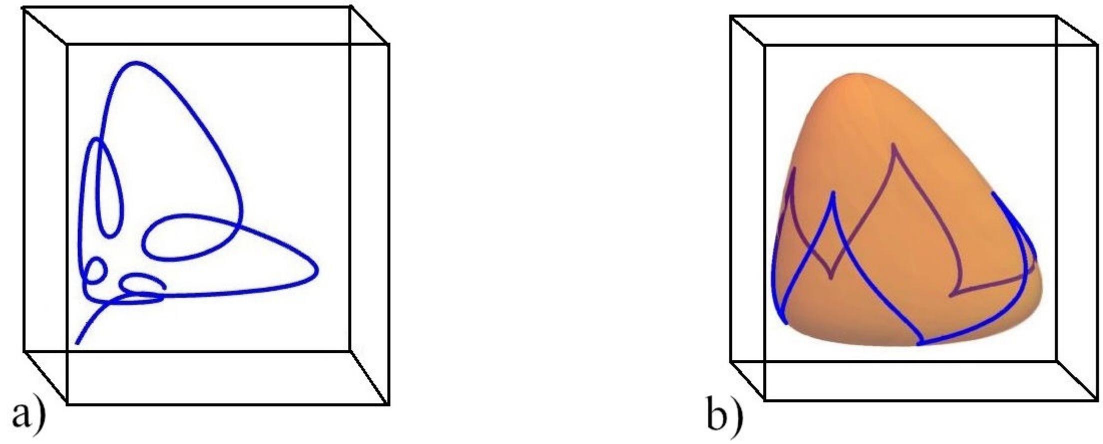

FIGURE 6. Multiplicative $\mathbf{Q}_t$ Rytov curve and multiplicative spherical indicatrix of $\mathbf{Q}$, respectively.

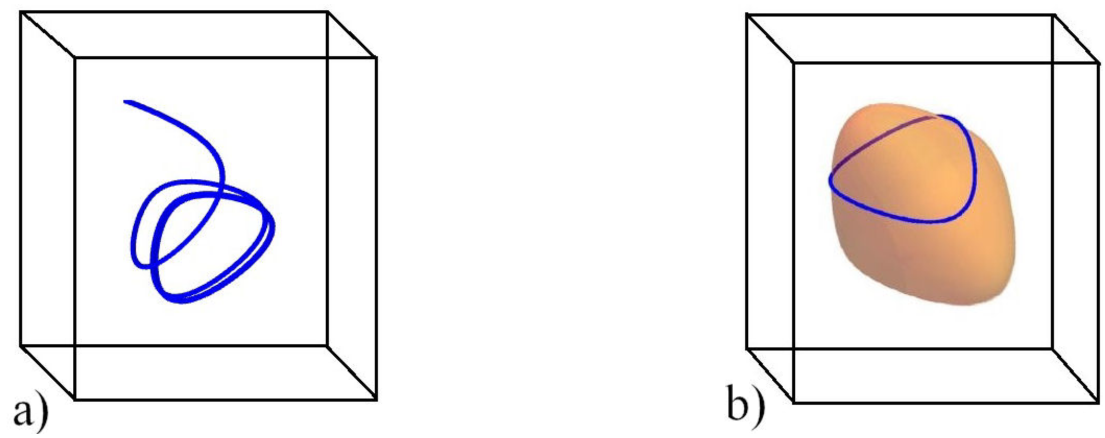

FIGURE 7. Multiplicative $\mathbf{Q}_n$ Rytov curve and multiplicative spherical indicatrix of $\mathbf{Q}$, respectively.

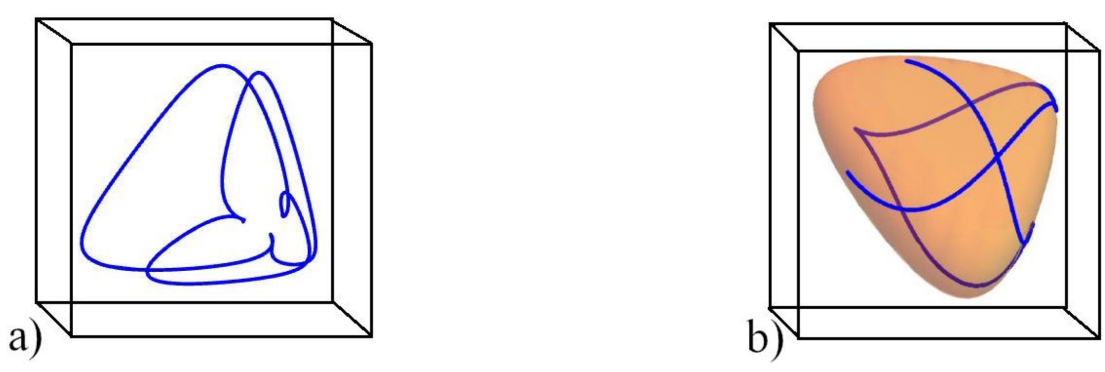

FIGURE 8. Multiplicative $\mathbf{Q}_b$ Rytov curve and multiplicative spherical indicatrix of $\mathbf{Q}$, respectively.

The multiplicative Frenet elements $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau\}$ of $\mathbf{x}(s)$ are obtained as follows

$$\mathbf{T} = (-_*(e^3/_*e^5) \cdot_* \sin_* s, (e^3/_*e^5) \cdot_* \cos_* s, (e^4/_*e^5)),$$

$$\mathbf{N} = (-_* \cos_* s, -_* \sin_* s, 0_*),$$

$$\mathbf{B} = (-_*(e^4/_*e^5) \cdot_* \sin_* s, -_*(e^4/_*e^5) \cdot_* \cos_* s, (e^3/_*e^5)),$$

and

$$\kappa = e^3/_*e^5,$$

$$\tau = e^4/_*e^5.$$

We present the multiplicative x curve in Fig. 5. The illustration in Fig. 6-8 depicts three cases of multiplicative Rytov curves connected with the multiplicative geometric phase model of the linearly polarized light wave within an optical fiber. Additionally, we present the multiplicative global indicator E of the polarization vector, as governed by the multiplicative derivative within the 3D multiplicative Riemann space.

5. Conclusion

In our present investigation, we delve into the geometric phase equations and the associated electromagnetic curves, employing multiplicative calculus within the framework of optical fibers. We extend our analysis to derive multiplicative Rytov curves, offering unique characteristics for traced curves during the rotational motion of the polarization vector along the optical fiber within the 3D multiplicative Riemannian manifold. Additionally, we provide examples and visualize their representations for various multiplicative values utilizing the Mathematica program. The novelty of this study lies in its incorporation of multiplicative calculi in analytical computations, setting it apart from other existing studies in the field.

The reason why multiplicative calculus is used in this study is that it has some advantages over Newton's calculus. Since the multiplicative space is produced with the help of the exponential function, it functions in the first quadrant of the traditional coordinate system. This transformation entails shifting the range $(-\infty, 0)$ to $(0, 1)$ and $(0, +\infty)$ to $(1, +\infty)$, effectively confining the multiplicative Euclidean space within the first quadrant region. Consequently, subjects analyzed using multiplicative arguments find themselves operating within a more constrained domain, facilitating a more streamlined examination process. Moreover, the proportional nature of measurements in multiplicative space enables a more efficient modeling of exponentially changing phenomena. Problems characterized by rapid exponential changes can be more effectively addressed with multiplicative arguments, allowing for a more numerical approach towards attaining real solutions compared to traditional methods. In scenarios where the velocity of a particle undergoes exponential fluctuations, as dictated by the Lorentz force, examining such phenomena within the framework of multiplicative Euclidean space yields more numerical solutions. However, it's worth noting that compressing multiplicative space into the first quadrant may pose disadvantages for certain problem structures. Depending on the nature of the problems under scrutiny, this aspect warrants careful consideration to ensure comprehensive analysis and accurate outcomes. There are also some algebraic advantages, which we can explain as follows. Taking into account multiplicative trigonometric functions, it is obvious that $e^{\sin \log x} = \sin_* x$. Similarly, multiplicative trigonometric functions all share the same situation. Thus, while multiplicative trigonometric functions are not integrable in Newtonian analysis, their multiplicative integrals are available. Evren M. E. et al. explained the importance of this situation in terms of differential geometry as follows [25]. It is not possible to interpret the geometric interpretation of some exponential expressions with the help of traditional analysis. More clearly, consider the following subset of $\mathbb{R}^2$.

$$C = \{(x, y) \in \mathbb{R}^2 : (\log x)^2 + (\log y)^2 = 1, x, y > 0\}.$$

We also can parameterize this set as $x(t) = e^{\cos(\log t)}$ and $y(t) = e^{\sin(\log t)}$, $t > 0$. If we use the usual arithmetic operations, derivative and integral, then it would not be easy to understand what the set C expresses geometrically. With or without the help of computer programs, we cannot even calculate its basic invariants, e.g., the arc length function $s(t)$ is given by a complicated integral

$$s(t) = \int^t \frac{1}{u}\bigg(\big[\sin(\log u) e^{\cos(\log u)}\big]^2 + \big[\cos(\log u) e^{\sin(\log u)}\big]^2\bigg)^{1/2} du.$$

However, applying the multiplicative tools, we see that C is indeed a multiplicative circle parameterized by the multiplicative arc length whose center is $(1, 1)$ and radius e, which is one of the simplest multiplicative curves. This is the reason why, in some cases, the multiplicative tools need to be applied instead of the usual ones.

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1. V. Volterra, B. Hostinsky, Operations Infinitesimales Lineares (Herman, Paris, 1938).
2. M. Grossman, R. Katz, Non-Newtonian Calculus, 1st ed. (Lee Press, Pigeon Cove Massachussets, 1972=.
3. M. Grossman, Bigeometric Calculus: A System with a Scale-Free Derivative (Archimedes Foundation, Massachusetts, 1983).
4. M. Rybaczuk, P. Stoppel, The fractal growth of fatigue defects in materials, *International Journal of Fracture* **103** (2000) 71.
5. M. Rybaczuk, W. Zielinski, The concept of physical and fractal dimension I. *The projective dimensions, Chaos, Solitons and Fractals* **12** (2001) 2517.
6. W. F. Samuelson, S. G. Mark, Managerial Economics (Wiley, New York, 2012).
7. H. H. Afrouzi *et al*., Statistical analysis of pulsating non-Newtonian flow in a corrugated channel using Lattice-Boltzmann method, *Physica A: Statistical Mechanics and its Applications* **535** (2019) 122486.
8. G. M. Othman, K. Yurtkan and A. Özyapıcı, Improved digital image interpolation technique based on multiplicative calculus and Lagrange interpolation, *SIViP* **17** (2023) 3953.
9. A. Uzer, Multiplicative type complex calculus as an alternative to the classical calculus, *Computers and Mathematics with Applications* **60** (2010) 2725.
10. A. Bashirov, M. Riza, On complex multiplicative differentiation, *TWMS Journal of Applied and Engineering Mathematics* **1** (2011) 75.
11. M. Yazici, H. Selvitopi, Numerical methods for the multiplicative partial differential equations, *Open Math.* **15** (2017) 1344.
12. K. Boruah, B. Hazarika, Some basic properties of bigeometric calculus and its applications in numerical analysis, *Afrika Matematica* **32** (2021) 211.
13. D. Aniszewska, Multiplicative Runge-Kutta methods, *Nonlinear Dynamics* **50** (2007) 262.
14. A. Bashirov, E. Mısırlı, Y. Tandogdu and A. Ozyapıcı, On modeling with multiplicative differential equations, *Appl. Math. J. Chinese Univ.* **26** (2011) 425.
15. N. Yalçın, E. Celik, Solution of multiplicative homogeneous linear differential equations with constant exponentials, *New Trends in Mathematical Sciences* **6** (2018) 58.
16. M. Waseem, M. A. Noor, F.A. Shah and K. I. Noor, An efficient technique to solve nonlinear equations using multiplicative calculus, *Turkish Journal of Mathematics* **42** (2018) 679.
17. A. E. Bashirov, E. M. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, *J. Math. Anal. Appl*. **337** (2008) 36.
18. T. Gulsen, E. Yılmaz and S. Goktas, Multiplicative Dirac system, *Kuwait J. Sci.* **49** (2022) 1.
19. S. Goktas, H. Kemaloglu and E. Yılmaz, Multiplicative conformable fractional Dirac system, *Turk. J. Math*. **46** (2022) 973.
20. S. Goktas, E. Yılmaz and A. Ç. Yar, Multiplicative derivative and its basic properties on time scales, *Math. Meth. Appl. Sci*. **45** (2022) 2097.
21. S. G. Georgiev, K. Zennir, Multiplicative Differential Calculus, 1st ed. (Chapman and Hall/CRC., New York, 2022).
22. S. G. Georgiev, Multiplicative Differential Geometry, 1st ed. (Chapman and Hall/CRC., New York, 2022).
23. S. G. Georgiev, K. Zennir and A. Boukarou, Multiplicative Analytic Geometry, 1st ed. (Chapman and Hall/CRC., New York, 2022).
24. S.K. Nurkan, I. Gurgil and M. K. Karacan, Vector properties of geometric calculus. *Math. Meth. Appl. Sci.* **46** (2023) 17672.
25. M. E. Aydin, A. Has, B. Yılmaz, A non-Newtonian approach in differential geometry of curves: multiplicative rectifying curves, *Bulletin of the Korean Mathematical Society* **61** (2024) 849.
26. A. Has, B. Yılmaz, On helices in multiplicative differential geometry, Preprint arxiv: 2403.11282 (2024).
27. A. Has, B. Yılmaz, A non-Newtonian magnetic curves in multiplicative Riemann manifolds, *Physica Scripta*, **99** (2024) 045239.
28. A. Has, B. Yılmaz, A non-Newtonian conics in multiplicative analytic geometry, *Turkish Journal of Mathematics* **48** (2024) 976.
29. Y. A. Kravtsov, Y. I. Orlov, Geometrical Optics of Inhomogeneous Medium (Springer-Verlag, Berlin, 1990).
30. M. Kugler, S. Shtrikman, Berry's phase, locally inertial frames, and classical analogues, *Phys. Rev. D.* **37** (1988) 934.
31. J. N. Ross, The rotation of the polarization in low briefrigence monomode optical fibres due to geometric effects, *Optical Quantum Electron* **16** (1984) 455.
32. Z. Özdemir, A new calculus for the treatment of Rytov's law in the optical fiber, *Optik Int. J. Light Electr. Opt.* **216** (2020) 164892.
33. Z. Bozkurt, I. Gok, Y. Yaylıand F. N. Ekmekçi, A new approach for magnetic curves in 3D Riemannian manifolds, *Journal of Mathematical Physics* **55** (2014) 053501.
34. S. K. Nurkan, H. Ceyhan, Z. Özdemir and I. Gök, Electromagnetic curves and Rytov's law in the optical fiber with Maxwellian evolution via alternative moving frame, *Rev. Mex. Fis.* **69** (2023) 061301.
35. T. KÖrpınar, R. C. Demirkol, Z. KÖrpınar, On the new conformable optical ferromagnetic and antiferromagnetic magnetically driven waves, *Optical and Quantum Electronics* **55** (2023) 496.
36. A. Has, B. Yılmaz, Effect of fractional analysis on magnetic curves, *Rev. Mex. Fís.* **68** (2022) 041401.
37. B. Yılmaz, A new type electromagnetic curves in optical fiber and rotation of the polarization plane using fractional calculus, *Optik - International Journal for Light and Electron Optics* **247** (2021) 168026.
38. B. Yılmaz, A. Has, A new type electromagnetic curves in optical fiber and rotation of the polarization plane using fractional calculus, *Optik - International Journal for Light and Electron Optics* **260** (2022) 169067.