THE PINCH EFFECT

(El efecto de constricción magnética)

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RESUMEN

Este artículo presenta una reseña de algunos trabajos recientes sobre el efecto de constricción magnética en plasmas gaseosos. Después de una discusión general de los fenómenos, se obtienen los criterios para la estabilidad de la columna contraida de plasma, utilizando el formulismo de Marshall y Rosenbluth.

1.- In simplest terms, the pinch effect is based on the well known fact that parallel wires carrying currents in the same direction attract each other.

The pinch effect, first produced experimentally only about a decade ago ,

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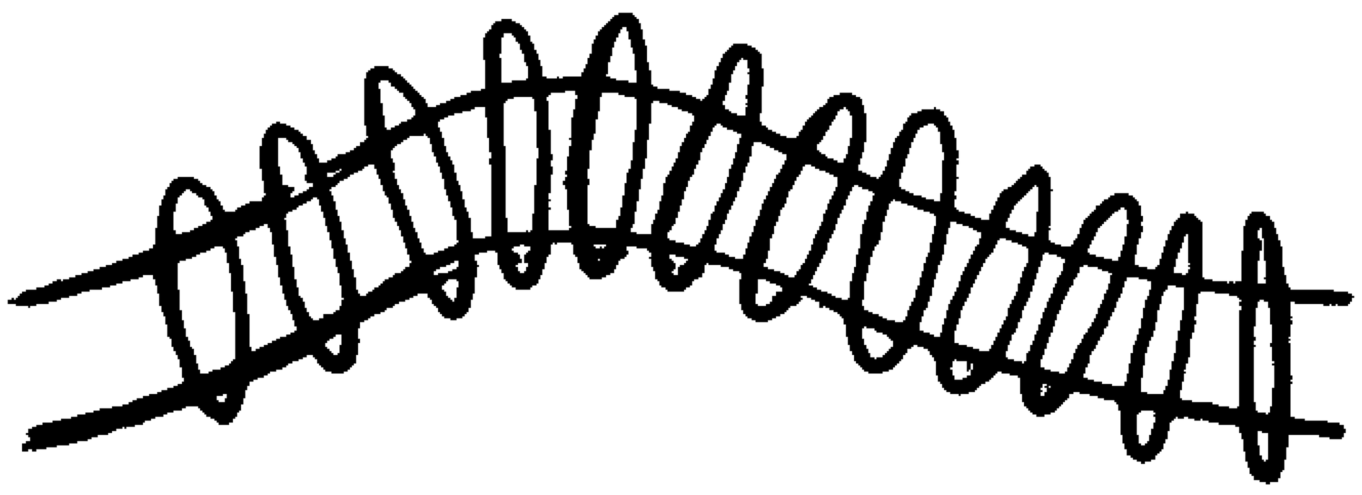
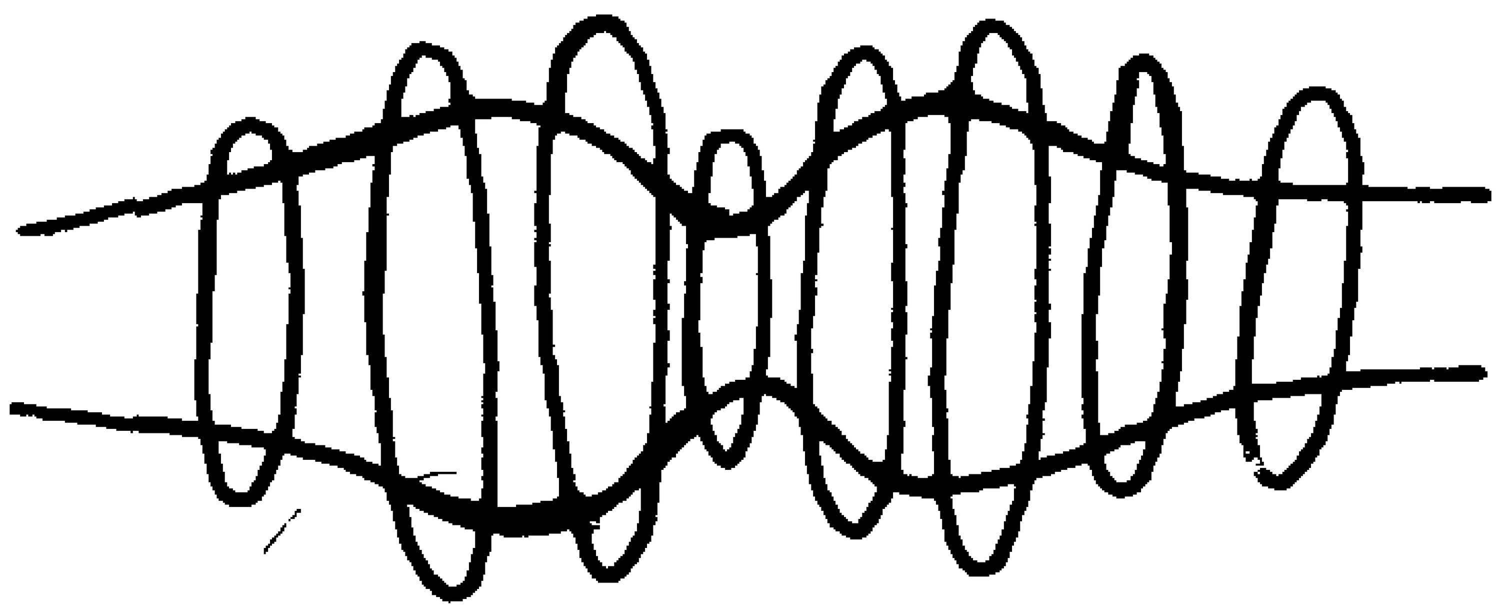
requires a very large electrical current. When such a current is passed through a conducting gas in a tube, it sets up a magnetic field which tends to pinch the gas and pull it away from the tube walls. The magnetic lines of force circling the gas compress it by their tension. Since a plasma is an excellent conductor of electricity, the pinch effect looked like an attractive and ready made means of forming a magnetic bottle.

Theoretical caculations showed that it would take a very large current indeed - millions of amperes - to confine a plasma of high temperature and low density. Not discouraged by this fact, investigators in many countries carried on experiments with simple pinch tubes. They applied a high voltage to a.low-pressure gas in an insulated tube and produced an electrical discharge. This ionized the gas, and heavy current then began to flow. The pinch made its dramatic appearence, but it lasted only a millionth of a second or so; no sooner had the column of plasma been compressed than it writhed violently and drove itself to the tube wall. Furthermore, the lighter the pinch the faster it destroyed itself.

This was not hard to understand and in fact was predictable theoretically. Two different types of instability can develop. In the first place, any small kink in the pinched column will grow rapidly, because the magnetic pressure is stronger on the concave side of the kink than on the convex side. The second cause of instability is a kind of "sausage" effect. The plasma tends to squeeze itself of at one or more points along the column and this cuts itself into pieces.



"kink"



"sausage"

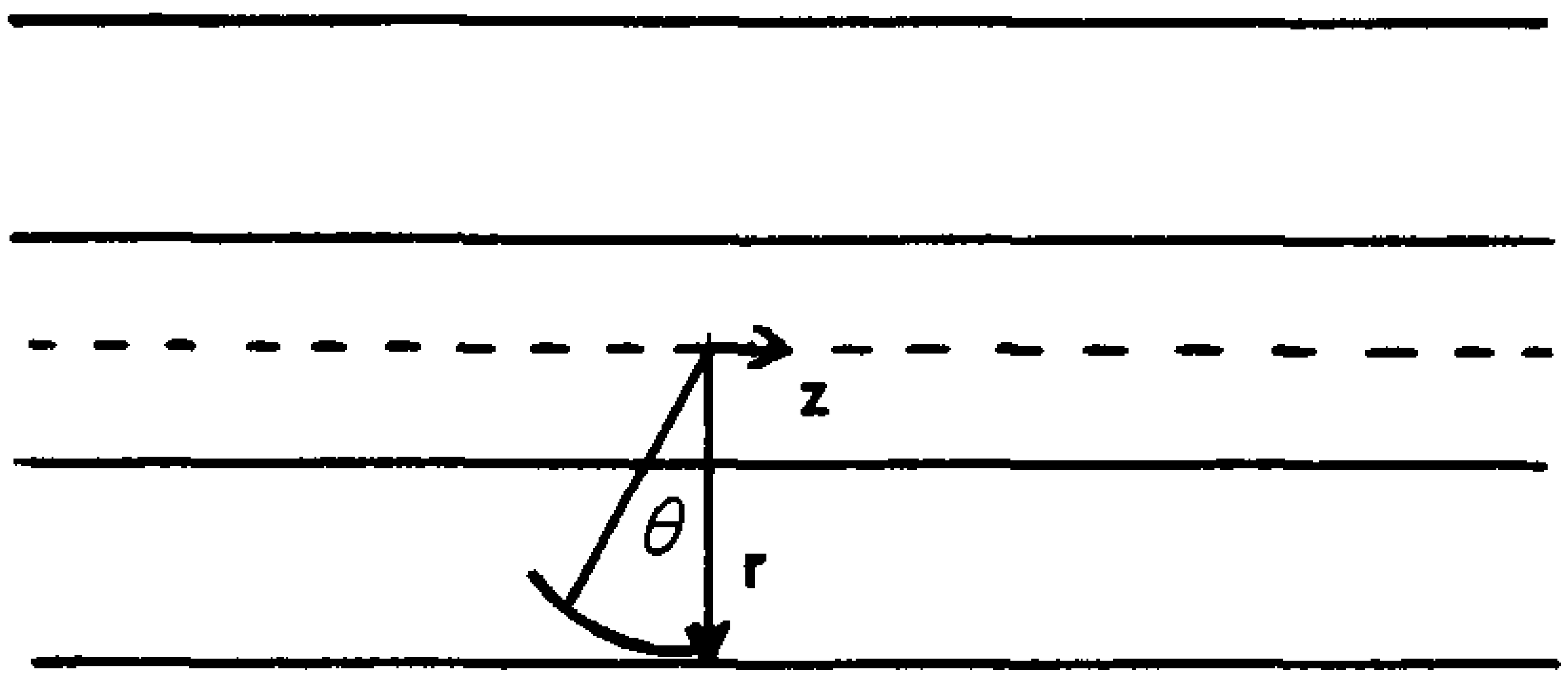
How could the pinch be stabilized? Theoretical investigations published in the U.S., U.K., and U.S.S.R have suggested a possible answer although they have not yet shown how the physical conditions necessary to make it work could be achieved. The idea is to create not a pinching magnetic field around the the plasma but also a strong longitudinal magnetic field within the plasma column. The internal magnetic field would act as a kind of Stiffener. If a kink started to develop, it would tend to stretch the interior lines of force and their elastic resistance would pull out the kink. Similarly, if the sausage type of constric - tion tried to pinch into the column, the internal lines of force would resist being squeezed together and this would prevent the collapse of the column.

There is a third type of instability which could destroy a plasma column. This is the long-wave instability, i. e. a long gentle bend of the column that would grow in strength and push the column to the chamber wall. However, this could be counteracted by using a conducting material for the walls of the tube. Since a conductor acts as a sort of barrier to a magnetic field, the magnetic field lines around the plasma column would be crowded against the wall where the bent column approached it, and the resulting pressure would push the column back toward the centre of the tube.

It should be made clear that the straight pinch columns illustrated here are simplified systems which merely exemplify the principles. In practise it would probably not be desirable to try to produce a stable pinch in a straight tube, for several reasons: among other things, the electrodes at the ends of the tube would have a cooling effect on the plasma. Pinch experiments have already been performed on other shapes. One of these is a doughnut-shaped tubes in which currents are induced and can circulate without bumping into a solid surface. A high voltage applied to the winding around a large iron transformer core in the tube produces an electrical discharge in the gas, which then functions as a one - turn secondary winding. Very heavy currents can be induced in the plasma by this means.

2.- The Stationary State

Consider now a column of plasma confined to a cylinder of radius r_o.



This cylindrical column of plasma is assumed to be surrounded by a metallic cylinder of radius R_o with perfectly conducting walls. Let $\vec{B}_p^0$ denote the magnetic field inside the plasma and $\vec{B}_v^0$ that in the vacuum surrounding it. We thus have

$$\vec{B}_p^o = \alpha_p B_\theta \vec{1}_z \quad (1)$$

$$\vec{B}_v^o = B_\theta \frac{r_o}{r} \vec{1}_\theta + \alpha_v B_\theta \vec{1}_z \quad (2)$$

where $\vec{1}_z$ and $\vec{1}_\theta$ are unit vectors in the corresponding directions, using cylindrical coordinates with z along the plasma axis, and B_θ is the tangential magnetic field at the plasma boundary. α_p and α_v are scalars characterizing the longitudinal field.

The condition of pressure balance at the surface of the plasma requires that

$$p_o + \frac{1}{8\pi} \left| \vec{B}_p^o \right|^2 = \frac{1}{8\pi} \left| \vec{B}_v^o \right|^2_{r = r_o} \quad (3)$$

where p_o is the gas pressure at the boundary of the plasma.

On substituting for $\vec{B}_p^o$ and $\vec{B}_v^o$ in accordance with equations (2) and (3), we obtain

$$\frac{8\pi}{B_\theta^2} p_o = 1 + \alpha_v^2 - \alpha_p^2 \tag{4}$$

Since the plasma pressure must be positive, we must have

$$1 + \alpha_v^2 - \alpha_p^2 \geqslant 0 \tag{5}$$

the equality sign corresponding to the vanishing plasma pressure and therefore to a vanishing plasma.

3.- Investigation of Stability.

We now wish to study the behaviour of the stationary state described in the preceding section for small departures from the state of equilibrium. We can then write down the linearized equations of hydromagnetics which govern the small departures from equilibrium. The perturbation can then be analyzed into normal modes as

$$f(r)\, e^{i(kz + m\theta + \sigma t)},$$

where k is the wave number of the perturbation, m the azimuthal number and σ the frequency of oscillation. The general perturbation will, of course, consist of a summation over all m and an integral over k. However, since the equations governing the perturbed state are linear, one can treat each of the modes separately. This leads to a characteristic value equation for σ. A particular mode is stable or unstable depending on whether σ is real or imaginary; if σ should happen to be complex, the imaginary part must be negative for the mode of oscillation to be unstable.

However, if one is only interested in the stability of the system, one can do so in a more direct manner by one of the following methods.

i) One can ask if there exist any deformations which result in the second order change in the potential energy to be negative. If there exist such deformations, the system will seek this state of lower energy and will be unstable. The energy principle for hydromagnetic stability problems has been devoloped by Bernstein, Frieman, Kruskal and Kulsrud at Project Matterhorn of Princeton University.

ii) The second approach is due to Marshall and Rosenbluth. It is based on the criterion that stability or instability of a system is a direct consequence of the pressure distribution at the boundary (perturbed) of the plasma. It states, if the normal component of the gradient of the perturbation in the total pressure acts in the direction of the normal component of the displacement, one has instability, whereas if it acts in the opposite direction to the normal component of the displacement, the system is stable.

It can be shown that this method is equivalent to the method of solving the characteristic value equation for σ and requiring that σ be real for stable configurations.

We shall follow Rosenbluth's treatment and use the hydromagnetic equations to discuss the stability of the pinch. We shall use a scalar pressure for the plasma, though strictly speaking one should take the pressure to be a tensor. However a scalar pressure simplifies the calculations a great deal.

4.- Stability

Consider a quasi-static displacement of the plasma. (Quasi-static means that during the displacement the equilibrium is preserved). Then for the plasma, we must have

$$4\pi \operatorname{grad} \delta p = (\vec{\nabla} \times \delta\vec{B}_p) \times \vec{B}_p^o \quad . \tag{6}$$

where δp and $\delta\vec{B}_p$ are the perturbations in the pressure p_o and the plasma field $\vec{B}_p^o$, respectively.

From this equation, it follows that

$$\vec{B}_p^o \cdot \vec{\nabla}\, \delta p = 0 \tag{7}$$

or

$$B_p^o \frac{\partial}{\partial z} \delta p = 0$$

or

$$i k \alpha_p B_\theta \delta p = 0 \tag{8}$$

$$\therefore \qquad \delta p = 0 \tag{9}$$

Thus inside the plasma, we must have

$$(\vec{\nabla} \times \delta \vec{B}_p) \times \vec{B}_p^o = 0 \quad . \tag{10}$$

$\delta \vec{B}_p$ is determined from the induction equation

$$\frac{\partial}{\partial t} \delta \vec{B}_p = \vec{\nabla} \times (\vec{u} \times \vec{B}_p^o) \quad , \tag{11}$$

$\vec{u}$ is the electron velocity in the plasma; the positive ion motion can be neglected in the first approximation, due to their very much larger mass.

Which by introducing the Lagrangian variable $\vec{w}$

$$\vec{u} = \partial \vec{w} / \partial t \tag{12}$$

can be integrated to give

$$\delta \vec{B}_p = \vec{\nabla} \times (\vec{w} \times \vec{B}_p^o) \tag{13}$$

$$= (\vec{B}_p^o \cdot \vec{\nabla})\, \vec{w} = i k \alpha_p B_\theta \vec{w} \tag{14}$$

Substituting (14) into (10), we obtain

$$(\vec{\nabla} \times \vec{w}) \times \vec{B}_p^o = 0 \quad . \tag{15}$$

Since $\vec{B}_p^o$ is in the z-direction, it follows from this equation that the r and θ components of $(\vec{\nabla} \times \vec{w})$ vanish. With the assumed form of θ and z dependence of u, $(\vec{u} \,||\, \vec{1}_z)$ one can convince oneself that the z- component of $(\vec{\nabla} \times \vec{w})$ also vanishes. Thus the displacements are irrotational, i.e.

$$\vec{\nabla} \times \vec{w} = 0 \tag{16}$$

$$\therefore \quad \vec{w} = \vec{\nabla} \psi \tag{17}$$

The equation of continuity gives

$$\frac{\partial}{\partial t} \delta\rho = - \vec{\nabla} \cdot (\rho_o \vec{u}) \quad . \tag{18}$$

This may be integrated to give

$$\delta\rho = - \rho_o \vec{\nabla} \cdot \vec{w} \tag{19}$$

and from the adiabatic relation

$$\delta p = \frac{\gamma p}{\rho} \delta\rho \tag{20}$$

Here ρ_o is the unperturbed plasma density, $\delta\rho$ its perturbation, and γ the ratio of specific heats.

Since δp vanishes for the plasma, it follows therefore that

$$\nabla \cdot \vec{w} = 0 \tag{21}$$

Thus the displacements of interest are both solenoidal and irrotational, i.e.

$$\vec{\nabla} \cdot \vec{w} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{w} = 0 \tag{22}$$

$$\therefore \quad \vec{w} = \vec{\nabla}\psi \quad \text{and} \quad \nabla^2 \psi = 0 \quad . \tag{23}$$

The solution of $\nabla^2 \psi = 0$ which is bounded on the axis of the cylinder is given by

$$\psi = C I_m (kr) e^{i(kz + m\theta)} \tag{24}$$

where C is an arbitrary constant and $I_m (x)$ is the Bessel function of order m of ix; primes indicate differentiation with respect to x. Then

$$\vec{w} = C \left[k I_m' (kr), \frac{im}{r} I_m (kr), ik I_m (kr) \right] e^{i(kz + m\theta)} \tag{25}$$

The boundary of the plasma is deformed from $r = r_o$ to one given by

$$r = r_o [1 + \delta e^{i(kz + m\theta)}] \, , \qquad \delta << 1 \tag{26}$$

the constant C in eq. (25) is determined from the condition that the displacement of the boundary given by (25) be compatible with that given by (26). Thus

$$C = \frac{r_o \delta}{k I_m' y} \qquad y = kr_o \tag{27}$$

$$\therefore \quad \delta \vec{B}_p = ik \alpha_p B_\theta \vec{w}$$

$$= \frac{i\,\alpha_p B_\theta r_o\,\delta}{I'_m(y)} \left[\, k\,I'_m(kr),\ + \frac{im}{r} I_m(kr),\ ik\,I_m(kr) \,\right]$$

$$\times\ e^{i(kz + m\theta)} \tag{28}$$

The change in the pressure inside the plasma is given by

$$\delta P_i = \frac{1}{4\pi}\, \vec{B}_p^o \cdot \delta\vec{B}_p$$

$$= \frac{1}{4\pi}\, \alpha_p\, B_\theta (\delta\vec{B}_p)_z \tag{29}$$

Substituting for δB_p form (28) into (29), we obtain

$$\delta P_i\,(r = r_o) = -\frac{1}{4\pi}\, \alpha_p^2\, B_\theta^2\, y^2\, \delta\, \frac{I_m(kr_o)}{y\,I'_m(y)}\, e^{i(kz + m\theta)} \tag{30}$$

where P_i is the total pressure (gas pressure + magnetic pressure) in the plasma.

5.- *The Perturbation in the Vacuum Field.*

Since no currents can flow in a vacuum,

$$\vec{\nabla} \times \delta\vec{B}_v = 0 \quad , \quad \therefore\ \delta\vec{B}_v = \vec{\nabla}\Psi \tag{31}$$

and

$$\vec{\nabla} \cdot \delta\vec{B}_v = 0 \quad , \quad \therefore\ \nabla^2 \Psi = 0 \tag{32}$$

$$\therefore \quad \Psi = [\, A I_m(kr) + B\,K_m(kr) \,]\ e^{i(kz + m\theta)} \tag{33}$$

$K_m(x)$ is the Neumann function of order m of ix.

$$\therefore \; \delta \vec{B}_v = \left(k\Psi', \; \frac{im}{r}\,\psi, \; ik\Psi\right) \quad . \tag{34}$$

Since the cylinder is supposed to be made of perfectly conducting material the normal component of the magnetic field must vanish at $r = R_o$.

$$\therefore \; A\, I'_m (k R_o) + B\, K'_m (k R_o) = 0$$

or

$$\frac{B}{A} = - \frac{I'_m (k R_o)}{K'_m (k R_o)} \tag{35}$$

The constant A will be determined by the boundary condition that the normal component of the magnetic field be continuous across the perturbed boundary of the plasma. Then

$$\vec{n} \cdot \Delta \, [\vec{B}] = 0 \quad , \tag{36}$$

where $\vec{n}$ denotes the unit normal to the surface and $\Delta\,[\vec{B}]$ is the jump experienced by $\vec{B}$ at the surface. The linearized from of (36) is

$$\vec{n}^o \cdot \Delta \, [\delta \vec{B}] + \delta \vec{n} \cdot \Delta \, [\vec{B}] = 0 \tag{37}$$

$$(\vec{n}^o = \vec{1}_r).$$

or

$$\vec{1}_r \cdot \delta \vec{B}_p + \delta \vec{n} \cdot \vec{B}_p = \vec{1}_r \cdot \delta \vec{B}_v + \delta \vec{n} \cdot \vec{B}^o_v \tag{38}$$

at $r = r_o$

The change in the unit normal is found from the equation of the unit normal.

$$\frac{d\vec{n}}{dt} = \vec{n} \times [\vec{n} \times \vec{\nabla}(\vec{n}\cdot\vec{u})]$$

$$= -\vec{\nabla}(\vec{n}\cdot\vec{u}) + [\vec{n}\cdot\vec{\nabla}(\vec{n}\cdot\vec{u})]\,\vec{n} \qquad (39)$$

$$\therefore \frac{\partial}{\partial t}\,\delta\vec{n} = -\vec{\nabla}(\vec{n}^{o}\cdot\vec{u}) + [\vec{n}^{o}\cdot\vec{\nabla}(\vec{n}^{o}\cdot\vec{u})]\,\vec{n}^{o} \qquad (40)$$

$$\therefore \delta\vec{n} = -\vec{\nabla}\zeta_r + \vec{1}r\,(\vec{1}r\cdot\vec{\nabla}\zeta_r), \text{ where } \zeta_r = \vec{n}^{o}\cdot\vec{w}$$

or

$$\delta\vec{n} = -(0, im, ikr_o)\,\delta e^{i(kz + m\theta)} \qquad (41)$$

Form (38) and (41), we obtain

$$ik\alpha_p B_\theta r_o\,\delta - ik\alpha_p B_\theta r_o\,\delta = 0 = k\Psi' - iB_\theta(\alpha_v y + m)\,\delta$$

Thus we obtain

$$\Psi' = \frac{iB_\theta}{k}\,\delta(\alpha_v y + m) \qquad \text{at } r = r_o \qquad (42)$$

and

$$\Psi = [\,AI_m(kr) + BK_m(kr)\,]\,e^{i(kz + m\theta)}$$

and

$$\frac{B}{A} = -\frac{I_m'(kR_o)}{K_m'(kR_o)}$$

$$\therefore\ A\left[I_m'(y) - \frac{I_m(kR_o)}{K_m(kR_o)} K_m'(y)\right] = \frac{iB_\theta}{k}\,\delta(\alpha_v y + m)$$

$$\therefore\quad A = \frac{iB_\theta(\alpha_v y + m)\,\delta/k}{I_m'(y) - \dfrac{I_m(\zeta y)}{K_m(\zeta y)} K_m'(y)} \tag{42}$$

where $\quad \zeta = \dfrac{R_o}{r_o}$.

6.- The Pressure change outside the Plasma

The total pressure in the vacuum is given by the magnetic pressure term:

$$P_o = \frac{1}{8\pi}\,|\vec{B}_v|^2 .$$

$$P_o(r_o+\delta) = \frac{1}{8\pi}\left[\,|\vec{B}_v|^2 + (\vec{w}\cdot\vec{\nabla})\,|\vec{B}_v|^2\right]_{r_o}$$

$$= \frac{1}{8\pi}\left[\,|\vec{B}_v^o|^2 + 2\vec{B}_v^o\cdot\delta\vec{B}_v + (\vec{w}\cdot\vec{\nabla})\,|\vec{B}_v'|^2\right]_{r_o}$$

$$\therefore\ \delta P_o(r_o) = \frac{1}{4\pi}\left[\,\vec{B}_v^o\cdot\delta\vec{B}_v + \frac{1}{2}(\vec{w}\cdot\vec{\nabla})\,|\vec{B}_v^o|^2\right]_{r_o}$$

$$= \frac{B_\theta}{4\pi}(\delta B_\theta + \alpha_v\,\delta B_z) - \frac{B_\theta^2}{4\pi}\,\delta e^{i(kz+m\theta)} \tag{43}$$

Substituting for $\delta\vec{B}_v$ in (43), we obtain

$$\delta P_o (r = r_o) = \frac{B_\theta^2}{4\pi} \delta e^{i(kz + m\theta)} \quad \times \tag{44}$$

$$[-1 - \frac{1}{y} (\alpha_v y + m)^2 \frac{I_m(y) K_m(\zeta y) - K_m(y) I_m(\zeta y)}{I_m'(y) K_m(\zeta y) - I_m(\zeta y) K_m'(y)}]$$

and we have earlier obtained (cf. eq. (30))

$$\delta P_i (r = r_o) = - \frac{1}{4\pi} \alpha_p^2 B_\theta^2 y^2 \delta \frac{I_m(k r_o)}{y I_m'(y)} e^{i(kz + m\theta)} \tag{45}$$

7.- *The Criterion for Stability*

For stability, we must have

$$\frac{\delta P_o - \delta P_i}{r - r_o} > 0 \tag{46}$$

Using the foregoing results we obtain for stability

$$\alpha_p^2 y^2 P_m(y) + (\alpha_v y \pm m)^2 \frac{G_{m,\zeta}(y) P_m(y) - Q_m(y)}{1 - G_{m,\zeta}(y)} > 1, \tag{47}$$

The negative sign in the bracket is obtained if $m < 0$. Here

$$P_m(y) = \frac{I_m(y)}{y I_m'(y)} \quad , \quad Q_m(y) = \frac{K_m(y)}{y K_m'(y)} \quad , \tag{48}$$

$$G_{m,\zeta}(y) = \frac{K_{m'}(\zeta y)}{K'_m(y)} \quad \frac{I_m'(y)}{I_m'(\zeta y)} \tag{48}$$

Equation (47) expresses the criterion for the stability of the pinch with a trapped axial magnetic field for a given k and m. It must be remembered that α_p, α_v ocurring in eq. (47) are not independent, for they must satisfy the condition

$$\alpha_p^2 \leqslant 1 + \alpha_v^2 \tag{49}$$

8.- *Discussion*

i) Absence of an axial magnetic field and conducting walls: The case of Kruskal & Schwarzschild.

If in eq. (47), we let

$$\alpha_v = \alpha_p = 0 \quad \text{and} \quad R_o \to \infty$$

then its can be readily seen that

$$G_{m,\zeta}(y) \to 0$$

and we find from (47) that for instability, we must have

$$m^2 Q_m(y) + 1 > 0$$

or explicitly

$$k r_o + m^2 \frac{K_m(k r_o)}{K'_m(k r_o)} > 0 \tag{50}$$

It is immediately clear from this result that the mode $m = 0$ is unstable to disturbances of all wave lengths. The same conclusion applies to the $m = 1$ mode as can be readily verified. For higher values of m, there is a range of wave numbers for which we obtain stability. For large values of $(k r_o)$,

$$\frac{K_m (k r_o)}{K'_m (k r_o)} \rightarrow -1 \tag{51}$$

and the largest value of $k r_o$ for which stability occurs is approximately m^2.

ii) Axial magnetic field but no conducting walls.

This is the case considered by Kruskal & Tuck. In the absence of conducting walls, $G_{m,\zeta} \rightarrow 0$ and (47) gives for *stability*

$$\alpha_p^2 y^2 P_m(y) - (\alpha_v y \pm m)^2 Q_m(y) > 1 \tag{52}$$

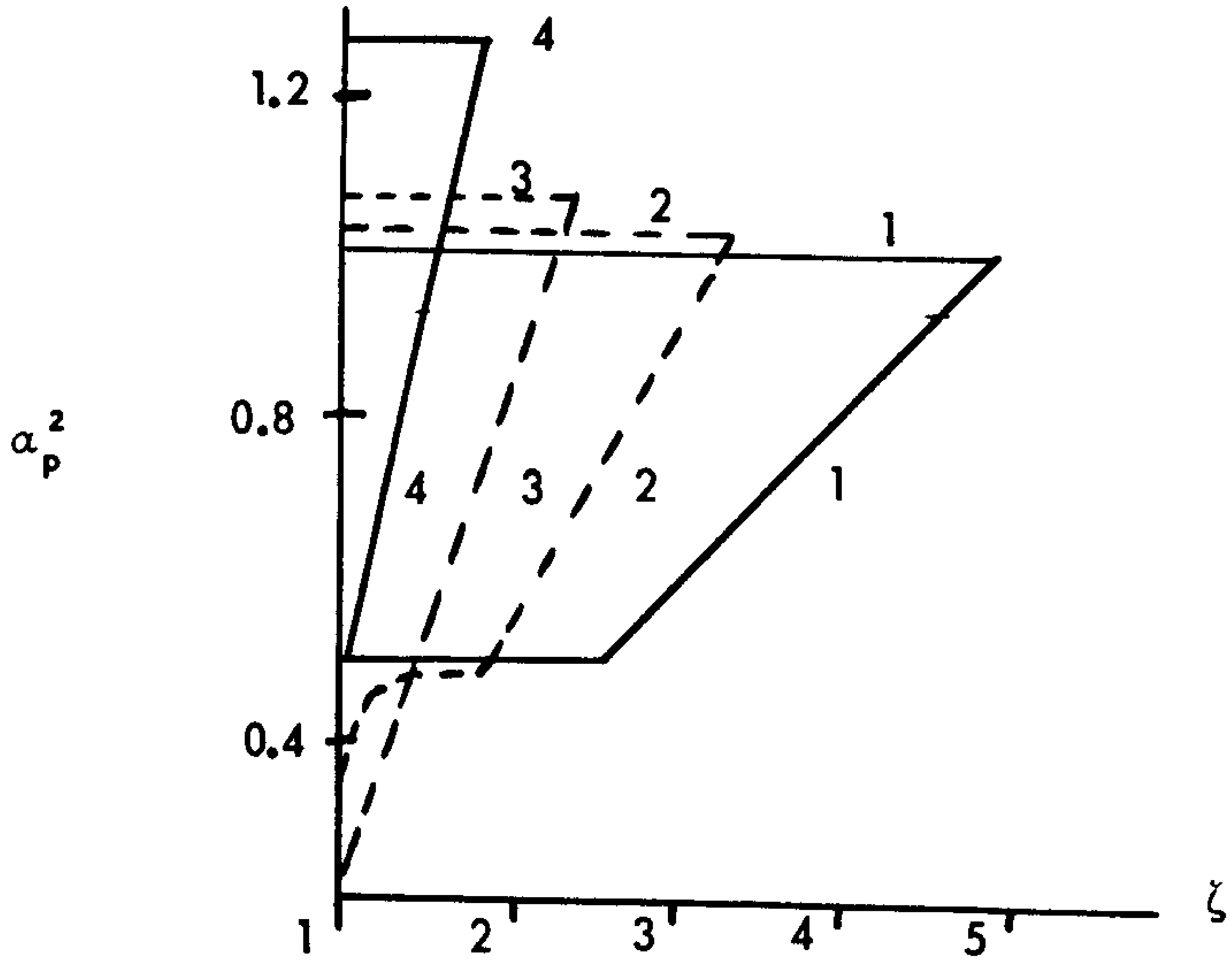
In this case we find that it is possible to stabilize both $m = 0$ modes and those with $m > 1$ for suitable values of α_p and α_v but that without the presence of conducting walls, it is not possible to stabilize the $m = 1$ modes.

iii) The general case.

We need consider only negative values of m as the additional term

$$4 m \alpha_v y \frac{G_{m,\zeta}(y) P_m(y) - Q_m(y)}{1 - G_{m,\zeta}(y)}$$

can be shown to be a positive definite quantity. So if the inequality (47) is satisfied for negative values of m, it is also satisfied for positive values of m. The stability diagram is shown below.



The region of stability in the (α_p^2, ζ) plane for $\alpha_v = 0, 0.1, 0.25$ and 0.5 (distinguished by the numbers 1, 2, 3, 4 respectively). For a given α_v a stable pinch occurs in a region bounded on the left by the α_p^2 axis, on the right by the marginal curve for $m = -1$, above by the equilibrium condition $\alpha_p^2 < (1 + \alpha_v^2)$ and below by the marginal curve for $m = 0$.

9.- *Concluding Remarks*

We have discussed the stability of the pinch on the basis of the equation of hydromagnetics using a scalar pressure. This is the treatment of Tayler, Shrafranov, Kruskal & Tuck and Rosenbluth.

However, the more realistic case is the one where one uses a tensor pressure. Then, of course, we have to abonden the adiabatic relation (or the equation of state) connecting the density and the pressure. Rosenbluth has also investigated the case of tensor pressure. Then to calculate the change in the total pressure, use is made of the two adiabatic invariants in the motion of charged particles:

i) The constancy of the magnetic moment, i. e.

$$\frac{E_{\perp}}{B}, \quad E_{\perp} = \frac{1}{2} m v_{\perp}^2, \quad v_{\perp} = \text{transverse velocity}$$

ii) The existence of the action integral

$$\oint v_{\parallel}\, dl = \oint \vec{v} \cdot \frac{d\vec{B}}{|\vec{B}|}, \quad v_{\parallel} = \text{paralell velocity}$$

The most rigorous account based on the solution of the Boltzmann Equation has been given by Chandrasekhar, Kaufman & Watson.

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