A novel symmetry property of the Fourier transform

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This manuscript presents and proves a reciprocity relation involving the Fourier transforms of a pair of square-integrable functions, expressed as a bilinear map. This reciprocity relation reveals a deep symmetry between the time (or spatial) and frequency domains. We explore its implications in theoretical and applied contexts such as signal processing, quantum mechanics, and computational physics. Additionally, we discuss the role of this relation in the bilinear nature of Fourier analysis.

Keywords: Fourier transform; signal processing; parseval theorem.

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1. Introduction

The Fourier transform (FT) is probably the most known and applied integral transform in sciences and engineering. It was introduced first by the French mathematician Joseph Fourier when he was characterizing the behavior of the heat along a rigid bar [1,2]. FT provides a connection between the time/spatial domain and the frequency domain. It has wide applications in various fields such as optics [3-6], quantum mechanics [7-10], and acoustics [11-13]. It is a very important mathematical tool for signal analysis [14,15] and digital image processing [16], where it is commonly used as a filter or feature extractor [17-19].

The FT takes a function $f(x)$ *living* in the space domain x and gets its representation in the spatial frequency domain k. The standard definitions of the one-dimensional FT and its inverse read as [20,21]

$$\mathscr{F}[f(x)] = F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x, \tag{1}$$

$$\mathscr{F}^{-1}[F(k)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\mathrm{d}k, \tag{2}$$

where (x, k) are the space and spatial frequency variables running along the real axes $(-\infty, \infty)$. The FT of the function $f(x)$ is defined only when the integral converges for all values of k.

In this manuscript, we investigate a symmetry in the Fourier transform that expresses a reciprocity relation between two square-integrable functions and their FTs. This symmetry is represented as a bilinear map, demonstrating an equivalence between the integrals of the Fourier transforms of two functions in different orders. To the best of our knowledge, this property has not been published anywhere else.

Therefore, the primary goal of this manuscript is to derive and prove a Fourier transform bilinear reciprocity relation of the form,

$$\int_{-\infty}^{+\infty} \mathscr{F}[f(x)]g(k)\mathrm{d}k = \int_{-\infty}^{+\infty} \mathscr{F}[g(x)]f(k)\mathrm{d}k. \tag{3}$$

We discuss the bilinear nature of this reciprocity and its potential applications. The manuscript is divided as follows. In Sec. 2, some properties of the Fourier transform that are related to Eq. (3) are discussed. In Sec. 3, the new property is proved and named. Also, this section discussed its practical implications with illustrative examples. Finally, the conclusions are given.

2. Properties of the Fourier transform

Before we introduce the bilinear reciprocity property between pairs of square-integrable functions, we introduce the most well-known properties of the Fourier transform that can be related to the mentioned property.

Linearity. The first property is linearity. Given the functions $f(x)$ and $g(x)$, and a and b are constants,

$$\mathscr{F}[af(x) + bg(x)] = aF(k) + bG(k). \tag{4}$$

The FT of a linear combination of functions is the same linear combination of their transforms. This property allows us to break down complex functions into simpler components [22].

Time shifting. the time shifting property considers $f(x)$ and its FT $F(k)$, then,

$$\mathscr{F}[f(x - x_0)] = e^{-jkx_0}F(k). \tag{5}$$

Shifting a signal in the time domain corresponds to multiplying its FT by a complex exponential. This is useful when analyzing delayed functions [15].

Frequency shifting. This property relates the function $f(x)$ and its Fourier $F(k)$ with,

$$\mathscr{F}[e^{jk_0t}f(x)] = F(k - k_0). \tag{6}$$

Multiplying a signal by a complex exponential shifts its spectrum in the frequency domain. This property is fundamental in modulation applications [15].

Symmetry. This property is stated as,

$$F(-k) = F^*(k). \tag{7}$$

The FT of a real signal exhibits conjugate symmetry.
Duality. The duality property of the FT for a single function $f(x)$ is written as,

$$\mathscr{F}[F(x)] = 2\pi f(-k). \tag{8}$$

The above equation means that FT of a time-domain signal corresponds to an inverse FT in the frequency domain and vice-versa [22].
Convolution theorem. The convolution theorem relates two given functions $f(x)$ and $g(x)$ and their Fourier transforms $F(k)$ and $G(k)$, with the following expression,

$$\mathscr{F}[f(x) * g(x)] = \mathscr{F}[f(x)] \cdot \mathscr{F}[g(x)]. \tag{9}$$

Convolution in the time domain corresponds to multiplication in the frequency domain.
Multiplication theorem. Similar to the convolution theorem, the multiplication theorem states,

$$\mathscr{F}[f(x) \cdot g(x)] = \frac{1}{2\pi}\mathscr{F}[f(x)] * \mathscr{F}[g(x)]. \tag{10}$$

Multiplication in the time domain corresponds to convolution in the frequency domain.
Parseval's theorem. The Parseval's theorem, an implication of the energy conservation principle, considers for any function $f(x)$,

$$\int |f(x)|^2 \, dx = \frac{1}{2\pi}\int |F(k)|^2 \, dk. \tag{11}$$

The total amplitude of a function remains the same whether computed in the time/space domain or frequency domain [23]. Once we have stated the properties that can be related to the proposed property, we can follow up with the proof of the bilinear reciprocity property in the following section.

3. The swapping symmetry property of Fourier transform

The swapping pairs symmetry property of the FT can be stated as the following theorem.

Theorem If $f(x)$ and $g(x)$ are two square integrable functions depending only on the real variable $x \in (-\infty, \infty)$, then the following equality holds,

$$\int_{-\infty}^{+\infty} \mathscr{F}[f(x)]g(k)\mathrm{d}k = \int_{-\infty}^{+\infty} \mathscr{F}[g(x)]f(k)\mathrm{d}k. \tag{3}$$

Above, equality implies that functions $f(x)$ and $g(x)$ can be interchanged in the integrands by keeping invariant the evaluation of the integral. Nevertheless, be aware that because the functions $f(x)$ and $g(x)$ are independent each other, then the integrands $\mathscr{F}[f(x)]g(k)$ and $\mathscr{F}[g(x)]f(k)$ will be different in general. To the best of my knowledge, the symmetry property of the FT in Eq. (3) has not been reported nor applied in the literature associated to FT.

Proof Let us apply the definition of the FT Eq.(1) in both sides of the equality in Eq.(3). Let us get first for the left side

$$\int_{-\infty}^{+\infty} \mathscr{F}[f(x)]g(k)\mathrm{d}k$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(k)e^{-ikx}\mathrm{d}x\mathrm{d}k, \tag{12}$$

and then for the right side,

$$\int_{-\infty}^{+\infty} \mathscr{F}[g(x)]f(k)\mathrm{d}k$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)f(k)e^{-ikx}\mathrm{d}x\mathrm{d}k. \tag{13}$$

Comparing the right sides of Eqs. (14) and (15) and observing that the reciprocal variables x and k run interchangeably over the real axis $(-\infty, \infty)$, it is easy to conclude that both integrals necessarily yield the same value.

Equation (3) expresses a duality of the FT of $f(x)$ paired with $g(k)$, which behaves the same as the FT of $g(x)$ paired with $f(k)$. This duality suggests that knowledge of a system in Fourier analysis can be equivalently represented in two complementary ways. Our knowledge of the system in Fourier analysis is invariant under a change of representation, which aligns with the epistemological idea that different perspectives of a system can lead to the same understanding, giving a global perspective on the system. Therefore in Fourier analysis, the knowledge of a system is not fragmented but interconnected. While the two integrals are equal, they represent different ways of understanding the system.

Notice that Eq. (3) holds for FT linearity property. Therefore, it is directly related to Eq. (4). Equations (5) and (6) are the relations of the space/time and frequency-shifting properties, respectively. They cannot be interchangeable in Eqs. (14) and (15) since they do not consider any shifting parameters x_0 or k_0. Therefore, Eq. (3) is not a consequence of Eq. (4) or Eq. (5).

Observe that the symmetry property expressed in Eq. (7) also applies in Eq. (3), replacing Eq. (7) in Eq. (14),

$$\int_{-\infty}^{+\infty} \mathscr{F}[f(-x)]g(k)\mathrm{d}k$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(k)e^{ikx}\mathrm{d}x\mathrm{d}k, \tag{14}$$

and then Eq. (7) in Eq. (15),

$$\int_{-\infty}^{+\infty} \mathscr{F}[g(-x)]f(k)\mathrm{d}k$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)f(k)e^{ikx}\mathrm{d}x\mathrm{d}k. \tag{15}$$

Convolution theorem and multiplication theorem consider shifting parameters which are not presented in Eq. (3). The computations between the functions in the mentioned theorem are only in space/time domain or on the frequency domain. On the other hand, the computation between the functions in Eq. (3) are mixed in space/time domain and frequency domain. Therefore, Eq. (3) is not directly related to the convolution theorem and multiplication theorem.

Equation (3) is not so related to the Parseval theorem in a direct way. Parseval's theorem states that the inner product of a function in the time domain is preserved in the frequency domain. In Eq. (3) considers two different independent square-integrable functions. Equation (3) cannot be seen as the inner product between functions $f(x)$ and $g(x)$ and the Fourier representation of the other. The inner product measures the extent to which one function projects onto another. If the inner product is large, it indicates that the two functions are similar in shape or behavior over the domain $x \in (-\infty, \infty)$. Conversely, if the inner product is zero, the functions are orthogonal. But Eq. (3) is not an inner product since it does not consider the conjugate of any of the functions of their Fourier representation. The operation computed in Eq. (3) is not an inner product but a bilinear map.

A bilinear map is an operation that maps two functions into a scalar through two linear operations [24]. In the context of Fourier transforms, we are dealing with a bilinear map because we are performing an integral that involves both $\mathscr{F}[f(x)]$ and $g(k)$, and similarly for the inverse. The key property of a bilinear map is that it is linear in both of its arguments. This can be seen in the following form for the bilinear map $B(f, g)$,

$$B(f,g) = \int_{-\infty}^{\infty} \mathscr{F}[f(x)]g(k)\,\mathrm{d}k. \tag{16}$$

A bilinear form B on L^2 is a map $B : L^2 \times L^2 \to \mathbb{C}$ such that for all functions $f, g, h \in L^2$ and scalars $\alpha, \beta \in \mathbb{C}$

Linearity in the first argument: $B(\alpha f + \beta g, h) = \alpha B(f, h) + \beta B(g, h)$ for all $f, g, h \in L^2$ and scalars α, β.

$$\begin{aligned} B(\alpha f + \beta g, h) &= \int_{-\infty}^{\infty} \mathscr{F}[\alpha f(x) + \beta g(x)]h(k)dk \\ &= \alpha \int_{-\infty}^{\infty} \mathscr{F}[f(x)]h(k)dk + \beta \int_{-\infty}^{\infty} \mathscr{F}[g(x)]h(k)dk \\ &= \alpha B(f, h) + \beta B(g, h). \end{aligned} \tag{17}$$

Linearity in the second argument: $B(f, \alpha g + \beta h) = \alpha B(f, g) + \beta B(f, h)$ for all $f, g, h \in L^2$ and scalars α, β.

$$\begin{aligned} B(f, \alpha g + \beta h) &= \int_{-\infty}^{\infty} \mathscr{F}[f(x)](\alpha g(x) + \beta h(k))dk \\ &= \alpha \int_{-\infty}^{\infty} \mathscr{F}[f(x)]g(k)dk + \beta \int_{-\infty}^{\infty} \mathscr{F}[f(x)]h(k)dk \\ &= \alpha B(f, g) + \beta B(f, h). \end{aligned} \tag{18}$$

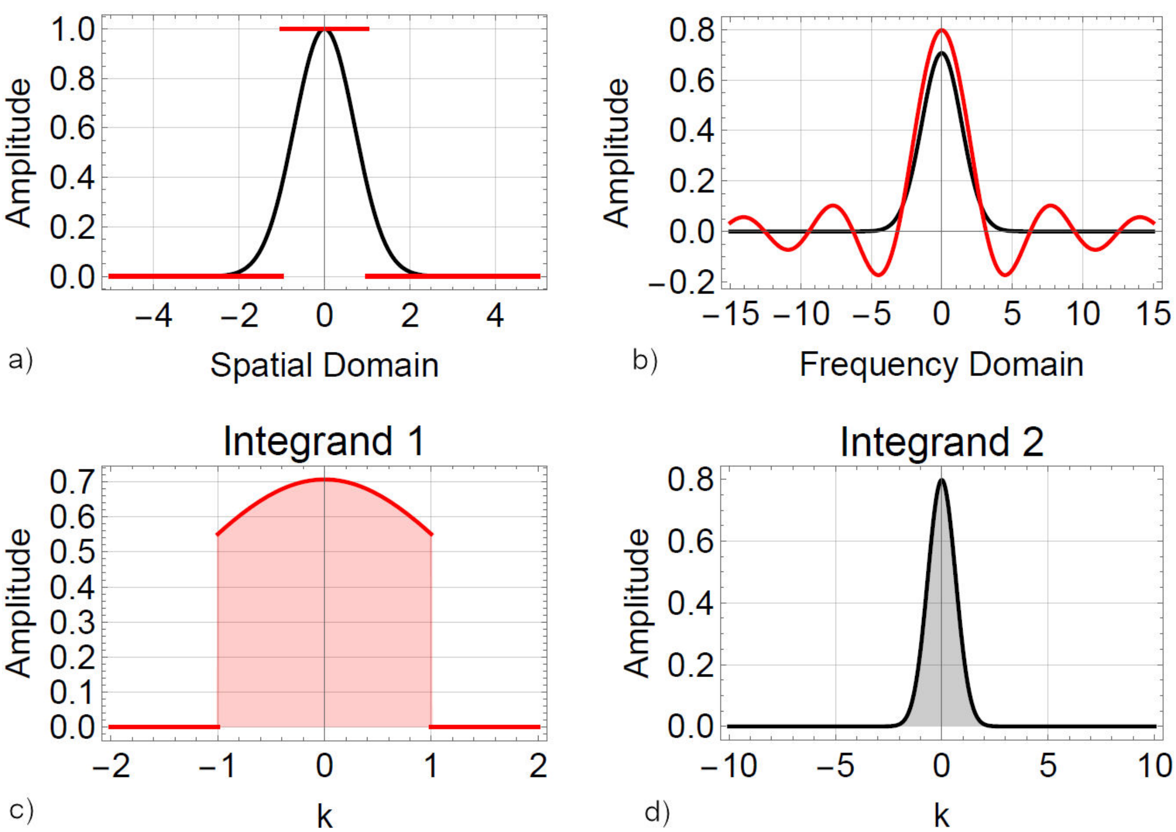

FIGURE 1. Plots for first example. a) The spatial domain of $f(x)$ and $g(x)$, in black and red, respectively. b) $F(k)$ and $G(k)$, in black and red, respectively. c) $\mathscr{F}[f(x)]g(k)$. d) $f(k)\mathscr{F}[g(x)]$.

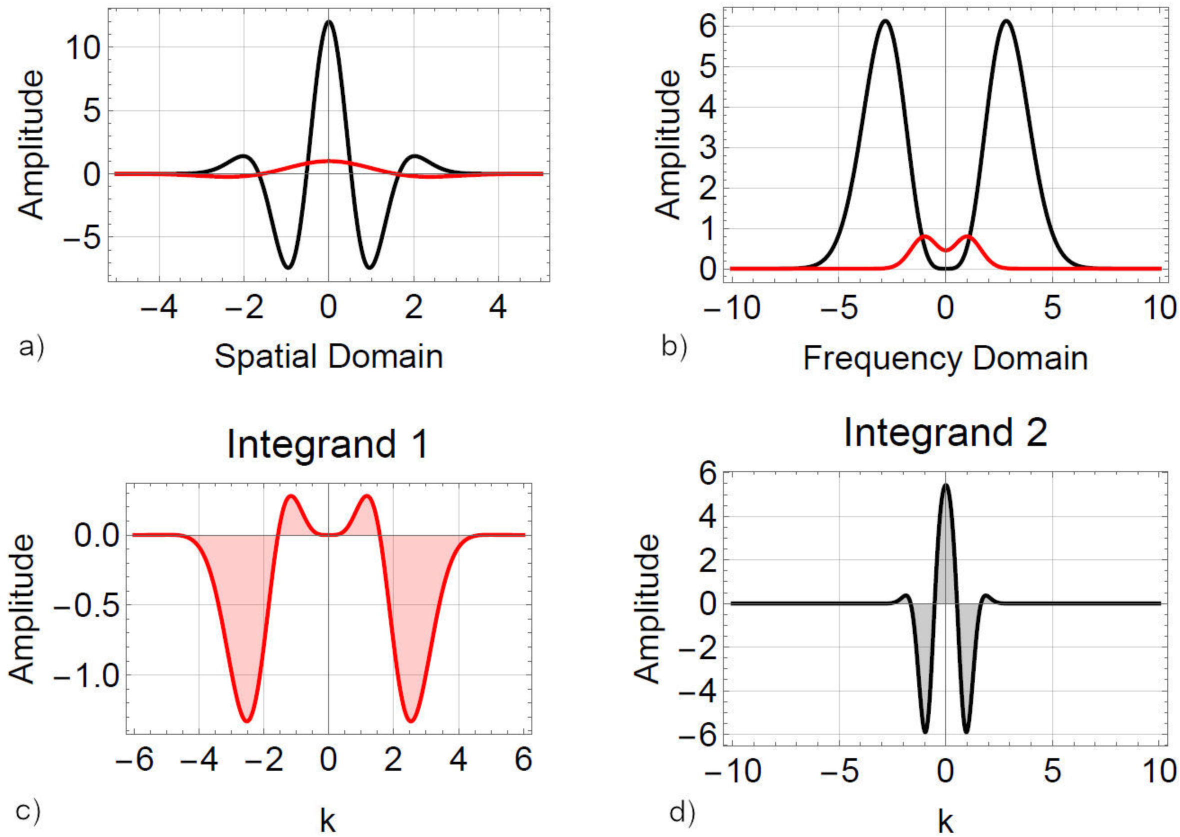

FIGURE 2. Plots for the second example. a) The spatial domain of $f(x)$ and $g(x)$, in black and red respectively. b) The spectral domain of $f(x)$ and $g(x)$, in black and red respectively. c) The integrant $\mathscr{F}[f(x)]g(k)$ and in d) The other integrate $f(k)\mathscr{F}[g(x)]$.

If both conditions are satisfied, then the map B is a bilinear form on the space L^2. In the case of Eq. (16) are satisfied since it is an integral and the FT is an integral too. Integrals are linear operators. Eq. (3) is a reflection of this bilinearity. Thus, both sides of the equation are equal, proving the reciprocity relation. This result demonstrates the bilinear symmetry inherent in the Fourier transform. Therefore, we shall call Eq. (3) the Fourier transform bilinear reciprocity property.

The following example considers the two classical functions in Fourier analysis, $f(x) = e^{-x^2}$ and $g(x)$ a step rectangle function. The plot of the $f(x)$ and $g(x)$ is in *up-left* of Fig. 1. On the same figure at the *up-right* is the spectral representation of $f(x)$ and $g(x)$. The integrants of Eq. (3) are plotted at *bottom-left bottom-right*. As predicted by Eq. (3), the area below the curve of integrants $\mathscr{F}[f(x)]g(k)$, $f(k)\mathscr{F}[g(x)]$ has the same value. The value is 1.3047.

The next example considers a more complicated pair of functions and their FTs. $f(x) = e^{-x^2}H_4(x)$, where $H_4(x)$ is a Hermite polynomial of 4-th order and $g(x) = \cos(x)e^{-x^2/5}$. The space domain and frequencies of $f(x)$ and $g(x)$ are presented in *up-left* and *up-right* of Fig. 2, respectively. $\mathscr{F}[f(x)]g(k)$ is ploted in the *bottom-left* and $f(k)\mathscr{F}[g(x)]$ in the *bottom-right* of the same figure. After numerically computing the integral over $(-\infty, \infty)$ over the two integrals, we conclude that they have the same value of -3.21866.

The final example considers the following functions and the frequency representation.

$$f(x) = \frac{1}{1+x^2}, \quad g(x) = \mathrm{Ai}(x), \tag{19}$$

where $\mathrm{Ai}(x)$ is an Airy function, which is a special function that is an independent solution of the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy = 0. \tag{20}$$

For this example, it is important to remark that $\mathrm{Ai}(x)$ is a square-integrable function. The FTs $f(x)$ and $g(x)$ are given by,

$$F(k) = \sqrt{\frac{\pi}{2}}e^{-|k|}, \quad G(k) = \frac{e^{-\frac{ik^3}{3}}}{\sqrt{2\pi}}, \tag{21}$$

in Eq. (19) is plotted in *up-left* of Fig. 3, $f(x)$ and $g(x)$ are in red and black, respectively. Equation (21) is plotted in *up-right* of Fig. 3. In black is $F(k)$, and in red and orange are the real and imaginary parts of $G(k)$. At *bottom-left* is the integrant $F(k)g(k)$ and at *bottom-right* is $f(k)G(k)$, where in black is the real part and in orange the imaginary. Numerically computing both integrals of Eq. (3) for this example, we get the result of 0.703319 in both cases.

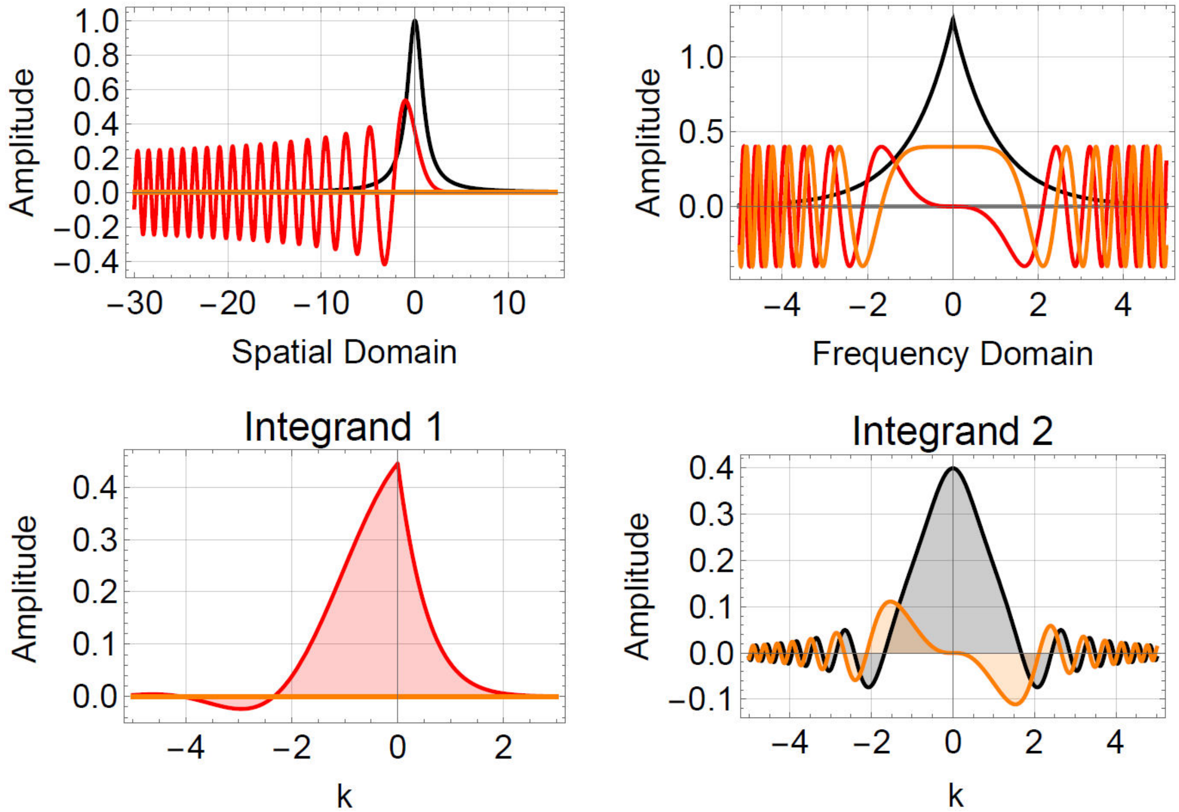

FIGURE 3. Plots for the third example. a) The spatial domain of $f(x)$ and $g(x)$, in black and red respectively. b) Spectral domain, $F(k)$ and $G(k)$. $F(k)$ is in black, the real part of $G(k)$ is red and it's imaginary is orange. c) The integrant $\mathscr{F}[f(x)]g(k)$. d) The real part of $f(k)\mathscr{F}[g(x)]$ is black and it's imaginary is orange.

We have computed three different examples, and the results are just as expected by Eq. (3). This should not be a surprise since the proof was done for a pair of square-integrable functions and the FTs. We had obtained a bilinear symmetry of the FT. The FT is widely applied in several fields in science and engineering. Therefore, this bilinear symmetry provides insights in all the fields where FT is applied.

For example, in signal processing, the bilinear map nature of the Fourier transform may allow us to manipulate signals by exchanging their roles in the time and frequency domains. The reciprocity relation can be used in filter design, where the relationship between time-domain signals and their frequency-domain counterparts is key. This symmetry could also simplify the design of algorithms for signal reconstruction or denoising [15]. In quantum mechanics, Fourier transforms are employed to relate wavefunctions in position space and momentum space. The reciprocity relation highlights the inherent symmetry in the exchange of spatial and momentum representations, which could help in simplifying problems involving wave-particle duality and quantum state representations [7,9]. The bilinear nature of the Fourier transform reciprocity relation has practical implications in computational physics and numerical methods. The ability to exchange functions and their Fourier transforms efficiently can improve algorithms for processing large datasets, especially when performing operations like filtering, deconvolution, and Fourier-domain analysis [16]. But for the moment, all the mentioned potential applications are considered to be the next step in this research and are out of the scope of this manuscript.

4. Conclusion and future directions

In this manuscript, we presented the Fourier transform bilinear reciprocity property that reveals the bilinear symmetry between the Fourier transforms of two square-integrable functions. This result highlights the deep connection between the spatial and frequency domains and provides new insights into the symmetry of Fourier analysis. We provided some examples that behaved as expected.

Future research shall explore the applications of this bilinear symmetry in fields where FT is applied, such as optics, signal processing to quantum mechanics, computational physics and numerical methods.

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1. K. B. Wolf, Construction and properties of canonical transforms, In Integral Transforms in Science and Engineering, (Springer, 1979) pp. 381-416 .
2. R. N. Bracewell and R. N. Bracewell, The Fourier transform and its applications, vol. **31999** (McGraw-Hill New York, 1986).
3. J. W. Goodman, Introduction to Fourier optics (Roberts and Company publishers, 2005).
4. H. M. Ozaktas and D. Mendlovic, Fractional fourier optics, *Journal of the optical society of America A* **12** (1995) 743
5. P. J. de Groot and X. Colonna de Lega, Fourier optics modeling of interference microscopes, *Journal of the Optical Society of America A* **37** (2020) B1
6. J. M. Boone, T. Yu, and J. A. Seibert, Sinusoidal modulation analysis for optical system MTF measurements, *Medical Physics* **23** (1996) 1955
7. M. Rezai and J. A. Salehi, Fundamentals of quantum Fourier optics, *IEEE Transactions on Quantum Engineering* **4** (2022) 1
8. H. De Bie, Clifford algebras, Fourier transforms, and quantum mechanics, *Mathematical Methods in the Applied Sciences* **35** (2012) 2198
9. L. P. Horwitz, Fourier transform, quantum mechanics and quantum field theory on the manifold of general relativity, *The European Physical Journal Plus* **135** (2020) 479
10. V. Lyubashenko and S. Majid, Fourier transform identities in quantum mechanics and the quantum line, *Physics Letters B* **284** (1992) 66
11. L. Kovalevsky, P. Ladeveze, and H. Riou, The Fourier version of the variational theory of complex rays for medium-frequency acoustics, *Computer Methods in Applied Mechanics and Engineering* **225** (2012) 142
12. A. Kuijpers, G. Verbeek, and J. Verheij, An improved acoustic Fourier boundary element method formulation using fast Fourier transform integration, *The Journal of the Acoustical Society of America* **102** (1997) 1394
13. M.-Y. Zhang, et al., An improvement of the generalized discrete Fourier series based patch near-field acoustical holography, *Applied Acoustics* **173** (2021) 107711
14. H. M. Ozaktas, Z. Zalevsky, and M. Kutay, The Fractional Fourier Transform with Applications in Optics and Signal Engineering (2001).
15. S. J. Orfanidis, Introduction to signal processing (Prentice-Hall, Inc., 1995).
16. R. C. Gonzalez *et al.*, Digital image processing using MATLAB., vol. **624** (Pearson-Prentice-Hall Upper Saddle River, New Jersey, 2004).
17. D. Mery, Computer Vision for X-Ray Testing, Switzerland: Springer International Publishing (2015).
18. C. T. Zahn and R. Z. Roskies, Fourier descriptors for plane closed curves, *IEEE Transactions on computers* **100** (1972) 269
19. H. Peng, F. Long, and C. Ding, Feature selection based on mutual information criteria of max-dependency, max-relevance, and min-redundancy, *IEEE Transactions on pattern analysis and machine intelligence* **27** (2005) 1226
20. A. D. Poularikas, Transforms and applications handbook (CRC press, 2010).
21. L. Debnath and D. Bhatta, Integral transforms and their applications (Chapman and Hall/CRC, 2006).
22. D. G. Duffy, Transform methods for solving partial differential equations (Chapman and Hall/CRC, 2004).
23. M. L. Boas, Mathematical methods in the physical sciences (John Wiley and Sons, 2006).
24. A. G. Myasnikov, The theory of models of bilinear mappings, *Siberian Mathematical Journal* **31** (1990) 439
25. O. Christensen, Functions, spaces, and expansions: mathematical tools in physics and engineering (Springer Science and Business Media, 2010).
26. D. G. Duffy, Advanced engineering mathematics with MATLAB (Chapman and Hall/CRC, 2016).
27. H. De Bie, R. Oste, and J. Van der Jeugt, Generalized Fourier Transforms Arising from the Enveloping Algebras of (2) and (1 2), *International Mathematics Research Notices* **2016** (2015) 4649.
28. R. Howe, The oscillator semigroup. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), 61-132, In Proc. Sympos. Pure Math, vol. **48**.
29. R. Klette, Concise Computer Vision-An Introduction into Theory and Algorithms. Undergraduate Topics in Computer Science (2014).
30. L. Gupta and M. D. Srinath, Contour sequence moments for the classification of closed planar shapes, *Pattern Recognition* **20** (1987) 267
31. R. Hartley and A. Zisserman, Multiple view geometry in computer vision (Cambridge university press, 2003).
32. M.-K. Hu, Visual pattern recognition by moment invariants, *IRE transactions on information theory* **8** (1962) 179
33. C.-H. Teh and R. T. Chin, On digital approximation of moment invariants, *Computer Vision, Graphics, and Image Processing* **33** (1986) 318.
34. K. Mao, Identifying critical variables of principal components for unsupervised feature selection, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* **35** (2005) 339.
35. R. O. Duda, P. E. Hart, and D. G. Stork, Pattern classification second edition John Wiley and sons, New York **58** (2001) 16.
36. R. G. González-Acuña and J. C. Gutierrez-Vega, A transition integral transform obtained from generalization of the Fourier transform, *Ain Shams Engineering Journal* **10** (2019) 841.