Further results on Sturm-Picone theorems for nonlinear fractional differential equations

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Received 19 June 2025; accepted 16 January 2026

The author obtains Sturm-Picone comparison results for fractional differential equations involving conformable and nonconformable fractional derivatives. Examples illustrate the conclusions.

Keywords: Fractional differential equations; nonlinear; Sturm-Picone theorem.

DOI: https://doi.org/10.31349/RevMexFis.72.030201

1. Introduction

In this paper we again examine a Sturm-Picone type oscillation comparison theorem for fractional differential equations (see [7] for results of a somewhat different nature from those obtained here). The basic idea is that the Sturm-Picone theorem compares the oscillatory behavior of two second order linear ordinary differential equations in the sense that under certain conditions, the oscillation of one equation implies the oscillation of the other one. The origin of this problem can be traced to the seminal results of Sturm [24] and the subsequent work of Picone [23] on this problem.

In this paper, we wish to obtain such a result for nonlinear differential equations involving fractional derivatives. Let us first recall the well known Sturm-Picone comparison theorem; see, *e.g.,* Sec. 9.1 in Ref. [2]) or Theorem 1.2 in Ref. [25]).

Theorem 1. *If the equation*

$$(p(t)x')' + q(t)x = 0$$

is oscillatory and

$$0 \le P(t) \le p(t) \quad \text{and} \quad Q(t) \ge q(t), \qquad \text{(SP)}$$

then the equation

$$(P(t)y')' + Q(t)y = 0$$

is also oscillatory.

Here, when we say that an equation is oscillatory, we mean that all solutions oscillate. We will consider the nonlinear fractional differential equations

$$D^\alpha(a(t)(D^\alpha y(t))(t) + F_1(t, y(t), D^\alpha y(t))(t) = 0 \qquad (1)$$

and

$$D^\alpha(A(t)(D^\alpha y(t))(t) + F_2(t, y(t), D^\alpha y(t))(t) = 0, \qquad (2)$$

where $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\alpha \in (0, 1]$, a, $A : \mathbb{R}_+ \to \mathbb{R}_+$ and $F_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are continuous functions, $uF_i(t, u, v) \ge 0$ for $i = 1, 2$. Here, D^α is either the conformable fractional derivative of Khalil *et al.* [14] (also see Abdeljawad [1] or Martinez *et al.* [16]) or the nonconformable fractional derivative due to Nápoles Valdés *et al.* [12, 17, 20–22]. These are each defined in the following section.

The problem studied in the paper [7] mentioned above was for the comparison problem involving the equations

$$N^\alpha(a(t)(N^\alpha y)(t))(t) + q(t)F_1(y) = 0$$

and

$$N^\alpha(A(t)(N^\alpha y)(t))(t) + Q(t)F_2(y) = 0,$$

where N^α is the nonconformable fractional derivative (defined in the next section) and where $\alpha \in (0, 1]$, a, A, q, $Q : \mathbb{R}_+ \to \mathbb{R}_+$ and $F_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are continuous, and $uF_i(u) \ge 0$ for $i = 1, 2$.

2. Some Preliminaries

Some basic concepts of conformable and nonconformable fractional derivatives and integrals are defined as follows.

Definition 1. (Def. 2.1 in Ref. [14], Def. 1 in Ref. [16], p. 57 in Ref. [1]) *Let* $f : [0, \infty) \to \mathbb{R}$. *The conformable fractional derivative of* f *of order* $\alpha \in (0, 1]$ *at* $t > 0$ *is defined by*

$$(D^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{t^{1-\alpha}}) - f(t)}{\epsilon},$$

for all $t > 0$, *and the conformable fractional derivative at* 0 *is defined as* $(D^\alpha f)(0) = \lim_{t \to 0}(D^\alpha f)(t)$.

Definition 2. (Def. 2.1 in Ref. [12], Def. 1 in Ref. [22]) *Let* $f : [0, \infty) \to \mathbb{R}$. *The nonconformable fractional derivative of* f *of order* $\alpha \in (0, 1]$ *is defined by*

$$(D^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{t^{-\alpha}}) - f(t)}{\epsilon},$$

for all $t > 0$.

It is important to note that both of these fractional derivatives satisfy the usual sum, product, and quotient rules as for ordinary (non-fractional) derivatives (see p. 66 in Ref. [14] and Theorem 2.3 in Ref. [12], respectively). The corresponding fractional integrals are given in the following definitions. The discussion of generalized fractional derivatives in Refs. [4,19] may be of interest to the reader on this point.

Definition 3. (p. 5016 in Ref. [15], p. 58 in Ref. [1], Def. 3 in Ref. [16]) *Let $f : [0,\infty) \to \mathbb{R}$. The conformable fractional integral of f of order $\alpha \in (0,1]$ on $[u,v]$ is given by*

$$J_u^\alpha f(x) = \int_u^x \frac{f(t)}{t^{1-\alpha}})dt,$$

where the integral is the usual Riemann improper integral.

Definition 4. (Def. 2 in Ref. [22]) *Let $f : [0,\infty) \to \mathbb{R}$. The nonconformable fractional integral of f of order $\alpha \in (0,1)$ is defined by*

$$J_{t_0}^\alpha f(t) = \int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds.$$

These types of fractional derivatives have been successfully used in the study of stability problems [3,5,10,11, 18,22], oscillation theory [13,22], the nonlinear limit-point/limit-circle problem [6], boundary value problems, and others.

3. Main Results

In what follows, we will assume that

$$a(t) \geq A(t) > 0 \tag{3}$$

and

$$\frac{F_1(t,u_1,u_2)}{u_1} \leq \frac{F_2(t,v_1,v_2)}{v_1}, \quad \text{for all} \quad t,u_1,u_2,v_1,v_2 \quad \text{with} \quad u_1 \neq 0 \neq v_1. \tag{4}$$

Moreover, we assume that the inequalities (3) and (4) are not simultaneously identities on any open interval.

If we define $H : \mathbb{R}_+ \to \mathbb{R}$ by

$$H(t)=D^\alpha \left\{\frac{x(t)[a(t)(D^\alpha x(t))y(t)-A(t)x(t)D^\alpha y(t)]}{y(t)}\right\}, \tag{5}$$

and if we suppress the argument t when no ambiguity exists, then

$$\begin{aligned}
H(t) &= D^\alpha\left(\frac{x}{y}\right)[a(D^\alpha x(t))y(t) - Ax(t)D^\alpha y(t)] + \frac{x}{y}D^\alpha\{a(D^\alpha x(t))y(t) - Ax(t)D^\alpha y(t)\} \\
&= \frac{(D^\alpha x(t))y(t) - x(t)D^\alpha y(t)}{y^2}\big[a(D^\alpha x(t))y(t) - Ax(t)D^\alpha y(t)\big] \\
&+ \frac{x}{y}\big[D^\alpha(a(D^\alpha x(t))y(t) + a(D^\alpha x(t))D^\alpha y(t) - D^\alpha(A(t)D^\alpha y(t))x(t) - A(t)(D^\alpha x(t))(D^\alpha y(t))\big] \\
&= \frac{1}{y^2}\big[a(D^\alpha x(t))^2y^2 - Axy(D^\alpha x(t))(D^\alpha y(t)) - axy(D^\alpha x(t))((D^\alpha y(t)) + Ax^2(D^\alpha y(t))^2\big] \\
&+ \frac{x}{y}\big[-F_1(t,x,D^\alpha x(t))y + a(D^\alpha x(t))D^\alpha y(t) + F_2(t,y,D^\alpha y(t))x - A(D^\alpha x(t))(D^\alpha y(t))\big] \\
&= \frac{a(D^\alpha x(t))^2y^2 - Axy(D^\alpha x(t))(D^\alpha y(t)) - axy(D^\alpha x(t))((D^\alpha y(t)) + Ax^2(D^\alpha y(t))^2}{y^2} \\
&+ \frac{x}{y}\big[a(D^\alpha x(t))(D^\alpha y(t)) - A(D^\alpha x(t))(D^\alpha y(t))\big] + \frac{x}{y}\big[-F_1(t,x,D^\alpha x(t))y + F_2(t,y,D^\alpha y(t))x\big] \\
&= \frac{a(D^\alpha x(t))^2y^2 - Axy(D^\alpha x(t))(D^\alpha y(t)) - ax(D^\alpha x(t))y((D^\alpha y(t)) + Ax^2(D^\alpha y(t))^2}{y^2} \\
&= \frac{1}{y^2}\big[a(D^\alpha x(t))^2y^2 - Axy(D^\alpha x(t))(D^\alpha y(t)) + Ax^2[D^\alpha y(t))]^2\big] - \frac{axy(D^\alpha x(t))(D^\alpha y(t))}{y^2} \\
&+ \frac{1}{y^2}[y^2x^2]\big[F_2(t,y,D^\alpha y(t))/y - F_1(t,x,D^\alpha x(t))/x\big] + \frac{x}{y}\big[a(D^\alpha x(t))(D^\alpha y(t)) - A(D^\alpha x(t))(D^\alpha y(t))\big] \\
&+ \frac{x}{y}\big[-F_1(t,x,D^\alpha x(t))y + F_2(t,y,D^\alpha y(t))x\big].
\end{aligned}$$

Since

$$\frac{x}{y}\big[-yF_1(t,x,D^\alpha x(t)) + xF_2(t,y,D^\alpha y(t))\big] = \frac{y^2x^2}{y^2}\left[\frac{F_2(t,y,D^\alpha y(t))}{y} - \frac{F_1(t,x,D^\alpha x(t))}{x}\right],$$

and

$$[yD^\alpha x(t) - xD^\alpha y(t)]^2 = [D^\alpha x(t)]^2 y^2 - 2xy[D^\alpha x(t)][D^\alpha y(t)] + x^2[D^\alpha y(t))]^2,$$

we arrive at the nonlinear fractional Picone type identity

$$H(t) = \left[\frac{F_2(t, y, D^\alpha y(t))}{y} - \frac{F_1(t, x, D^\alpha x(t))}{x}\right] x^2 + A\frac{[yD^\alpha x(t) - xD^\alpha y(t)]^2}{y^2} + (a - A)[D^\alpha x(t)]^2. \tag{6}$$

Suppose that $x(t)$ is solution of (1) with $x(t_1) = x(t_2) = 0$, where $t_1 < t_2$ and y is a solution of (2) that does not have a zero in (t_1, t_2). Integrating (5) from t_1 to t_2 gives 0 but from conditions (3) and (4) the integral of the right hand side of (6) is positive. This contradiction proves the following result.

Theorem 2. *If equation* (1) *is oscillatory and conditions* (3) *and* (4) *hold, then Eq.* (2) *is oscillatory.*

If Eq. (1) is replaced by

$$D^\alpha(a(t)(D^\alpha y(t))(t) + F_1(t, y(t), D^\alpha y(t))(t) = E(t, x), \tag{7}$$

where $E : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and all other quantities are the same, it is not hard to see that (6) becomes the nonlinear fractional Picone type identity

$$H(t)=\left[\frac{F_2(t, y, D^\alpha y(t))}{y}-\frac{F_1(t, x, D^\alpha x(t))}{x}\right] x^2+A\frac{[yD^\alpha x(t)-xD^\alpha y(t)]^2}{y^2}+(a - A)[D^\alpha x(t)]^2 + E(t, x(t))x(t). \tag{8}$$

The presence of the term $E(t, x)$ changes the dynamics of the solutions of equation (1). For example, it is possible for equation (7) to have oscillatory solutions (have arbitrarily large zeros) that do not change signs. To see this, consider the very simple example of a second order linear ordinary differential equation (say, $\alpha = 1$ in Definition 1)

$$x'' + x = 1,\ t \geq 0.$$

Here $x(t) = 1 + \sin t$ is an oscillatory solution but $x(t) \geq 0$ for all $t \geq 0$. Such solutions have been referred to as Z-type solutions. As strange as it may seem, this type of solution may have the same asymptotic properties as do nonoscillatory solutions (see, for example, [8, 9]).

Therefore, in what follows we will classify solutions as being *nonoscillatory* if there exists $T > 0$ such that $x(t) \neq 0$ for $t \geq T$, *oscillatory* if for every $t_1 \in \mathbb{R}_+$ there exists $t_2 \in \mathbb{R}_+$ with $t_2 > t_1$ such that $x(t_1)x(t_2) < 0$, or *Z-type* if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

Notice that in an argument used to prove Theorem 2 above in the case where (1) is replaced by (7) and $x(t)$ is a nonnegative Z-type solution, the same contradiction is reached. However, if $x(t)$ is a nonpositive Z-type solution, no contradiction is immediately obtained. This situation can be remedied by the addition of the condition

$$xE(t, x) \geq 0 \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{9}$$

It would then be possible to obtain some corollaries like the following.

Corollary 1 *If conditions* (3), (4), *and* (9) *hold and Eq.* (7) *has an oscillatory or Z-type solution, then every solution of* (2) *is oscillatory or Z-type.*

Corollary 2 *If conditions* (3), (4), *(and* (9)*) hold and Eq.* (2) *has a nonoscillatory solution, then every solution of* (1) *((*7*)) is nonoscillatory.*

4. Conclusions

As we wished to do, we have obtained a generalization of the famous Sturm-Picone theorem to nonlinear fractional differential equations. we were also able to extent the results to equations involving a perturbation term E.

It would be interesting to see if the results here could be obtained for equations of higher order. Additional results for non-homogeneous or perturbed equations would also be of interest. Also, notice that if equations (1) and (2) are linear in u and v, respectively, say $F_1(t, u_1, u_2) = q(t)u(t)g(D^\alpha u)$ and $F_2(t, v_1, v_2) = Q(t)v(t)G(D^\alpha v)$, then condition (4) becomes $q(t)g(D^\alpha u) \leq Q(t)G(D^\alpha v)$, which are seemingly unconnected. This could lead to some interesting situations.

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