Formation of optical solitons to the nonlinear Kariat-X Equation via analytical techniques

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In this research, exact traveling wave solutions of nonlinear Kairat-X model are derived using modified F-expansion method and extended hyperbolic function method. Different solutions to the proposed model has been constructed using these methods. Trigonometric function solutions, soliton solutions, rational solutions and exponential solutions are obtained using modified F-expansion method. The solutions obtained by extended hyperbolic function method are periodic, singular, bright, dark and periodic singular soliton solutions. The obtained results are explained by plotting some graphs in $3D$, $2D$ (line plots) and contour plots using Mathematica which demonstrates the structure of solutions.

Keywords: Solitary waves; modified F-expansion method; extended hyperbolic function method; exact solutions.

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1. Introduction

Nonlinear partial differential equations (PDEs) [1] and linearization techniques are now being studied due to developments in mathematical physics, energy problems and other fields. To study complex nonlinear physical phenomena [2], nonlinear PDEs are crucial mathematical tools, where relationship between variable is not linear. Nonlinear PDEs are commonly used in different scientific fields such as mathematical physics, telecommunication engineering, plasma physics [3, 4] and fiber optics [5]. Recently, in both pure and applied mathematics nonlinear PDEs have become more important. This increasing significant is linked to developments in computer technology, which have made it possible for mathematicians to explore new horizons [6] while working in a variety of applied disciplines.
Many researchers have paid their attention in finding the exact soliton solutions of nonlinear PDEs. The aim of this paper is to investigate nonlinear Kairat-X (K-X) equation [7, 8], which considers the group velocity dispersion and second-order spatiotemporal dispersion. This model has application in many fields such as non-linear optics, optical fibers and ferromagnetic materials. The K-X equation is used to produce soliton solutions as they preserve their shape during propagation, have applications ranging from telecommunications to quantum physics and are stable. This equation explains how optical solitons propagate in non-linear mediums. Prior to this study, there was no research in which such solutions could be discovered. Several mathematical techniques have been developed to solve nonlinear PDEs including generalized Kudryashov's method [9], tanh method [10], sinh-Gordon equation expansion method [11] and φ^6-model expansion method [12], etc. In this paper the purposed model has explored using two analytical techniques, modified F-expansion method (MFEM) [13–15] and extended hyperbolic function method (EHFM) [16–18].

MFEM is a powerful mathematical technique used to find approximate solutions to nonlinear PDEs. On the other hand EHFM provides variety of solutions such as hyperbolic solutions, rational functional solutions bright and dark soliton solutions and jacobi elliptic solutions. Both the extended hyperbolic function approach (EHFM) and the modified F-expansion approach offer more expansive and adaptable solution structures than the conventional tanh method, φ^6-model expansion method and G'/G-expansion method. For producing trigonometric, hyperbolic, rational, and Jacobi elliptic solutions, the Modified F-expansion approach is especially effective; however, its algebraic structure may become complicated. The EHFM is less efficient for elliptic or multi-soliton solutions, but it is easier to use and produces a variety of hyperbolic-type solutions (solitons, Kinks, singular waves). As a result, both techniques improve upon current methods by providing richer solution families at the expense of increased algebraic complexity.

The remaining detail steps of attaining a solution to the above-mentioned problem have been segmented into following subsequent sections: Section 2 gives governing model. Description and application of MFEM are discussed in Sec. 3 and Sec. 4 respectively. Description and application of EHFM are discussed in Sec. 5 and Sec. 6 respectively. Section 7 describes results and discussions. The conclusion of the article is in last section.

2. Govering model

The K-X equation [7,19,20] is given as follows:

$$W_{tt} + W_{xxt} - 3(W_x W_t)_x = 0, \tag{1}$$

where $W = W(x,t)$. In both time (t) and space (x) the function changes and describes the behavior of system. The considered model defines curve's surface geometry and has applications in optical communication, quantum mechanics, optical fibers and more.

Following transformation can be used to drive the soliton solutions of Eq. (1):

$$W(x,t) = V(\xi), \xi = kx + \lambda t, \tag{2}$$

where $V(\xi)$ is pulse rate, wave number is denoted by k and λ represents the wave velocity. Following equation is obtained on substituting Eq.(2) into Eq.(1):

$$k^3 V''' + \lambda V'' - 3k^2 {(V'^2)}' = 0. \tag{3}$$

By integrating Eq.(3) with respect to λ, the obtained result is:

$$k^3 V''' + \lambda V' - 3k^2 V'^2 + c = 0, \tag{4}$$

where c is the real integration constant.

Using the substitution $V'(\xi) = U(\xi)$, where $U(\xi)$ is real valued function, Eq. (4) transforms into following ordinary differential equation (ODE):

$$k^3 U'' + \lambda U - 3k^2 U^2 + c = 0. \tag{5}$$

3. Modified F-expansion method

A general nonlinear PDE with dependent variable u and independent variables $(x_1, x_2, ..., x_i, t)$ is following:

$$H(u, D_t u, D_x u, D_t D_t u, D_t D_{x_1} u, ...) = 0, \tag{6}$$

where the polynomial function of $u(x_1, x_2, ..., x_i, t)$ is represented by H. Following steps can be taken to find the exact solution using modified F-expansion method:

Step 1. Take following transformation to find a traveling wave solution of Eq. (6)

$$u(x_1, x_2, ..., x_i, t) = u(\xi),$$
$$\xi = c_1 x_1 + c_2 x_2 + ... + c_i x_i + \nu t, \tag{7}$$

where $c_1, c_2, ..., c_i, \nu$ are constants, need to be determined. Substituting Eq. (7) into Eq.(6), an ODE for $u(\xi)$is obtained

$$D(u, u', u'', ...) = 0. \tag{8}$$

Step 2. Assume following algebraic expression for $u(\xi)$.

$$u(\xi) = h_0 + \sum_{j=1}^{M} \left(h_j R^j(\xi) + h_{-j} R^{-j}(\xi) \right),$$
$$(h_M \neq 0), \tag{9}$$

where $h_0,\ h_j,\ h_{-j}$ are constants. The balancing number that results from balancing the nonlinear terms in Eq. (7) and the highest order derivatives of $u(\xi)$ is represented by the integer M. The Riccati equation is satisfied by $R(\xi)$

$$R'(\xi) = E + FR(\xi) + GR^2(\xi), \quad (G \neq 0), \tag{10}$$

where $E,\ F,\ G$ are constants.

Step 3. Substitute Eq. (9) into Eq. (8) using Eq. (10) to turn the left-hand side of Eq. (8) into a finite series in $R^n(\xi), n = 0, \pm1, \pm2, ..., \pm M$. $R^n(\xi)$ has coefficients of power that can all be equal to zero to produce an algebraic system of equations.

Step 4.The resulting system of algebraic equations with unknown parameters $h_j,\ h_{-j},\ c_r(r = 1, 2, ..., i),\ u$ was solved in multiple families using **MATHEMATICA** or **MAPLE**.

Step 5. From Table.1, the exact solutions such as trigonometric function, soliton-like, rational and exponential solutions of Eq. (6) can be obtained.

4. Application of MFEM

MFEM is applied in this section to derive travelling wave solutions for Eq. (5). Considering homogeneous balance principle in Eq. (5), it yields to $M = 2$ and therefore, the solution to Eq. (5) is supposed to be:

$$q(\xi) = h_0 + h_1 R(\xi) + \frac{h_{-1}}{R(\xi)} + h_2 R(\xi)^2 + \frac{h_{-2}}{R(\xi)^2},$$

where $h_0,\ h_1,\ h_{-1}, h_{-2}$ are constants to be determined and $\xi = kx + \lambda t$. For simplicity, take $h_{-1} = h_2$ and $h_{-2} = h_3$. So the above equation becomes:

$$q(\xi) = h_0 + h_1 R(\xi) + \frac{h_2}{R(\xi)} + \frac{h_3}{R(\xi)^2}. \tag{11}$$

Equation (5) is transformed into a finite series in $R^n(\xi)(n = -4, -3, -2, -1, 0, 1, 2, 3, 4)$ by substituting Eq. (11) into Eq. (5) and using Eq. (10). Equating every coefficient of power of $R^n(\xi)$ to zero results in the system of algebraic equations that follow:

$$R^0 : -3h_0^2 k^2 + h_0 \lambda = 0,$$
$$R^1 : -6h_0 h_1 k^2 + g h_1 k^3 + h_1 \lambda = 0,$$
$$R^2 : -3h_1^2 k^2 - 6h_0 h_2 k^2 + 4g h_2 k^3 + h_2 \lambda = 0,$$
$$R^3 : -6h_1 h_2 k^2 + 2h h_1 k^3 = 0,$$
$$R^4 : -3h_2^2 k^2 + 6h h_2 k^3 = 0.$$

MATHEMATICA is used to solve above system to get following solutions for $h_0,\ h_1,\ h_2$:

Set 1.

$$h_0 = 2EGk, \quad h_1 = 2FGk, \quad h_2 = 2c^2k,$$
$$g_1 = 0, \quad g_2 = 0, \quad \lambda = -(F^2 - 4EG)k^3.$$

Set 2.

$$h_0 = \frac{1}{3}(F^2 + 2EG)k, \quad h_1 = 2FGk, \quad h_2 = 2c^2k,$$
$$g_1 = 0, \quad g_2 = 0, \quad \lambda = F^2k^3 - 4EGk^3.$$

The following section provides solutions to Eq. (5) for **sets** 1 and 2. As a result, Eq. (1) yields numerous soliton-like, trigonometric function, rational, and exponential solutions.

4.1. The soliton-like solutions to K-X equation

- When $E = 0, F = 1, G = -1$ then from Table I, $R(\xi) = (1/2) + (1/2)\tanh([1/2]\xi)$. Using **set** 1 and **set** 2, the exact solutions to Eq. (1) are found as:

$$Y_1 = \frac{k}{3} - 2k\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right) \pm 2k\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right)^2.$$

- When $E = 0, F = -1, G = 1$ then from Table I, $R(\xi) = (1/2) - (1/2)\coth([1/2]\xi)$. Employing **set** 1, exact solution to Eq. (1) is extracted as:

$$Y_2 = -2k\left(\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right) + 2k\left(\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right)^2.$$

- When $E = 1/2$, $F = 0$, $G = -1/2$ then from Table I, $R(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi)$ or $\tanh(\xi) \pm \imath\operatorname{sech}(\xi)$. By **set** 1, the exact solution to Eq. (1) are found as:

$$Y_3 = -\frac{k}{2} + \frac{1}{2}k\left(\coth(\xi) + \operatorname{csch}(\xi)\right)^2,$$
$$Y_4 = -\frac{k}{2} + \frac{1}{2}k\left(\tanh(\xi) \pm \imath\operatorname{sech}(\xi)\right)^2.$$

Employing **set** 2, the exact solutions to Eq. (1) are given as:

$$Y_5 = -\frac{k}{6} + \frac{1}{2}k\left(\coth(\xi) + \operatorname{csch}(\xi)\right)^2,$$
$$Y_6 = -\frac{k}{6} + \frac{1}{2}k\left(\tanh(\xi) \pm \imath\operatorname{sech}(\xi)\right)^2.$$

- When $E = 1, F = 0, G = -1$ then from Table I, $R(\xi) = \tanh(\xi)$ or $\coth(\xi)$. By **set** 1, the exact solutions to Eq. (1) are given as:

$$Y_7 = -2k + 2k\tanh(\xi)^2,$$
$$Y_8 = -2k + 2k\coth(\xi)^2.$$

By **set** 2, the exact solutions to Eq.(1) are obtained as:

$$Y_9 = -\frac{2k}{3} + 2k\tanh(\xi)^2,$$
$$Y_{10} = -\frac{2k}{3} + 2k\coth(\xi)^2,$$

where $\xi = kx + \lambda t$.

4.2. The trigonometric function solutions to K-X equation

When $E = 1/2$, $F = 0$, $G = 1/2$ then from Table I, $R(\xi) = \sec(\xi) + \tan(\xi)$ or $\csc(\xi) - \cot(\xi)$. Using **set** 1, the exact solutions to Eq. (1) are obtained as:

TABLE I. Relation between the values of parameters: E, F, G and the corresponding solutions of $R(\xi)$ in Eq. (10)(the Riccati's equation).

E	F	G	$R(\xi)$
0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$
0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$
$\frac{1}{2}$	0	-$\frac{1}{2}$	$\coth(\xi) \pm \operatorname{csch}(\xi)$ or $\tanh(\xi) \pm \imath\operatorname{sech}(\xi)$
1	0	-1	$\tanh(\xi)$ or $\coth(\xi)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec(\xi) + \tan(\xi)$ or $\csc(\xi) - \cot(\xi)$
-$\frac{1}{2}$	0	-$\frac{1}{2}$	$\sec(\xi) - \tan(\xi)$ or $\csc(\xi) + \cot(\xi)$
1(-1)	0	1(-1)	$\tan(\xi)(\cot(\xi))$
0	0	$\neq 0$	$-\frac{1}{C\xi+\lambda}$(λ is an arbitrary constant)
arbitrary constant	0	0	$A\xi$
arbitrary constant	$\neq 0$	0	$\frac{\exp(B\xi)-A}{B}$

$$Y_{11} = \frac{k}{2} + \frac{1}{2}k\left(\sec(\xi) + \tan(\xi)\right)^2,$$

$$Y_{12} = \frac{k}{2} + \frac{1}{2}k\left(\csc(\xi) - \cot(\xi)\right)^2.$$

set 2 yields the exact solutions to Eq. (1) as follows:

$$Y_{13} = \frac{k}{6} + \frac{1}{2}k\left(\sec(\xi) + \tan(\xi)\right)^2,$$

$$Y_{14} = \frac{k}{6} + \frac{1}{2}k\left(\csc(\xi) - \cot(\xi)\right)^2.$$

When $E = -1/2$, $F = 0$, $G = -1/2$ then from Table I, $R(\xi) = \sec(\xi) - \tan(\xi)$ or $\csc(\xi) + \cot(\xi)$. Using **set** 1, the exact solutions to Eq. (1) are derived as:

$$Y_{15} = \frac{k}{2} + \frac{1}{2}k\left(\sec(\xi) - \tan(\xi)\right)^2,$$

$$Y_{16} = \frac{k}{2} + \frac{1}{2}k\left(\csc(\xi) + \cot(\xi)\right)^2.$$

set 2 yields the exact solutions to Eq. (1) as follows:

$$Y_{17} = \frac{k}{6} + \frac{1}{2}k\left(\sec(\xi) - \tan(\xi)\right)^2,$$

$$Y_{18} = \frac{k}{6} + \frac{1}{2}k\left(\csc(\xi) + \cot(\xi)\right)^2.$$

When $E = -1, F = 0$, $G = -1$ then from Table I, $R(\xi) = \cot(\xi)$.

set 1 yields the exact solutions to Eq. (1) as follows:

$$Y_{19} = 2k + 2k\cot(\xi)^2.$$

By **set** 2, the exact solution to Eq. (1) is found as:

$$Y_{20} = \frac{2k}{3} + 2k\cot(\xi)^2,$$

where $\xi = kx + \lambda t$.

4.3. The rational solutions to K-X equation

- When $E = F = 0$, $G \neq 0$ then from Table I, $R(\xi) = -(1/[C\xi + \lambda])$ (λ is an arbitrary constant). Employing **set** 1 and **set** 2 , the exact solution to Eq. (1) is found as:

$$Y_{21} = \frac{Gk}{\xi^2},$$

where G is an arbitrary constant and $\xi = kx + \lambda t$.

4.4. The exponential solutions to K-X equation

- When $F \neq 0$, $G = 0$, and E is an arbitrary constant then from Table I, $R(\xi) = \exp(B\xi) - A/B$. By **set** 2, the exact solution to Eq. (1) is extracted as:

$$Y_{22} = \frac{Fk}{3},$$

where $\xi = kx + \lambda t$.

5. Extended hyperbolic function method

This section contains a detailed description of the extended hyperbolic function (EHFM) approach.

Consider the nonlinear PDE and the wave transformation that converts it into an ODE, as described in Eqs. (6)-(8). The EHFM describes the solution of Eq. (8) as:

$$u(\xi) = \sum_{j=0}^{M} h_j N(\xi), \tag{12}$$

where M is obtained by homogeneous balancing rule and $h_j \neq 0$, $F(\xi)$ is a real function.

Now, the following types of ODEs are satisfied by $N(\xi)$:

Type 1:

$$N'(\xi) = N\sqrt{r + sN^2}, \quad r, s \in \mathbb{R}. \tag{13}$$

For above equation following solutions are obtained:

Case 1: When $r > 0$, $s > 0$, then

$$N_1(\xi) = -\sqrt{\frac{r}{s}}\operatorname{csch}\sqrt{r}(\xi).$$

Case 2: When $r < 0$, $s > 0$, then

$$N_2(\xi) = \sqrt{\frac{-r}{s}}\sec\sqrt{-r}(\xi).$$

Case 3: When $r > 0$, $s < 0$, then

$$N_3(\xi) = \sqrt{\frac{r}{-s}}\operatorname{sech}\sqrt{r}(\xi).$$

Case 4: When $r < 0$, $s < 0$, then

$$N_4(\xi) = \sqrt{\frac{-r}{s}}\csc\sqrt{-r}(\xi).$$

Case 5: When $r < 0$, $s > 0$, then

$$N_5(\xi) = \cos\sqrt{-r}(\xi) + \iota\sin\sqrt{-r}(\xi).$$

Case 6: When $r = 0$, $s > 0$, then

$$N_6(\xi) = \frac{1}{\sqrt{s}(\xi)}.$$

Case 7: When $r = 0$, $s < 0$, then

$$N_7(\xi) = \frac{1}{\sqrt{-s}(\xi)},$$

where $\xi = kx + \beta t$.

Type 2:

$$N'(\xi) = r + sN^2, \quad r, s \in \mathbb{R}. \tag{14}$$

For above equation following solutions are obtained:

Case 1: When $rs > 0$, then

$$N_8(\xi) = sgn(r)\sqrt{\frac{r}{s}}\tan(\sqrt{rs}(\xi)).$$

Case 2: When $rs > 0$, then

$$N_9(\xi) = -sgn(r)\sqrt{\frac{r}{s}}\cot(\sqrt{rs}(\xi)).$$

Case 3: When $rs < 0$, then

$$N_{10}(\xi) = sgn(r)\sqrt{\frac{-r}{s}}\tanh(\sqrt{-rs}(\xi)).$$

Case 4: When $rs < 0$, then

$$N_{11}(\xi) = sgn(r)\sqrt{\frac{-r}{s}}\coth(\sqrt{-rs}(\xi)).$$

Case 5: When $r = 0,\ s > 0$, then

$$N_{12}(\xi) = -\frac{1}{s(\xi)}.$$

Case 6: When $r < 0,\ s = 0$, then

$$N_{13}(\xi) = s(\xi),$$

where $\xi = kx + \lambda t$.

Insert Eq. (12) into Eq. (8) and use Eqs. (13) and (14) to obtain the system of algebraic equations. **Mathematica** may be used to solve the system and extract constant values.

6. Application of EHFM

EHFM is applied to Eq. (5) in this section.

Type 1: By applying the homogeneous balancing rule, yields $M = 2$ for Eq. (5). The solution of the Eq. (5) using EHFM takes the form

$$q(\xi) = h_0 + h_1 N(\xi) + h_2 N^2(\xi), \tag{15}$$

where $h_0,\ h_1$ and h_2 are constants to be determined and $\xi = kx + \lambda t$. Substituting Eq. (13) and Eq. (15) into Eq. (5), Eq. (5) is converted into a finite series in $N^i(\xi)$ $(i = 1, 2, 3)$, by equating each coefficient of power of $N^i(\xi)$ equal to zero yields a system of algebraic equations for $h_0,\ h_1,\ h_2$. After solving this system, following set is obtained:

Set 3.

$$h_0 = \frac{4rk}{3}, \quad h_1 = 0, \quad h_2 = 2sk, \quad \lambda = 4rk^3.$$

By using these values of $h_0,\ h_1,\ h_2$ in the solutions of Eq. (12), following results are obtained:

Case 1: When $r > 0,\ s > 0$, then

$$Y_{23} = \frac{4rk}{3} + 2rk\,\text{csch}(\sqrt{r}\xi)^2.$$

Case 2: When $r < 0,\ s > 0$, then

$$Y_{24} = \frac{4rk}{3} - 2rk\sec(\sqrt{-r}\xi)^2.$$

Case 3: When $r > 0,\ s < 0$, then

$$Y_{25} = \frac{4rk}{3} - 2rk\,\text{sech}(\sqrt{r}\xi)^2.$$

Case 4: When $r < 0,\ s < 0$, then

$$Y_{26} = \frac{4rk}{3} - 2rk\csc(\sqrt{-r}\xi)^2.$$

where $\xi = kx + \lambda t$.

Type 2: Similarly, by substituting Eq. (14) and Eq. (15) into Eq. (5), a system of equations is obtained. By solving this system, the following set is obtained:

Set 4.

$$h_0 = 2rsk, \quad h_1 = 0, \quad h_2 = 2s^2k, \quad \lambda = 4rk^3.$$

Using these values of $h_0,\ h_1,\ h_2$ in the solutions of Eq. (15), following results are obtained:

Case 1: When $rs > 0$, then

$$Y_{27} = 2rsk + 2rsk\tan(\sqrt{rs}x)^2.$$

Case 2: When $rs > 0$, then

$$Y_{28} = 2rsk + 2rsk\cot(\sqrt{rs}x)^2.$$

Case 3: When $rs < 0$, then

$$Y_{29} = 2rsk - 2rsk\tanh(\sqrt{-rs}x)^2.$$

Case 4: When $rs < 0$, then

$$Y_{30} = 2rsk - 2rsk\coth(\sqrt{-rs}x)^2.$$

Case 5: When $r = 0,\ s > 0$, then the following rational solution is calculated as:

$$Y_{31} = 2rsk + \frac{2s^2k}{sx^2}.$$

Case 6: When $r < 0,\ s = 0$, then

$$Y_{32} = 2rsk + 2rx^2s^2k,$$

where $\xi = kx + \lambda t$.

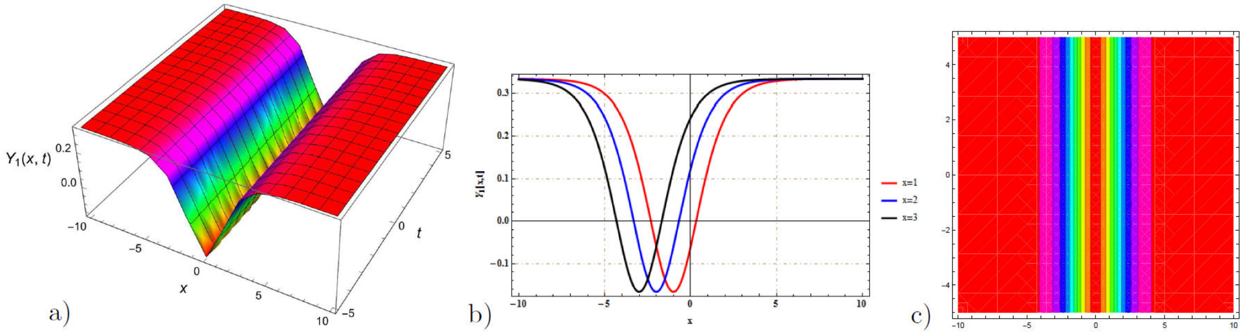

FIGURE 1. 3D, 2D and contour plot representations of $Y_1(x,t)$ for $k = 1$.

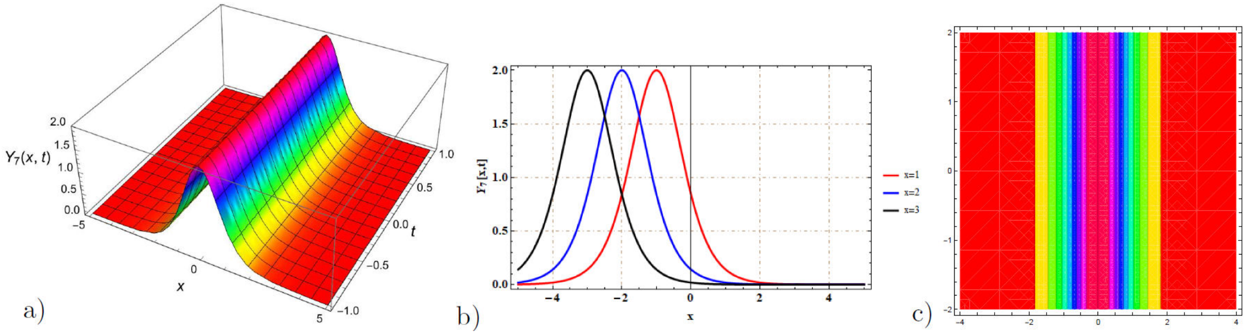

FIGURE 2. 3D, 2D and contour plot representations of $Y_7(x,t)$ for $k = -1$.

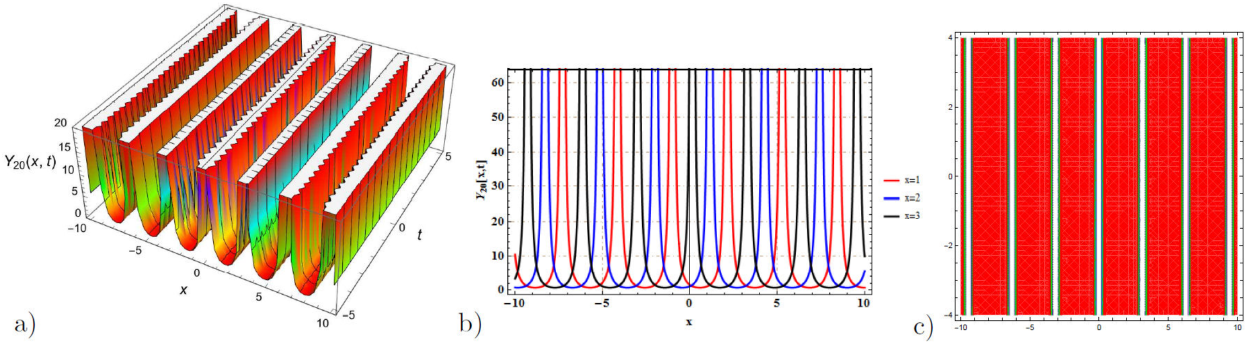

FIGURE 3. 3D, 2D and contour plot representations of $Y_{20}(x,t)$ for $k = 1$.

7. Results and discussion

This section offers a thorough explanation of the outcomes that motivated this work. Firstly, MFEM and EHFM are used in Sec. 4 and Sec. 6 respectively to get the traveling wave solutions of nonlinear kariat-X equation. These techniques have free or relevant arbitrary constants and comprises various solutions in the form of mathematical expressions. This section also contains graphs of some solutions. Figures 1 to 3 represent the $3D$, $2D$ and contour plots corresponding to $Y_1(x,t)$, $Y_7(x,t)$ and $Y_{20}(x,t)$. Subgraph (a) represents the $3D$ plot while the subgraph (b) represents the $2D$ plot and subgraph (d) represents the contour plot. Similarly Figs. 4-6 represent the $3D$, $2D$ and contour plots corresponding to $Y_{23}(x,t)$, $Y_{25}(x,t)$ and $Y_{27}(x,t)$. In Fig. 1, periodic soliton solutions are obtained corresponding to $Y_1(x,t)$. Bright soliton solutions corresponding to $Y_3(x,t)$ are given in Figs. 2 and 3 gives periodic singular soliton solutions that indicate a repeating process corresponding to $Y_{20}(x,t)$. In Fig. 4 singular soliton solutions corresponding to $Y_{23}(x,t)$ are given. Figure 5 represents bright soliton solutions against $Y_{25}(x,t)$. Bright soliton solutions describes the localised intensity peak above the continuous wave background. Periodic singular soliton solutions against $Y_{27}(x,t)$ are obtained in Fig. 6. The solutions generated for the solitons are novel, unique, and extremely successful, and they have never been used in the existing literature. The suggested methods are also simple and useful for investigating various nonlinear equations. They highlight the solution's physical properties and are being used for the first time on the suggested model.

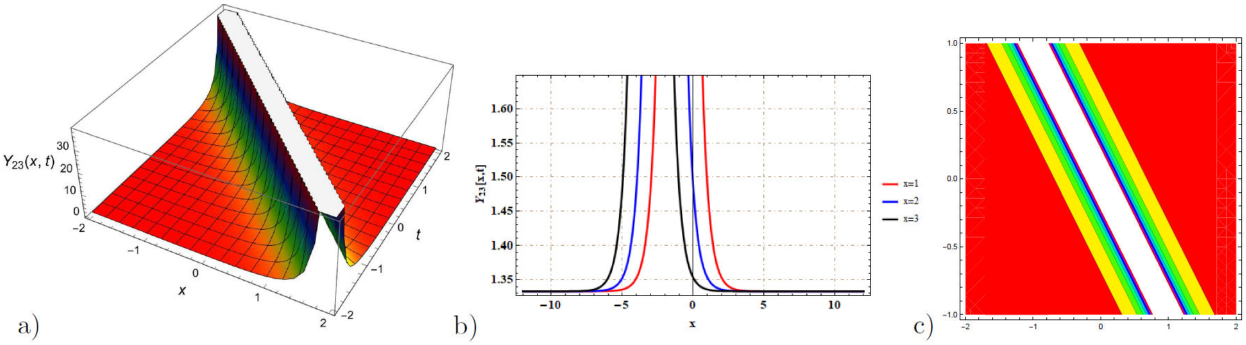

FIGURE 4. 3D, 2D and contour plot representations of $Y_{23}(x,t)$ for $k = 1$, $g = 1$.

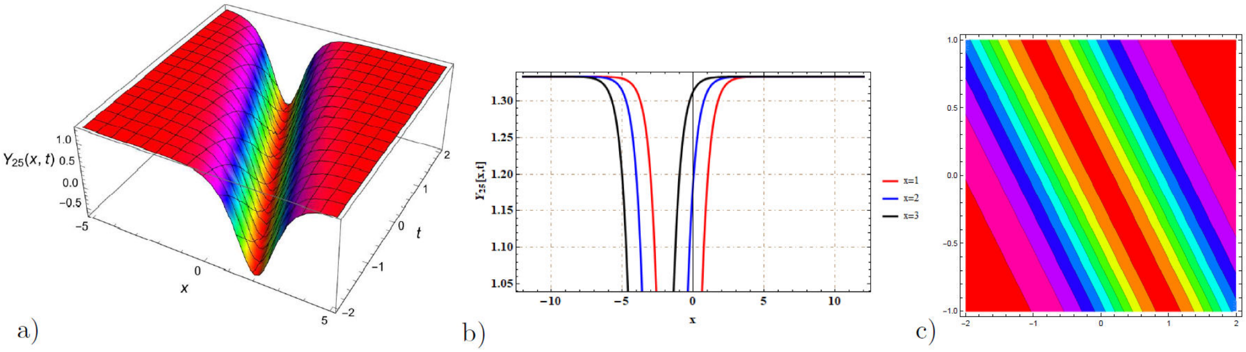

FIGURE 5. 3D, 2D and contour plot representations of $Y_{25}(x,t)$ for $k = 1$, $g = 1$.

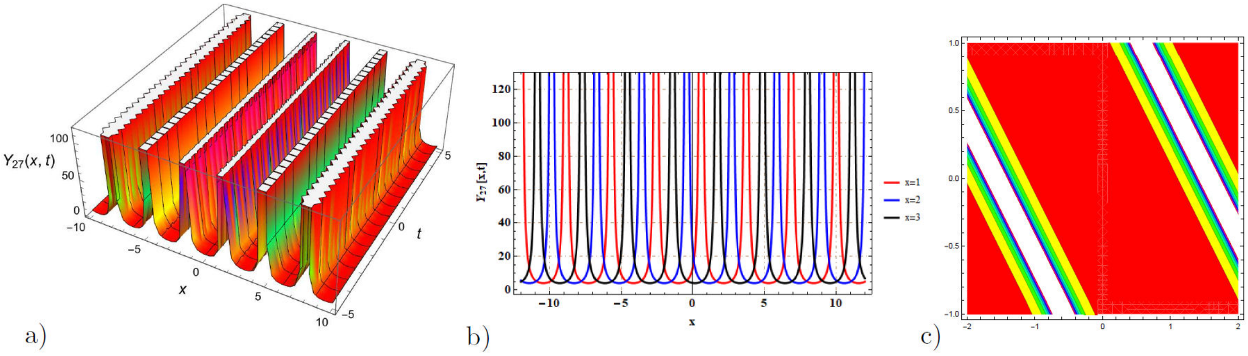

FIGURE 6. 3D, 2D and contour plot representations of $Y_{27}(x,t)$ for $k = 2$, $g = 1, h = 1$.

8. Conclusion

In this paper, two precise approaches, modified F-expansion method and extended hyperbolic function method, have been used to get exact traveling wave solutions to the nonlinear Kairat-X problem. These methods have not previously been applied to this model and are regarded as the most modern strategies for getting the suggested model's soliton solutions. These solutions add to one's understanding of soliton phenomena and are novel contributions to the field. This research has uncovered a mesmerizing range of solitons, each with unique features. The approaches used are straightforward and steady. Many new solutions including hyperbolic, trigonometric, exponential functions and rational are discovered. Graphical behaviour of some of generated solutions are explored using $3D$, $2D$ and contour plots. The exact solutions of nonlinear Kairat-X equations comprises periodic, singular, dark, and periodic singular soliton solutions. In telecommunications industry and optical fibers, the findings of this paper are extremely important. In future, these strategies can be used to get solutions for numerous models. This model can be studied by different ways in future, thus there is still some gaps have to be covered.

Declarations

Ethics approval

Not applicable.

Competing interests

The authors declare that they have no conflict of interests.

Authors' contributions

GA participated in the conceptualization, investigation, methodology, supervision, validation and visualization. MS participated in the conceptualization, administration, validation and visualization of the manuscript. SA participated in the investigation, software implementation, methodology, review and editing of the manuscript. PB participated in the data curation, formal analysis, methodology, software implementation, and writing of the manuscript. AB participated in the investigation, software implementation, review and editing of the manuscript. All authors read and approved the final manuscript.

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